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BROWNIAN MOTION PARAMETRIZED WITH METRIC SPACE OF CONSTANT CURVATURE

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Introduction

P. Lévy introduced a generalized notion of Brownian motion in his monograph "Processus stochastiques et mouvement brownien" by taking the time parameter space to be a general metric space. Let (M, d) be a metric space and let O be a fixed point of M called the origin. Following his definition, *a Brownian motion parametrized with the metric space* (M, d) is a Gaussian system $\mathscr{B} = \{B(m); m \in M\}$ such that the difference $B(m) - B(m')$ is a random variable with mean zero and variance $d(m, m')$, and that $B(O) = 0$.

There does not always exist a Brownian motion with (M, d)-parameter for an arbitrary metric space (M, d). To show the existence of a Brownian motion there exist two methods (A) and (B) as follows:

(A) A Method Based upon the Positive Definiteness of the Function v Defined in (i). Applying the general theory of Gaussian systems to our problem, a necessary and sufficient condition for a Brownian motion $\mathscr{B}$ to exist is that the function

$$\text{(i)} \qquad v(m, m') = \tfrac{1}{2}(d(O, m) + d(O, m') - d(m, m'))$$

is positive definite.

In case the n-dimensional Euclidean space R^n is taken to be the parameter space (M, d), P. Lévy showed that the function v in (i) is positive definite (P. Lévy [8]). R. Gangolli discussed the case where M is a homogeneous space. He has given an equivalent condition of the positive definiteness of v in terms of spherical harmonic functions and has shown that the function v is positive definite in the case of n-dimensional sphere S^n(R. Gangolli [3]). According to his line, Г. М. Молчан has shown that in the case of hyperbolic space the function v is positive definite (Г. М.

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Молчан [10]).

(B) A Constructive Method Using White Noise.

The ordinary 1-parameter Brownian motion $\{B(t); t \geq 0\}$ can be represented in the following integral form with respect to the *white noise*;

$$\text{(ii)} \qquad B(t) = \int_0^t dB(s) .$$

The *white noise* $\{dB/dt\}$ has the simplest dependency, that is, the white noise is *independent at every point.* Availing this property, P. Lévy constructed the Brownian motion with S^n-parameter from the white noise on S^n. H. H. Ченцов also constructed the Brownian motion with R^n-parameter. In this case, the white noise is defined on the set

$$\text{(iii)} \qquad \mathscr{H} = \{\text{hyperplane of codimension 1 of } R^n\}$$

(H. H. Ченцов [12]).

In this paper three Riemannian spaces Q with constant curvature—the sphere S^n, the Euclidean space R^n and the hyperbolic space H^n are taken as the parameter space (M, d). By the natural way these three spaces can be realized in the real projective space P^n. Our main result is that the Brownian motion with parameter space S^n, R^n, or H^n can be represented by an integration of a white noise in the same manner. The key point of our method is to construct the white noise using the fact that the set of all totally geodesic submanifolds of codimension 1 of Q can be identified with a subset of the set of all hyperplanes in P^n by the natural way. For any point A of Q, set

$$\text{(iv)} \qquad S_A = \begin{Bmatrix} \text{hyperplane which separates the point } A \text{ of } Q \text{ and} \\ \text{the origin } O \text{ in } P^n - \ell_\infty \end{Bmatrix},$$

where ℓ_∞ denotes the hyperplane of P^n at infinity. Then the Brownian motion can be represented by

$$\text{(v)} \qquad B(A) = \int_{S_A} \xi_x \sqrt{d\mu(x)} ,$$

where $\{\xi_x \sqrt{d\mu(x)}\}$ is a white noise defined on the set $\mathscr{H}$. If Q is S^n or R^n, this representation coincides with that of P. Lévy or that of H. H. Ченцов.

As an application of the representation (v), we obtain the canonical representation of the M_t-process which is defined as

$$M_t = \int_{|x|=t} B(x)dS(x) ,$$

where dS is the normalized uniform measure on the sphere S^{n-1}.

