J. Laub
Nagoya Math. J.
Vol. 83 (1981), 1–4

A SINGULAR CONVOLUTION KERNEL WITHOUT PSEUDO-PERIODS

JESPER LAUB

Let G be a locally compact abelian group and N a non-zero convolution kernel on G satisfying the domination principle. We define the cone of N-excessive measures $E(N)$ to be the set of positive measures ξ for which N satisfies the relative domination principle with respect to ξ. For $\xi \in E(N)$ and $\Omega \subseteq G$ open the reduced measure of ξ over Ω is defined as

$$R_\xi^\Omega = \inf\{\eta \in E(N) \mid \eta \geq \xi \text{ in } \Omega\}.$$

Further discussion of excessive and reduced measures is given in [4] and [5].

Let $\mathcal{V}$ denote the set of compact neighbourhoods of O, the neutral element of G. The convolution kernel N is said to be *singular* if

$$R_N^{\complement V} = N \text{ for all } V \in \mathcal{V}.$$

A point $x \in G$ is called a *pseudo-period* of N if there exists a number $c > 0$ such that

$$N * \varepsilon_x = cN,$$

where ε_x denotes the Dirac-measure at x. The set of pseudo-periods of N is a closed subgroup of G.

In [3] Itô gave the following result (Corollaire 2):

A convolution kernel N satisfying the domination principle is singular if and only if the group of pseudo-periods of N is non-compact.

The "if" part of the statement is easy to prove (cf. e.g. [1]), but the "only if" statement is false in general, although it seems reasonable due to obvious examples. It is our purpose to give a counterexample to this statement.

Suppose that there exists a strictly decreasing sequence $(G_n)_{n \in N}$ of closed non-compact subgroups of G

$$G = G_1 \supset G_2 \supset G_3 \supset \cdots$$

satisfying $\bigcap_{n=1}^{\infty} G_n = \{0\}$. We denote by ω_{G_n} a Haar-measure on G_n. Let φ be a fixed non-zero positive continuous function with compact support and put $a_n = \sup_{x \in G} \omega_{G_n} * \varphi(x)$, $n \in N$.

The convolution kernel, which we will consider, is

$$\kappa = \sum_{n=1}^{\infty} \frac{1}{2^n a_n} \omega_{G_n} .$$

Since every positive continuous function with compact support can be majorized by a finite linear combination of translates of φ, it follows that the series converges vaguely. Furthermore κ is shift-bounded.

1°. The only pseudoperiod of κ is 0.

Since κ is shift-bounded, we have $c = 1$ for a pseudo-period $x \in G$ of κ. If $x \neq 0$, then we can find $i \in N$ such that $x \in G_i \backslash G_{i+1}$ and therefore

$$\kappa * \varepsilon_x = \sum_{n=1}^{i} \frac{1}{2^n a_n} \omega_{G_n} + \sum_{n=i+1}^{\infty} \frac{1}{2^n a_n} \omega_{G_n} * \varepsilon_x$$

$$\kappa = \sum_{n=1}^{i} \frac{1}{2^n a_n} \omega_{G_n} + \sum_{n=i+1}^{\infty} \frac{1}{2^n a_n} \omega_{G_n} .$$

These two expressions cannot be equal, since x belongs to the support of the second term of $\kappa * \varepsilon_x$, but not to support of the second term of κ.

2°. κ satisfies the domination principle.

We shall need the following two lemmas, which are both easily proved

LEMMA 1 (Itô [2]). *Let N be a shift-bounded convolution kernel and ω_G a Haar-measure on G. If N satisfies the domination principle, then $N + \omega_G$ satisfies the domination principle.*

LEMMA 2. *Let N be a convolution kernel on G and H a closed subgroup of G such that* $\operatorname{supp} N \subseteq H$. *Then N satisfies the domination principle as convolution kernel on G if and only if N satisfies the domination principle as convolution kernel on H.*

By repeated use of these lemmas it follows, that the partial sum

$$\kappa_k = \sum_{n=1}^{k} \frac{1}{2^n a_n} \omega_{G_n} , \qquad k \in N$$

satisfies the domination principle. Since the set of convolution kernels satisfying the domination principle is vaguely closed and $\kappa = \lim_{k \to \infty} \kappa_k$, we have that κ satisfies the domination principle.

