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INFINITE DIMENSIONAL LANGEVIN EQUATION AND FOKKER-PLANCK EQUATION

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§ 0. Introduction

Stochastic processes on a Hilbert space have been discussed in connection with quantum field theory, theory of partial differential equations involving random terms, filtering theory in electrical engineering and so forth, and the theory of those processes has greatly developed recently by many authors (A. B. Balakrishnan [1, 2], Yu. L. Daletskii [7], D. A. Dawson [8, 9], Z. Haba [12], R. Marcus [18], M. Yor [26]).

The most basic concept arising there is the so-called *cylindrical Brownian motion*, abbr. c.B.m., (see Definition 1.2). It is thought of as a natural generalization of a finite dimensional Brownian motion, and it can be formed from multi-parameter white noise as is briefly illustrated in what follows.

First we introduce a (Gaussian) white noise μ indexed by a space-time parameter running through $D \times T$, where D is a domain of the d-dimensional Euclidean space $\boldsymbol{R}^d$ and T is $\boldsymbol{R}^1$ on which the time t runs. Namely, μ is the standard Gaussian measure on $\mathscr{E}^*$ determined by the characteristic functional

$$C_\mu(\eta) = \exp\left\{-\frac{1}{2}\|\eta\|^2\right\}, \quad \|\eta\|^2 = \int_{D\times T}|\eta|^2 dx, \quad \eta \in \mathscr{E} ,$$

where $\mathscr{E}^*$ is the dual of $\mathscr{E}$ forming a Gelfand triple:

$$\mathscr{E}^* \subset \mathscr{H} = L^2(D \times T) \subset \mathscr{E}^* .$$

We are now given a generalized stochastic process in the sense that $\langle \eta, \omega \rangle$, $\eta \in \mathscr{E}$, $\omega \in \mathscr{E}^*$, is an ordinary random variable, where $\langle\ ,\ \rangle$ is the canonical bilinear form connecting $\mathscr{E}$ and $\mathscr{E}^*$ (I.M. Gelfand and N. Ya. Vilenkin [11]). The bilinear form $\langle\ ,\ \rangle$ extends to the case where η is of the form $\xi \otimes \chi_{[s,t]}$

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in such a way that $\langle \xi \otimes \chi_{[s,t]}, \omega \rangle$ is still a random variable on $(\mathscr{E}^*, \mu)$. With this remark in mind, set

$$B_t(\xi) = \langle \xi \otimes \chi_{[0\wedge t, 0\vee t]}, \omega \rangle, \ t \in \boldsymbol{R}^1 .$$

Then it is a Brownian motion with parameter space $\boldsymbol{R}^1$ with variance $\|\xi\|^2_{L^2(D)}$. In view of this, $B_t(\cdot)$ is called a *cylindrical Brownian motion.* Stochastic integrals with respect to c.B.m. can be defined in a usual manner. Details will be discussed in § 1 to some extent. It is noted that white noise is given as soon as we take the time-derivative $(d/dt)B_t(\cdot)$ of $B_t(\cdot)$, and it is indexed by ξ.

In § 2 we shall summarize some known results, which have been obtained by the author ([19] and [20]) about stochastic differential equations on a Hilbert space. In § 3 we shall focus our attention on the most basic equation

$$(1) \qquad dX_t = -\hat{\omega} X_t dt + dB_t$$

on $H = L^2([0, \pi])$, where B_t is a c.B.m. on H and $\hat{\omega} = \sqrt{-\Delta}$ ($\Delta = d^2/d\sigma^2$: Laplacian on $L^2([0, \pi])$ with Neumann boundary condition). Unfortunately the equation (1) has no solution in H, and so we are led to extend the state space. Put $\tilde{H} = \{\tilde{X} \in H; (\tilde{X}, 1)_H = 0\}$. If $\hat{\omega}$ is restricted to $\tilde{H}$, it is a strictly positive operator such that $\hat{\omega}^{-1}$ is of Hilbert-Schmidt type. We are therefore able to construct a Hilbert scale derived from $\hat{\omega}$ (for definition see Yu. L. Daletskii [7]):

$$\tilde{H}_\infty \subset \cdots \subset \tilde{H}_1 \subset \tilde{H}_{1/2} \subset \tilde{H} \subset \tilde{H}_{-1/2} \subset \tilde{H}_{-1} \subset \cdots \subset \tilde{H}_{-\infty} ,$$

and hence the equation (1) is viewed as an equation on $\tilde{H}$ or on this Hilbert scale instead of on the original H.