§1. Preliminaries

Let (M, d) be a metric space with metric $d(\cdot, \cdot)$ and O be an arbitrary but fixed origin of M.

DEFINITION. A Gaussian system $\{B(m, \omega); m \in M, \omega \in \Omega\}$ defined on a probability space (Ω, P) is called *a Brownian motion parametrized with the metric space* (M, d), or simply *a Brownian motion with* (M, d)*-parameter*, if it satisfies the following two conditions B1) and B2);

B1) $B(O) \equiv 0$,

B2) $B(m) - B(m')$ is subject to the Gaussian law $N(0, d(m, m'))$ with mean zero and variance $d(m, m')$.

That is, $\{B(m); m \in M\}$ is a Gaussian system with mean 0 and covariance

(1) $$v(m, m') = \tfrac{1}{2}(d(m, O) + d(m', O) - d(m, m')).$$

We are going to show the existence of Brownian motion by the constructive method (B) mentioned in the introduction. To this end, we prepare some notations.

Let $(E, \mathscr{B}, \mu)$ be an abstract σ-finite positive measure space and $\mathscr{B}_0$ be the family of the $\mathscr{B}$-sets of finite measure, $\mathscr{B}_0 = \{V \in \mathscr{B}; \mu(V) < \infty\}$. Since $\mu(V_1 \cap V_2)$, $V_1, V_2 \in \mathscr{B}_0$, is positive definite, there exists a Gaussian system $\mathscr{X} = \{X(V, \omega); V \in \mathscr{B}_0\}$ with mean zero and covariance function $E[X(V_1,\omega) X(V_2, \omega)] = \mu(V_1 \cap V_2)$. The Gaussian system is called *a Gaussian random measure* associated with the measure space (E, μ). The *stochastic integral*

$$\int f(t)dX(t, \omega)$$

of an element $f \in L^2(E, \mu)$ by the random measure $\mathscr{X}$ gives an isometry from $L^2(E, \mu)$ onto the closed linear hull of $\{X(V, \omega); V \in \mathscr{B}_0\}$ in $L^2(\Omega)$ (see J.L. Doob [2], Chap. IX). It is more suggestive to use the following P. Lévy's notation

(2) $$dX(\alpha, \omega) = \xi_\alpha \sqrt{d\mu(\alpha)} .$$

Using this notation, we can regard that $\{\xi_\alpha\}$ is a Gaussian system which is independent at every point and that the element ξ_α is subject to the law $N(0, 1)$ (P. Lévy [8]). It is known that in the following two cases the

Brownian motion can be expressed as a stochastic integral by a certain random measure.

Case 1. The case of n-dimensional sphere S^n.

Let d_s be the distance along the geodesic on S^n and set

(S1) $S_A = \{B \in S^n ; d_s(A, B) \leq \pi/2\}$.

Let $\{\xi_A \sqrt{dS(A)}\}$ be the Gaussian random measure associated with the measure space (S^n, dS), where dS is the normalized uniform measure on S^n. Fix an origin O of S^n arbitrarily. Then the Gaussian system

$$\text{(B1)} \quad B_1(A) = \sqrt{\pi} \cdot \left\{ \int_{S_A} \xi_M \sqrt{dS(M)} - \int_{S_O} \xi_M \sqrt{dS(M)} \right\}$$

is the Brownian motion with (S^n, d_s)-parameter (P. Lévy [8]).

Case 0. The case of n-dimensional Euclidean space R^n.

Consider the Euclidean metric $d(\cdot, \cdot)$ on R^n and let O be the origin of R^n. Let $\{\xi_A \sqrt{dч(A)}\}$ be the Gaussian random measure associated with the measure space $(\mathscr{H}, dч)$, where $\mathscr{H}$ is the set of all hyperplanes of R^n and $dч$ denotes the canonical measure on $\mathscr{H}$ (c.f. H. H. Ченцов [12] and S. Takenaka [11]). For any point A of R^n, set

(S0) $S_A = \{h \in \mathscr{H} ; h$ separates the point A and the origin $O\}$.