3°. κ is singular.

Let $V \in \mathcal{V}$ be given and choose for $i \in N$ a point $x_i \in G_i \backslash G_{i+1}$ such that $x_i \notin V - \operatorname{supp} \varphi$. Then we have

$$\kappa * \varepsilon_{x_i} * \varphi = \sum_{n=1}^{i} \frac{1}{2^n a_n} \omega_{G_n} * \varphi + \sum_{n=i+1}^{\infty} \frac{1}{2^n a_n} \varepsilon_{x_i} * \omega_{G_n} * \varphi$$
$$\leq R_{\kappa*\varphi}^{\mathscr{C}V} + 2^{-i} \text{ in } \mathscr{C}V$$

However since $\operatorname{supp}(\varepsilon_{x_i} * \varphi) \subseteq \mathscr{C}V$ and $R_{\kappa*\varphi}^{\mathscr{C}V} + 2^{-i} \in E(\kappa)$ we obtain

$$\sum_{n=1}^{i} \frac{1}{2^n a_n} \omega_{G_n} * \varphi \leq \kappa * \varepsilon_{x_i} * \varphi \leq R_{\kappa*\varphi}^{\mathscr{C}V} + 2^{-i},$$

and by letting i tend to infinity we get $R_{\kappa*\varphi}^{\mathscr{C}V} = \kappa * \varphi$. Finally Lemma 1.8 in [5] gives

$$\kappa * \varphi = \lim_{V \uparrow G} R_{\kappa*\varphi}^{\mathscr{C}V} = \left(\lim_{V \uparrow G} R_{\kappa}^{\mathscr{C}V}\right) * \varphi$$

which shows that $R_{\kappa}^{\mathscr{C}V} = \kappa$ for all $V \in \mathcal{V}$.

EXAMPLE. For $G = Z$, $G_n = 2^{n-1} Z = \{2^{n-1}k \mid k \in Z\}$ and φ the function which takes the value 1 at 0 and 0 elsewhere we get

$$\kappa(\{0\}) = 1;\ \kappa(\{m\}) = 1 - 2^{-i-1},\ m \neq 0$$

where i is the largest non-negative integer for which 2^i divides m.

Remark. If a singular convolution kernel N satisfies the balayage principle for all open sets, then the group of pseudo-periods of N is non-compact, because if $\varepsilon'_{\mathscr{C}V}$ denotes a N-balayaged measure of ε_0 on $\mathscr{C}V$, $V \in \mathcal{V}$, then we have $N = N * \varepsilon'_{\mathscr{C}V}$. Consequently N has a pseudo-period in $\operatorname{supp} \varepsilon'_{\mathscr{C}V} \subseteq \overline{\mathscr{C}V}$ by Proposition 7 in [3].

REFERENCES

[1] Berg, C. and Laub, J., The resolvent for a convolution kernel satisfying the domination principle, Preprint Series 1977 No. 41, Dept. of Math., Univ. of Copenhagen.
[2] Itô, M., Une caractérisation du principe de domination pour les noyaux de convolution, Japan J. Math., New series **1**, No. 1 (1975), 5–35.
[3] ——, Caractérisation du principe de domination pour les noyaux de convolution non-bornés, Nagoya Math., J., **57** (1975), 167–197.
[4] ——, Sur le principe relatif de domination pour les noyaux de convolution, Hiroshima Math. J., **5** (1975), 293–350.

[5] Laub, J., On unicity of the Riesz decomposition of an excessive measure, Math. Scand., to appear.

Matematisk Institut
Universitetsparken 5
DK-2100 København Ø
Denmark