We are now ready to state our main results. The equation (1) has a unique solution on $\tilde{H}_{-1/2}$, which comes from the discussion in § 2. There exists a unique invariant probability measure ν of $\tilde{X}_t$ given by (1) on $\tilde{H}_{-1/2}$ and it is proved that the transition probability $P(t, \tilde{X}, d\tilde{Y})$ of the Markov process $\tilde{X}_t$ is always equivalent to $\nu(d\tilde{Y})$. As for the Radon-Nikodym derivative $P(t, \tilde{X}, \tilde{Y}) = dP(t, \tilde{X}, \cdot)/d\nu(\cdot)$ we have a version, still denote by the same symbol, satisfying enough analytic properties as is prescribed by the following theorem.

THEOREM 3.4. *The function* $P(t, \tilde{X}, \tilde{Y})$ *has the following properties*:
(i) $P(t, \tilde{X}, \tilde{Y})$ *is a continuous function on* $(0, \infty) \times \tilde{H}_{-1/2} \times \tilde{H}_{-1/2}$.

(ii) *$P(t, \tilde{X}, \tilde{Y}) = P(t, \tilde{Y}, \tilde{X})$, that is $P(t, \tilde{X}, \tilde{Y})$ is symmetric.*

(iii) *For fixed $t > 0$ and $\tilde{X} \in \tilde{H}_{-1/2}$, $P(t, \tilde{X}, \tilde{Y})$ is an $L^p(d\nu)$-function of $\tilde{Y}$ for every $p > 0$. But $P(t, \tilde{X}, \tilde{Y})$ is not bounded.*

(iv) *$P(t, \tilde{X}, \tilde{Y})$ is the fundamental solution of the following equation*

$$\frac{\partial P(t, \tilde{X}, \tilde{Y})}{\partial t} = \frac{1}{2}\Delta_V P(t, \tilde{X}, \tilde{Y}) - \int_0^\pi \hat{\omega}\tilde{Y}(\sigma)\frac{\delta P(t, \tilde{X}, \tilde{Y})}{\delta \tilde{Y}(\sigma)}d\sigma\,, \tag{3.14}$$

$$P(t, \tilde{X}, \tilde{Y}) \longrightarrow \delta_{\tilde{X}}(\tilde{Y}) \qquad (t \downarrow 0)\,, \tag{3.15}$$

where Δ_V denotes the Volterra Laplacian and $\delta P/\delta\tilde{Y}(\sigma)$ denotes the functional derivative (see I. Kubo [16] § 7 *and Remark* 3.1 *given below), and where the precise meaning of* (3.15) *is that*

$$\int_{\tilde{H}_{-1/2}} P(t, \tilde{X}, \tilde{Y})f(\tilde{Y})d\nu(Y) \longrightarrow f(\tilde{X}) \qquad \textit{as } t \downarrow 0$$

for any continuous bounded function $f(\tilde{Y})$ of $\tilde{H}_{-1/2}$.

Another main theorem is concerned with the generator L, associated with $\tilde{X}_t$, acting on the Hilbert space $L^2(\tilde{H}_{-1/2}, d\nu)$. Let $\Phi_{\{n_j\}}$ be a Fourier-Hermite polynomial (see § 3). Having extended L to be a closed operator on $L^2(d\nu)$, we have

THEOREM 3.6. (i) *The functions $\{\Phi_{\{n_j\}}(\tilde{Y})\}$, $n_j = 0, 1, \cdots, \sum n_j < \infty$, satisfy the following formula*

$$L\Phi_{\{n_j\}}(\tilde{Y}) = -\Big(\sum_j jn_j\Big)\Phi_{\{n_j\}}(\tilde{Y})\,. \tag{3.20}$$

The operator L, acting on $L^2(d\nu)$, is non-positive definite and self-adjoint.

(ii) *The function $P(t, \tilde{X}, \tilde{Y})$ satisfies the equation*

$$\frac{dP}{dt}(t, \tilde{X}, \tilde{Y}) = LP(t, \tilde{X}, \tilde{Y}), \quad t > 0, \quad \textit{for fixed } \tilde{X}\,. \tag{3.21}$$

These results are interesting in connection with the theory of string model as well (C. Rebbi [22], M. Kaku and K. Kikkawa [15], and Z. Haba and J. Lukierski [13]). In fact, the equation (3.14) is similar to (3.4) of [15] and the formula (3.13), which is given in § 3, is similar to (4.2) of [15] in appearance. Actually, our approach was inspired by these works.

In the last part of § 3, we shall briefly discuss equations of the form $d\tilde{X}_t = -\hat{\omega}\tilde{X}_t dt + BdB_t$, where B is a bounded linear operator on $\tilde{H}$.