Then the Brownian motion $B_0(A)$ with R^n-parameter is represented as the following integral form;

$$\text{(B0)} \quad B_0(A) = \int_{S_A} \xi_A \sqrt{dч(A)} \,.$$

These two cases have a common character that S^n and R^n are the Riemannian manifolds of constant curvature, although one is positive constant and the other is constant zero. The remained case, the *Case* -1, is the Riemannian manifold of constant negative curvature, that is, the hyperbolic space H^n. In the following section we shall give a unified method of integral representations of the Brownian motions for these three Riemannian manifolds of constant curvature.

§ 2. Projective geometry and the integral representation

In this section we want to treat Riemannian space Q of constant curvature—S^n, R^n or H^n, as the parameter space of Brownian motion, and to construct the integral representation of Brownian motion will be discussed in a unified manner. It is convenient to consider Q as a subset of the projective space $P^n(R)$.

I. We define three submanifolds Q_κ, $\kappa = 1, 0, -1$, of R^{n+1}:

$$
(3) \quad \begin{aligned}
Q_1 &= \{x \in R^{n+1};\, x = (x_1, \cdots, x_n, x_0),\ x_1^2 + \cdots + x_n^2 + x_0^2 = 1\}\,, \\
Q_0 &= \{x \in R^{n+1};\, x_0 = 1\} \quad \text{and} \\
Q_{-1} &= \{x \in R^{n+1};\, x_1^2 + \cdots + x_n^2 - x_0^2 = -1,\, x_0 \geq 1\}\,.
\end{aligned}
$$

Introduce the Riemannian metric d_κ of Q_κ given by the quadratic form;

$$
(4) \qquad d_\kappa s^2 = dx_1^2 + \cdots + dx_n^2 + \kappa dx_0^2\,.
$$

Then the number κ is the sectional curvature of (Q_κ, d_κ), and the Riemannian metric space (Q_κ, d_κ) is the n-dimensional unit sphere S^n, the n-dimensional Euclidean space R^n or the n-dimensional hyperbolic space H^n according to $\kappa = 1$, 0 or -1.

Let G_κ be the subgroup of all elements of $SL(n+1, R)$ which keep the quadratic form $d_\kappa s^2$ invariant. Then the action of G_κ on R^{n+1} leaves Q_κ stable and acts transitively on Q_κ. Furthermore, for any pairs (A, A′) and (B, B') in Q_κ with $d_\kappa(A, A') = d_\kappa(B, B')$, there exists an element $g \in G_\kappa$ such that

(P1) $gA = B$ and $gA' = B'$.

It is well known that:

$$
(5) \qquad G_1 = SO(n+1) \quad \text{and} \quad Q_1 = SO(n+1)/SO(n)\,.
$$

$$
(6) \qquad G_0 = M(n) \qquad \text{and} \quad Q_0 = M(n)/SO(n)\,,
$$

where $M(n)$ denotes the Euclidean motion group

$$
(7) \quad \begin{gathered}
M(n) = \left\{ g \in SL(n+1, R);\, g = \left[\begin{array}{c|c} \tilde{g} & a \\ \hline 0, \cdots, 0 & 1 \end{array}\right],\ \tilde{g} \in SO(n),\ a \in R^n \right\}\,. \\
G_{-1} = L_n \quad \text{and} \quad Q_{-1} = L_n/SO(n)\,,
\end{gathered}
$$

where L_n is the n-dimensional Lorentz group

$$
L_n = \left\{ g \in SL(n+1, R);\ \begin{array}{l} gI(n,1)^t g = I(n,1),\ I(n,1) \text{ is the diagonal} \\ \text{matrix with the diagonal } (1, \cdots, 1, -1) \end{array} \right\}.
$$

II. Let π be the projection from $R^{n+1} - \{O\}$ into the projective space P^n;

$$
(8) \quad \begin{aligned}
\pi;\ R^{n+1} - \{O\} &\longrightarrow P^n \\
(x_1, x_2, \cdots, x_n, x_0) &\longmapsto [(x_1, x_2, \cdots, x_n, x_0)]\,,
\end{aligned}
$$

where the right hand side of (8) means the line passing through $(x_1, x_2, \cdots,$