As we have mentioned above, c.B.m. $B_t(\cdot)$ is derived from a white noise μ. We therefore expect some intimate connections between the space

$L^2(\mathscr{E}^*\to K)$ arising from white noise μ, K being a Hilbert space, and stochastic integrals with respect to $B_t(\cdot)$. In fact, we shall be able to see them in § 4 in the Wiener's direct sum decomposition of $L^2(\mathscr{E}^*\to K)$ and its representation formula by means of multiple Wiener integrals or of iterated stochastic integrals w.r.t. c.B.m. In the case of $K=\boldsymbol{R}^1$, the above results are known (see T. Hida [14]), however our generalization, where K is infinite dimensional, requires to establish some basic techniques such as iterated stochastic integrals, the correspondence between $\sigma_2(L^2((D\times T)^n), K)$ (= a Hilbert space consisting of all Hilbert-Schmidt operators from $L^2((D\times T)^n)$ to K) and $L^2((D\times T)^n\to K)$, and etc.

We then come to another important topic to discuss the equation of the form

$$(2)\qquad dX_t=-\hat{\omega}X_t dt+X_t\cdot dB_t\,,$$

where $X_t\cdot$ is a multiplicative operator defined by $(X_t\cdot h)(\sigma)=X_t(\sigma)\times h(\sigma)$. Similar equations, but somewhat different equations of this type have been investigated by many authors (D.A. Dawson [8, 9, 10], A.V. Balakrishnan [2], A. Shimizu [23]), however we can show that (2) can be dealt with in line with the analysis on $L^2(\mathscr{E}^*\to K)$.

The equation (2), describing actual phenomenon, is itself interesting, but unfortunately it has no solution on H, and even not on a Hilbert scale. We shall therefore consider in § 5 a modified equation of the form

$$(4)\qquad dX_t=-\hat{\omega}X_t dt+\left(\int_0^\pi \Gamma(\sigma,\sigma')X_t(\sigma')d\sigma'\right)\cdot dB_t$$

where $\Gamma(\ ,\ )$ is an element of $H\times H_1$. This equation does have a unique solution in H_{-1} (Theorem 5.1). Our discussions in § 5 are based on the theory developed in § 4, and the main results are Theorem 5.2 and Theorem 5.5. The former (Theorem 5.2) gives a system of equations in terms of the kernels of the integral representation of X_t, while the latter (Theorem 5.5) gives a system of ordinary differential equations on H_{-1} which is proved to be equivalent to (3).

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§ 1. Multi-parameter white noise and cylindrical Brownian motion

Let D be a domain of the d-dimensional Euclidean space $\boldsymbol{R}^d$. Let H be a real Hilbert space $L^2(D)$ and let $\mathscr{H}$ be a real Hilbert space $L^2(D\times T)$, where $T = \mathbf{R}^1$. We start with a Gelfand triple

$$\mathscr{E} \subset \mathscr{H} = L^2(D \times T) \subset \mathscr{E}^* ,$$

where $\mathscr{E}$ is a nuclear space and $\mathscr{E}^*$ is the dual space of $\mathscr{E}$. Given a characteristic function

$$C_\mu(\eta) = \exp\left\{-\frac{1}{2}\|\eta\|^2\right\}, \qquad \|\eta\|^2 = \int_{D\times T} |\eta|^2 dx$$

we are given a probability space $(\mathscr{E}^*, \mathscr{B}, \mu)$ such that

$$C_\mu(\eta) = \int_{\mathscr{E}^*} e^{i\langle\eta,\omega\rangle} d\mu(\omega) ,$$

where $\langle\ ,\ \rangle$ is the canonical bilinear form connecting $\mathscr{E}$ and $\mathscr{E}^*$.

DEFINITION 1.1. The probability space $(\mathscr{E}^*, \mathscr{B}, \mu)$ is called a (Gaussian) white noise on H or a (Gaussian) white noise with parameter space $D \times T$.

In this paper we adopt the white noise space $(\mathscr{E}^*, \mathscr{B}, \mu)$ as the basic probability space, and we denote by $\mathscr{B}_t$ the σ-field generated by $\{\langle\eta, \omega\rangle;$ $\eta \in \mathscr{E}, \omega \in \mathscr{E}^*, \operatorname{supp}\{\eta\} \subset D \times (-\infty, t]\}$.

We next will give a definition of a cylindrical Brownian motion. Assume that a probability space (Ω, F, P) and an increasing family of σ-field F_t, $t \geqq 0$, $F_t \subset F$, are given.

DEFINITION 1.2 (M. Yor [26]). A mapping $B_t(h, \omega)$: $[0, \infty) \times H \times \Omega \to \mathbf{R}^1$ is called a cylindrical Brownian motion (abbr. c.B.m.) on H if it satisfies the following conditions:

(i) $B_0(h, \cdot) = 0$ and $B_t(h, \cdot)$ is F_t-adapted.

(ii) For any $h \in H$, $h \neq 0$, $(1/\|h\|)B_t(h, \cdot)$ is a one-dimensional Brownian motion.

(iii) For any $t \in [0, \infty)$ and α, $\beta \in \mathbf{R}^1$ and h, $k \in H$, the following formula holds

$$B_t(\alpha h + \beta k) = \alpha B_t(h) + \beta B_t(k), \qquad (P\text{-a.s.}) .$$

Remark 1.1. If necessary, we can take a continuous version of $B_t(h)$. The process B_t can not be regarded as a process on H.