$x_n, x_0)$ and the origin O. Take a local coordinate (U, ϕ) around the origin as follows;

$$\begin{array}{lcl} \phi;\ U & \longrightarrow & R^n \\ [(x_1, \cdots, x_n, x_0)] & \longmapsto & (x_1/x_0, \cdots, x_n/x_0) \end{array}, \tag{9}$$

where $U = \{[(x_1, x_2, \cdots, x_n, x_0)] \in P^n;\ x_0 \neq 0\}$.

Then

$$\phi\cdot\pi(Q_k) = R^n, \qquad \text{for } \kappa = 1 \text{ or } 0 \quad \text{and}$$
$$\phi\cdot\pi(Q_{-1}) = \{z \in R^n;\ |z|^2 = z_1^2 + \cdots + z_n^2 < 1\}.$$

Let $\mathscr{H}$ be the set of all hyperplanes of codimension 1 in P^n. Then there exists a one-to-one map $*$ between P^n and $\mathscr{H}$ called the projective inversion (c.f. W. Blaschke [1]);

$$P^n \longrightarrow \mathscr{H}$$
$$x \longmapsto x^* = \{y \in P^n;\ \langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n + x_0y_0 = 0\}.$$

By this map $*$, we can identify P^n with $\mathscr{H}$. Let

$$\mathscr{H}_\kappa = \{h \in \mathscr{H};\ h \cap Q_\kappa \neq \emptyset\}. \tag{10}$$

Then for any element $h \in \mathscr{H}_\kappa$, the intersection $\ell = h \cap Q_\kappa$ is a hyperplane of Q_κ, that is, a totally geodesic submanifold of Q_κ of codimension 1. Conversely, for any hyperplane ℓ of Q_κ there exists an element h of $\mathscr{H}_\kappa$ such that h includes ℓ (c.f. M. Kurita [7]).

Applying the inversion map $*$, we can identify $\mathscr{H}_\kappa$ with a subset $\mathscr{H}_\kappa^*$ of P^n;

$$\mathscr{H}_\kappa^* = P^n, \qquad \text{for } \kappa = 1,\ 0 \text{ and}$$
$$\mathscr{H}_{-1}^* = \{x \in P^n;\ (x_1/x_0)^2 + \cdots + (x_n/x_0)^2 > 1\}. \tag{11}$$

The action of G_κ on Q_κ induces the action of the group $G_\kappa^* = \{{}^tg;\ g \in G_\kappa\}$ on $\mathscr{H}_\kappa$ as follows;

$$g^*x^* = {}^tgx^* = (gx)^*,\ g \in G_\kappa,\ x \in Q_\kappa. \tag{12}$$

It is well known that there exists a measure μ_κ on $\mathscr{H}_\kappa^*$ which is invariant under the action of G_κ^*. Using the local coordinate (ϕ, U), the measure μ_κ is given explicitely in the following form $\phi_*(\mu_\kappa)$ in $\phi(\mathscr{H}_\kappa^*) \subset R^n$;

$$\phi_*(\mu_1) = \frac{dx}{(|x|^2 + 1)^{(n+1)/2}},$$

$$(13)\qquad \phi_*(\mu_0) = \frac{dx}{|x|^{n+1}} \quad \text{and}$$

$$\phi_*(\mu_{-1}) = \frac{dx}{(|x|^2 - 1)^{(n+1)/2}}\,,$$

where $x \in R^n$, dx is the Lebesgue measure on R^n and $|x|$ is the Euclidean norm of x.

III. In this part we construct the Brownian motion parametrized with space (O_κ, d_κ) in the sense of P. Lévy. For a point A of Q_κ, set

(S) $S_A = \{h^* \in \mathscr{H}^*;\ h$ separates the point A and the origin O $= [(0, \cdots, 0, 1)]\}$.

Note that the set $\phi(S_A)$ is one of the two connected components of $R^n - \{A^*\}$ which does not contain the origin O.