We will now form a c.B.m. on $H = L^2(D)$ from a white noise. Put $B_t(\xi) = \langle \xi \otimes \chi_{[0,t]}, \omega\rangle$, where $\xi \in H$ and $\chi_{[0,t]}$ is the defining function of the interval $[0, t]$ and $\omega \in \mathscr{E}^*$. The function $\langle \xi \otimes \chi_{[0,t]}, \omega\rangle$ is not well-defined on $\mathscr{E}^*$, because $\xi \otimes \chi_{[0,t]}$ may not belong to $\mathscr{E}$. But, we know that if $\eta_j \to$

η in $L^2(D\times T)$, then $\langle\eta_j, \omega\rangle \to \langle\eta, \omega\rangle$ in $L^2(\mathscr{E}^*, \mu)$. By using this fact, we are able to define a random variable $\langle\xi\otimes\chi_{[0,t]}, \omega\rangle$ as the mean square limit of a sequence $\langle\eta_j, \omega\rangle$, $j=1, 2, \cdots$, where $\eta_j\in\mathscr{E}$ and $\eta_j\to\xi\otimes\chi_{[0,t]}$. It is easy to see that the mapping $B_t(\xi)=\langle\xi\otimes\chi_{[0,t]}, \omega\rangle$ satisfies the conditions (i)—(iii) in Definition 1.2.

We will define stochastic integrals with respect to the c.B.m. B_t constructed above. Let $\phi(t, \omega)$ be a $\mathscr{B}_t$-adapted measurable function on $[0, \infty)\times\mathscr{E}^*$ into H such that

$$E\left[\int_0^t \|\phi(s)\|^2 ds\right] < \infty \qquad \text{for any } t>0\,,$$

where $E[\ ]$ means the expectation with respect to μ. Then a real valued martingale $\sum_{n=1}^{\infty}\int_0^t (\phi(s), e_n)dB_s(e_n)$ is well-defined in the ordinary sence, where $\{e_n\}$, $n=1, 2, \cdots$, is an orthonormal base in H.

DEFINITION 1.3. The stochastic integral $\int_0^t \langle\phi(s), dB_s\rangle$ of ϕ is the martingale given by

$$\int_0^t \langle\phi(s), dB_s\rangle = \sum_{n=1}^{\infty}\int_0^t (\phi(s), e_n)dB_s(e_n).$$

Remark 1.2. We can easily prove that the definition of stochastic integrals does not depend on the choice of a base $\{e_n\}$, and that the following equation holds:

$$E\left[\left|\int_0^t \langle\phi(s), dB_s\rangle\right|^2\right] = E\left[\int_0^t \|\phi(s)\|^2 ds\right].$$

Given two Hilbert spaces H and K, we denote by $\sigma_2(H, K)$ the Hilbert space consisting of all Hilbert-Schmidt operators from H into K. Let $\Phi(t, \omega)$ be a $\sigma_2(H, K)$-valued $\mathscr{B}_t$-adapted function defined on $[0, \infty)\times\mathscr{E}^*$ into $\sigma_2(H, K)$ such that

$$E\left[\int_0^t \|\Phi(s)\|^2_{\sigma_2(H,K)} ds\right] < \infty\,.$$

Then the integral

$$\int_0^t \langle\Phi^*(s)y, dB_s\rangle, \qquad \Phi^*(s) \text{ the dual operator of } \Phi(s)\,,$$

is well-defined for every $y\in K$, and it is linear in y.

DEFINITION 1.4. The stochastic integral of Φ is the K-valued martingale M_t which is uniquely determined by

$$(y, M_t)_K = \int_0^t \langle \Phi^*(s)y, dB_s\rangle, \qquad y \in K,$$

and it is denoted by $\int_0^t \Phi(s)dB_s$.

Remark 1.3. It is easy to verify

$$E\left[\left\|\int_0^t \Phi(s)dB_s\right\|^2\right] = E\left[\int_0^t \|\Phi(s)\|^2_{\sigma_2(H,K)}ds\right].$$

§ 2. Stochastic differential equations on a Hilbert space

Given two real separable Hilbert spaces H and K, we consider a stochastic integral equation on K

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t G(X_s)dB_s, \tag{2.1}$$

where a and G are mappings such that a: $K \to K$, G: $K \to \sigma_2(H, K)$, and B_s is a c.B.m. on H. For simplicity, we write the equation (2.1) in the form of stochastic differential equation

$$dX_t = a(X_t)dt + G(X_t)dB_t. \tag{2.2}$$

In the case where $K = H$ and where a and G have suitable properties, the equations of this type have been well investigated (e.g., M. Yor [26]). This paper deals with such equations without much restriction on G but with a specific drift term. Namely, we consider the equation of the form

$$dX_t = -AX_tdt + G(X_t)dB_t \tag{2.3}$$

where A is an unbounded linear operator with the domain $D(A)$ of dense in K. We will summarize the results obtained by the author in [19] and [20] for the equation (2.3).