Consider three points A_1, A_2 and A_3 lying in a same geodesic line in this order, then the following equalities hold;

(P2) $g^*(S_{A_1} \Delta S_{A_2}) = S_{gA_1} \Delta S_{gA_2},\ g \in G$,

(P2) $S_{A_1} \Delta S_{A_2} + S_{A_2} \Delta S_{A_3} = S_{A_1} \Delta S_{A_3}$ (mode μ_κ),

where Δ denotes the symmetric difference.

THEOREM 1. *In the case of* $\kappa = 0$ *or* -1, *the Gaussian system*

(B) $B_\kappa(A) = c_\kappa \int_{S_A} \xi_x \sqrt{d\mu_\kappa(x)}$

is the Brownian motion of (Q_κ, d_κ)*-parameter, where* $\{\xi_x \sqrt{d\mu_\kappa(x)}\}$ *is a Gaussian random measure associated with the invariant measure* μ_κ *of* $\mathscr{H}^*_\kappa$ *and* c_κ *is the normalizing constant given by the equation* (18) *or* (19). *In case of* $\kappa = 1$ *we restrict the variable* A *in a hemisphere and apply the formula* (B). *Set* $B_1(\tilde{A}) = -B_1(A)$, *where* $\tilde{A}$ *denotes the antipodal point of* A, *then we obtain the Brownian moiton on the whole sphere.*

Proof. The condition B1) in the definition of the Brownian motion in § 1, is obvious. The condition B2) is that the variance

$$(14)\qquad E(B_\kappa(A) - B_\kappa(A'))^2 = c_\kappa^2 \mu_\kappa(S_A \Delta S_{A'})$$

is a linear function of the geodesic distance $s = d_\kappa(A, A')$. Let (A, A') and (A_1, A_1') be any pairs in Q_κ with $d_\kappa(A, A') = d_\kappa(A_1, A_1') = s$. Then, by the virtue of (P1) there exists an element $g \in G_\kappa$ which satisfies that $gA = A_1$, $gA' = A_1'$. Since μ_κ is invariant under g^*, by (P2) we have;

$$(15)\qquad \mu_\kappa(S_A \Delta S_{A'}) = \mu_\kappa(g^*(S_A \Delta S_{A'})) = \mu_\kappa(S_{gA} \Delta S_{gA'}) = \mu_\kappa(S_{A_1} \Delta S_{A_1'})\,.$$

Therefore, $\mu_\kappa(S_A \Delta S_{A'})$ depends only on $d_\kappa(A, A')$, say $\mu_\kappa(S_A \Delta S_{A'}) = u(d_\kappa(A, A'))$, where u is a function. Let A_1, A_2, A_3 be three points lying on a geodesic line in this order. Then by (P3)

$$
\begin{aligned}
u(d_\kappa(A_1, A_3)) &= \mu_\kappa(S_{A_1} \Delta S_{A_3}) = \mu_\kappa(S_{A_1} \Delta S_{A_2}) + \mu_\kappa(S_{A_2} \Delta S_{A_3}) \\
&= u(d_\kappa(A_1, A_2)) + u(d_\kappa(A_2, A_3)) \,.
\end{aligned} \tag{16}
$$

Since the function $u(s)$ is continuous in s, $u(s)$ is a linear function. The normalizing constants are taken as the following equalities hold;

$$
\frac{\pi}{4} = \tan^{-1} 1 = c_1^2 \int_{S_{A(1)}} d\mu_1(x) \,, \tag{17}
$$

$$
1 = c_0^2 \int_{S_{A(1)}} d\mu_0(x), \quad \text{and} \tag{18}
$$

$$
\frac{1}{2} \ln 3 = \tanh^{-1} \frac{1}{2} = c_{-1}^2 \int_{S_{A(1/2)}} d\mu_{-1}(x), \tag{19}
$$

where $A(t) = [(t, 0, \cdots, 0, 1)]$. All the coefficients c_κ are finite, because

$$
S_A \subset \{A'; |\phi(A')| \geq \inf_{A'' \in S_A} |\phi(A'')|\} \,, \tag{20}
$$

and the definition of the measures μ_κ in (13). q.e.d.