We assume that $-A$ is the infinitesimal generator of a semi-group $\{T_t\}$, $t \geqq 0$, of class C_0 and that G satisfies the following condition

$$\|G(X)\|_{\sigma_2(H,K)} \leqq c_1 + c_2\|X\|_K,$$

where c_1 and c_2 are constants. The equation (2.3) is expressed in the form

$$X_t = X_0 + \int_0^t -AX_sds + \int_0^t G(X_s)dB_s. \tag{2.4}$$

Since A is unbounded, the condition $X_s \in D(A)$ is not always satisfied. While, by the assumptions on $-A$, the conjugate operator $-A^*$ of $-A$ is the infinitesimal generator of the conjugate semi-group $\{T_t^*\}$, which is of class C_0, and the domain $D(A^*)$ of A^* is dense in K. Taking these facts into consideration, two different kinds of solutions can be defined.

DEFINITION 2.1 (w-solution). A weak solution of (2.4) is a $\mathscr{B}_t$-adapted process X_t on K which satisfies the following conditions:

(i) $E\left[\int_0^t \|X_t\|^2 ds\right] < \infty \qquad$ for any $t \geqq 0$.

(ii) For any $Y \in D(A^*)$, the equality

$$(2.5) \qquad (Y, X_t) = (Y, X_0) + \int_0^t (-A^* Y, X_s) ds + \int_0^t \langle G^*(X_s) Y, dB_s \rangle, \qquad \text{a.e.}$$

holds for any $t \geqq 0$.

DEFINITION 2.2 (e-solution). An evolutional solution of (2.4) is a $\mathscr{B}_t$-adapted process on K which satisfies the condition (i) in Definition 2.1 and satisfies the following equation

$$(2.6) \qquad X_t = T_t X_0 + \int_0^t T_{t-s} G(X_s) dB_s .$$

The equivalence of the above two definitions of solutions is not valid in general, but the following theorems can be proved.

THEOREM 2.1 (Y. Miyahara [20]). *An* e-solution of (2.4) *is a w-solution of* (2.4).

THEOREM 2.2 (Y. Miyahara [19]). *Suppose that A has point spectrums and that AX is expanded in such a form as*

$$(2.7) \qquad AX = \sum \lambda_n (X, e_n) e_n ,$$

where $\{e_n\}$, $n = 1, 2, \cdots$, is an orthonormal base of K consisting of eigenvectors of A. Then a w-solution of (2.4) *is an e-solution of* (2.4).

The existence and the uniqueness of the solution of (2.4) is given in the form of the following theorem.

THEOREM 2.3 (Y. Miyahara [20]). *If G: $K \rightarrow \sigma_2(H, K)$ satisfies the Lipschitz condition, then the equation* (2.4) *has a unique e-solution.*

Remark 2.1. These theorems are proved in the case of $K = H$ in [19] and [20]. The methods used there can be applied in the case of $K \neq H$ without any difficulties.

The equation (2.3) is considered as an equation obtained from a differential equation $dX(t)/dt = -AX(t)$ by random perturbation. From this point of view, we can treat the stability problem of the equation (2.3), and indeed they are treated in [20] (the main result is Theorem 4.2 of [20]).

§ 3. Infinite dimensional Ornstein-Uhlenbeck process and Fokker-Planck equation

We are interested in an equation formally described as

$$(*) \qquad \frac{\partial X(t,\sigma)}{\partial t} = -\hat{\omega}X(t,\sigma) + \chi(t,\sigma), \quad t \geqq 0, \quad \sigma \in [0,\pi] ,$$

where $\hat{\omega} = \sqrt{-\Delta} = \sqrt{-(d^2/d\sigma^2)}$ and $\chi(t, \sigma)$ is a 2-parameter white noise. The precise definition of which will be given later.

In order to give a realization of this formal equation $(*)$ as a stochastic differential equation on a Hilbert space, we will proceed in the following manner.

Put $H = L^2([0, \pi])$. The Laplacian $\Delta = d^2/d\sigma^2$ on H with the Neumann boundary condition is well defined; indeed Δ is a non-positive self-adjoint operator for which $\{-j^2; j = 0, 1, 2, \cdots\}$ and $\{\xi_0 = 1/\sqrt{\pi},\ \xi_j = \sqrt{2/\pi}\cos j\sigma; j = 1, 2, \cdots\}$ form the eigensystem. We then define an operator $\hat{\omega}$ by $\hat{\omega} = \sqrt{-\Delta}$. It is a non-negative self-adjoint operator on H, and $\{j\}$ and $\{\xi_j\}$, $j = 0, 1, 2, \cdots$, form the eigensystem of $\hat{\omega}$.

Now we can regard the stochastic differential equation

$$(3.1) \qquad dX_t = -\hat{\omega}X_t dt + dB_t$$

as a realization of the formal equation $(*)$, where B_t is the c.B.m. on H given by $B_t(\xi) = \langle \xi \otimes \chi_{[0,t]}, \omega \rangle$ (see § 1).

It is easy to see that any solution to the equation (3.1) is not found in H (see, e.g., Y. Miyahara [19]), so we must extend the Hilbert space H to a larger Hilbert space K in which the solution lives.

Since the constant part of X_t (i.e., $(X_t, \xi_0)\xi_0$) and the remaining part are treated separately, we focus our attention on a subspace $\tilde{H} = \{\tilde{X} \in H; (\tilde{X}, \xi_0) = 0\}$ of H. The operator $\hat{\omega}$ is strictly positive on $\tilde{H}$ and $\hat{\omega}^{-1}$ is of Hilbert-Schmidt type, so we obtain a Hilbert scale derived from $\hat{\omega}$

$$\tilde{H}_\infty \subset \cdots \subset \tilde{H}_\alpha \subset \cdots \subset \tilde{H}_1 \subset \tilde{H} \subset \tilde{H}_{-1} \subset \cdots \subset \tilde{H}_{-\alpha} \subset \cdots \subset \tilde{H}_{-\infty}$$

where $\tilde{H}_\alpha$ $(0<\alpha<\infty)$ is a Hilbert space with the inner product $(\xi,\xi')_\alpha=(\hat{\omega}^\alpha\xi,\hat{\omega}^\alpha\xi')_{\tilde{H}}$, and $\tilde{H}_{-\alpha}$ is the dual space of $\tilde{H}_\alpha$ (see Yu. L. Daletskii [7] for details).

We are now ready to rephrase (3.1) so as to be an equation on the extended space $\tilde{H}_{-\alpha}$,

$$d\tilde{X}_t=-\hat{\omega}\tilde{X}_t dt+I_{0,-\alpha}dB_t\,, \tag{3.1'}$$

where $I_{0,-\alpha}$ is the injection of $\tilde{H}$ into $\tilde{H}_{-\alpha}$. If $\alpha>1/2$, then $I_{0,-\alpha}\in\sigma_2(\tilde{H},\tilde{H}_{-\alpha})$ and we can apply the results in § 2 to (3.1′) to see that (3.1′) has a unique solution. To fix the idea, we take α to be 1 and we have

$$d\tilde{X}_t=-\hat{\omega}\tilde{X}_t dt+I_{0,-1}dB_t\,. \tag{3.2}$$

As was seen in § 2, the unique solution of (3.2) is given by

$$\tilde{X}_t=T_t\tilde{X}_0+\int_0^t T_{t-s}dB_s\,, \tag{3.3}$$

which is called an infinite dimensional Ornstein-Uhlenbeck process.

We use the notation $\langle\xi,\tilde{X}\rangle$ to denote the dual bilinear form on $\tilde{H}_\alpha\times\tilde{H}_{-\alpha}$, i.e. $\langle\xi,\tilde{X}\rangle=(\hat{\omega}^\alpha\xi,\hat{\omega}^{-\alpha}\tilde{X})_{\tilde{H}}$ for $\xi\in\tilde{H}_\alpha$ and $\tilde{X}\in\tilde{H}_{-\alpha}$.

Recall that a solution to a stochastic differential equation is often made to be stationary with a suitable choice of initial probability distribution at $t=0$. This distribution is called the invariant measure.

THEOREM 3.1. *The process $\tilde{X}_t$ has a unique invariant probability measure ν, supported by $\tilde{H}_{-1/2}$, with the characteristic functional*

$$\begin{aligned} c_\nu(\xi) &= \int_{\tilde{H}_{-1/2}} e^{i\langle\xi,\tilde{X}\rangle}d\nu(\tilde{X}) \\ &= \exp\left\{-\frac{1}{4}\|\xi\|^2_{-1/2}\right\}=\exp\left\{-\frac{1}{4}\|\hat{\omega}^{-1/2}\xi\|^2\right\}, \qquad \xi\in\tilde{H}_{1/2}\,. \end{aligned} \tag{3.4}$$

Proof. By definition, the solution X_t satisfies the following equation:

$$\begin{aligned} d\langle\xi_j,\tilde{X}_t\rangle &= -\langle\hat{\omega}\xi_j,\tilde{X}_t\rangle dt+dB_t(\xi_j) \\ &= -j\langle\xi_j,\tilde{X}_t\rangle dt+dB_t(\xi_t), \qquad j=1,2,\cdots\,. \end{aligned} \tag{3.5}$$

Since $\{B_t(\xi_j)\}$, $j=1,2,\cdots$, are mutually independent 1-dimensional Brownian motions, $\{\langle\xi_j,\tilde{X}_t\rangle\}$, $j=1,2,\cdots$, are mutually independent 1-dimensional Ornstein-Uhlenbeck processes. Therefore each $\langle\xi_j,\tilde{X}_t\rangle$ has the invariant Gaussian measure ν_j with mean 0 and variance $1/2j$.