Remark. Our representation (B) coinsides with (B1) in the case of $\kappa = 1$ or with (B0) in the case of $\kappa = 0$.

§ 3. The M_t-process

Define a new Gaussian process L_t^κ from the white noise

$$
L_t^\kappa = \int_{|x| \geq 1/t} \xi_x \sqrt{d\mu_\kappa(x)} \,, \tag{21}
$$

for $t \in [0, \infty)$ in the case $\kappa = 1$ or 0, and for $t \in [0, 1)$ in the case of $\kappa = -1$, where, $|x|^2 = (x_1/x_0)^2 + \cdots + (x_n/x_0)^2$. Then the process L_t^κ becomes a process with *independent increments* and its covariance is

$$
E(L_t^\kappa \cdot L_s^\kappa) = \int_{|x| \geq (1/t \vee 1/s)} d\mu_\kappa = \begin{cases} \Omega_{n-1} \displaystyle\int_0^{\tan^{-1}(t \wedge s)} \cos^{n-1} \theta d\theta, & \text{for } \kappa = 1 \,, \\ \Omega_{n-1} \cdot (t \wedge s), & \text{for } \kappa = 0 \,, \\ \Omega_{n-1} \displaystyle\int_0^{\tanh^{-1}(t \wedge s)} \cosh^{n-1} u \, du, & \\ & \text{for } \kappa = -1 \,, \end{cases} \tag{22}
$$

where Ω_{n-1} is the volume of $n-1$ sphere. Note that L_t^0 is the ordinary (1-parameter) Brownian motion.

Take continuous versions of $B_\kappa(x)$ and L_t^κ. And consider the process M_t^κ which is the mean of the process $B_\kappa(x)$ on the sphere of radius t and center O in $\phi(U)=R^n$, that is,

$$M_t^\kappa = \int_{|x|=t} B_\kappa(x) dS(x) , \tag{23}$$

where dS is the normalized uniform measure on $\{t\cdot x; x\in S^{n-1}\}$. By the change of the order of integrations of the expression (23), we get the following;

$$M_t^\kappa = c_\kappa \int_{|x|=t} dS(x)\cdot\int_{S_x} \xi_y\sqrt{d\mu_\kappa(y)} = c_\kappa\int_0^t P\left(\frac{s}{t}\right)dL_s^\kappa , \tag{24}$$

where

$$P(u) = \Omega_{n-2}\int_0^{\cos^{-1}u}\sin^{n-2}\theta d\theta. \tag{25}$$

In the case of odd n, it is well known that P is a polynomial of order $(n+1)/2$ and that for any $\alpha(s)\in L^2(R_+, ds)$, if

$$f(t) = \int_0^t P\left(\frac{s}{t}\right)\alpha(s)ds = 0, \qquad \text{for any } t\geq 0 , \tag{26}$$

then $\alpha(\cdot)=0$. As the measure $dE(L_s^\kappa)^2$ has non zero density, in case the measure $dE(L_s^\kappa)^2$ is taken in (26) in stead of ds, this fact is also true. That is, the representation (24) is canonical, and the process M_t^κ has $(n+1)/2$-ple Markov property (see T. Hida [4a]). Thus, we obtain

THEOREM 2. *In the case of odd dimension n, the process M_t^κ is a $(n+1)/2$-ple Markov process and its canonical representation is*

$$M_t^\kappa = \int_0^t P\left(\frac{s}{t}\right)dL_s^\kappa.$$

Recently, an advanced investigation of M_t-process has been done by A. Noda is connection with Theorem 2. A new characterization of Brownian motion B_κ with (Q_κ, d_κ)-parameter has been given by K. Inoue. They have investigated interesting properties of conjugate sets of the parameter in the sense of P. Lévy ([8]) for B_κ. Their results were reported in the General Meeting of the Japanese Mathematical Society, April 1979 (c.f. [13], [14]).

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