Let ν be the measure on $\tilde{H}_{-1}$ with characteristic functional (3.4), then the distribution of the random variable $\langle \xi_j, \tilde{X} \rangle$ is equal to ν_j, which proves that ν is an invariant measure of $\tilde{X}_t$. The uniqueness of the invariant probability measure follows from that of ν_j.

Finally we must prove that the measure ν is supported by $\tilde{H}_{-1/2}$. Consider a functional $C(\eta)$ on $\tilde{H}_{-1/2}$ given by

$$C(\eta) = \exp\left\{-\frac{1}{4}\|\hat{\omega}^{-1}\eta\|^2_{-1/2}\right\}, \qquad \eta \in \tilde{H}_{-1/2} .$$

Since $\hat{\omega}^{-1}$ is a Hilbert-Schmidt operator, there exists a Gaussian measure $\tilde{\nu}$ on $\tilde{H}_{-1/2}$ such that

$$\int_{\tilde{H}_{-1/2}} e^{i(\eta, \tilde{X})_{-1/2}} d\tilde{\nu}(\tilde{X}) = C(\eta) .$$

Put $\eta = \hat{\omega}\xi$ for $\xi \in \tilde{H}_{1/2}$, then $(\eta, \tilde{X})_{-1/2} = (\hat{\omega}^{1/2}\xi, \hat{\omega}^{-1/2}\tilde{X}) = \langle \xi, X \rangle$. We therefore have

$$\int_{\tilde{H}_{-1/2}} e^{i\langle \xi, \tilde{X} \rangle} d\tilde{\nu}(\tilde{X}) = \int_{\tilde{H}_{-1/2}} e^{i(\eta, \tilde{X})_{-1/2}} d\tilde{\nu}(\tilde{X})$$
$$= \exp\left\{-\frac{1}{4}\|\hat{\omega}^{-1}\eta\|^2_{-1/2}\right\} = \exp\left\{-\frac{1}{4}\|\xi\|^2_{-1/2}\right\} .$$

This implies that $\nu = \tilde{\nu}$. (Q.E.D.)

COROLLARY 3.1. (i) *The invariant measure ν is an $\tilde{H}_{1/2}$—standard Gaussian measure supported by $\tilde{H}_{-1/2}$.*

(ii) *The space $\tilde{H}_{1/2}$ is equal to the set of all $\xi \in \tilde{H}_{-1/2}$ such that ν is quasi-invariant under the translation by ξ.*

Proof. Consider a Gelfand triple

$$\tilde{H}_{3/2} \subset \tilde{H}_{1/2} \subset \tilde{H}_{-1/2}$$

where the space $\tilde{H}^*_{-1/2}$ (the dual space of $\tilde{H}_{-1/2}$) is identified with $\tilde{H}_{3/2}$ under the isomorphism $\tilde{H}^*_{1/2} \cong \tilde{H}_{1/2}$. The canonical bilinear form $\langle\!\langle \xi, \tilde{X} \rangle\!\rangle$ is expressible as

$$\langle\!\langle \xi, \tilde{X} \rangle\!\rangle = (\hat{\omega}\xi, \hat{\omega}^{-1}\tilde{X})_{1/2} = (\hat{\omega}^{3/2}\xi, \hat{\omega}^{-1/2}\tilde{X}) = \langle \hat{\omega}\xi, \tilde{X} \rangle .$$

Thus it holds that

$$\int_{\tilde{H}_{-1/2}} e^{i\langle\!\langle \xi, \tilde{X} \rangle\!\rangle} d\nu(\tilde{X}) = \int_{\tilde{H}_{-1/2}} e^{i\langle \hat{\omega}\xi, \tilde{X} \rangle} d\nu(\tilde{X})$$
$$= \exp\left\{-\frac{1}{4}\|\hat{\omega}\xi\|^2_{-1/2}\right\} = \exp\left\{-\frac{1}{4}\|\xi\|^2_{1/2}\right\} .$$

The last formula proves the assertion (i). The assertion (ii) follows from (i) by using Corollary 5.3.2 of H. Xia [25]. (Q.E.D.)

Theorem 3.1 assures that the equation (3.2) determines a stationary process on $\tilde{H}_{-1/2}$. In fact, a process Y_t given by $Y_t = \int_{-\infty}^{t} T_{t-s} dB_s$ is a realization of such a process.

We now turn to the investigation of the continuity of X_t. The key theorem of our approach is the following (see P. Billingsley [3] or K. R. Parthasarathy [21]).

THEOREM (Kolmogorov-Prokhorov). *If there are two constants* $\alpha > 0$ *and* $\beta > 1$ *such that*

$$E[\|\tilde{X}_t - \tilde{X}_s\|^{\alpha}] \leqq c|t-s|^{\beta}, \qquad \text{for } 0 \leqq s < t \leqq T\,,$$

then $\tilde{X}_t$ *is continuous on* $[0, T]$ *with probability* 1.

Our result is

THEOREM 3.2. *The solution of* (3.2) *is continuous with respect to* $\tilde{H}_{-1/2}$ *norm with probability* 1.

Proof. First we assume $\tilde{X}_0 = 0$, that is $\tilde{X}_t = \int_0^t T_{t-s} dB_s$, and for notational convenience we put $Y = \tilde{X}_t - \tilde{X}_s$, $0 \leqq s < t$. In order to apply the Kolmogorov-Prokhorov theorem to $\tilde{X}_t$ with $\alpha = 4$, we will calculate $E[\|Y\|_{-1/2}^4]$.

Put $\eta_j = \sqrt{j}\,\xi_j$, $j = 1, 2, \cdots$. Then $\{\eta_j, j = 1, 2, \cdots\}$ is an orthonormal base in $\tilde{H}_{-1/2}$. Hence

$$\begin{aligned}[E\|Y\|_{-1/2}^4] &= E\Big[\Big(\sum_n (Y, \eta_n)_{-1/2}^2\Big)^2\Big] \\ &= \sum_n E[(Y, \eta_n)_{-1/2}^4] + \sum_{n \neq m} E[(Y, \eta_n)_{-1/2}^2] E[(Y, \eta_m)_{-1/2}^2]\end{aligned}$$

where it is noted that $(Y, \eta_n)_{-1/2}$ and $(Y, \eta_m)_{-1/2}$ are mutually independent if $n \neq m$. The random variable $(Y, \eta_n)_{-1/2}$ is a Gaussian random variable with mean 0. Its variance is calculated as follows

$$\begin{aligned}E[(Y, \eta_n)_{-1/2}^2] &= E\Big[\Big(\Big\langle \int_0^t T_{t-u}^* \eta_n, dB_u \Big\rangle - \Big\langle \int_0^s T_{s-u}^* \eta_n, dB_u \Big\rangle\Big)^2\Big] \\ &= \tfrac{1}{2} n^{-2} \{2(1 - e^{-n(t-s)}) - e^{-2ns}(1 - e^{-n(t-s)})^2\} \\ &\leqq n^{-2}(1 - e^{-n(t-s)})\,.\end{aligned}$$

By using

$$E[(Y,\eta_n)^4_{-1/2}] = 3E[(Y,\eta_n)^2_{-1/2}]^2 \leqq 3n^{-4}(1-e^{-n(t-s)})^2 ,$$

we have

$$E[\|Y\|^4_{-1/2}] \leqq 3\sum_n n^{-4}(1-e^{-n(t-s)})^2 + \Big\{\sum_n n^{-2}(1-e^{-n(t-s)})\Big\}^2 .$$

We will prove that there exists a constant $\delta > 1$ such that

$$\sum_n n^{-4}(1-e^{-n(t-s)})^2 \leqq \text{const.}\,|t-s|^{\delta} , \tag{3.6.1}$$

$$\sum_n n^{-2}(1-e^{-n(t-s)}) \leqq \text{const.}\,|t-s|^{\delta/2} . \tag{3.6.2}$$

The inequality (3.6.1) follows from (3.6.2). Therefore we have only to prove (3.6.2).

Put $\gamma = t-s\ (>0)$ and $f(z) = z^{-2}(1-e^{-\gamma z})$, $z>0$. Then $f(z)>0$, $f'(z) < 0$, and so $f(z)$ is a decreasing function. Therefore we get

$$\sum_{n=1}^{\infty} f(n) \leqq f(1) + \int_1^{\infty} f(z)dz = (1-e^{-\gamma}) + \int_1^{\infty} f(z)dz . \tag{3.7}$$

It is easy to see that

$$\int_1^{\infty} f(z)dz = (1-e^{-\gamma}) + \gamma\int_{\gamma}^{\infty} \frac{e^{-v}}{v}dv ,$$

and to see that

$$\text{if } \gamma \geqq 1, \int_1^{\infty} f(z)dz \leqq \text{const.}\,\gamma \tag{3.8}$$

$$\begin{aligned} \text{if } 0<\gamma<1, \int_1^{\infty} f(z)dz &\leqq (1-e^{-\gamma}) + \gamma\int_{\gamma}^{1} \frac{e^{-v}}{v}dv + \text{const.}\,\gamma \\ &\leqq c(\varepsilon)\gamma^{\varepsilon}, \text{ for arbitrary } \varepsilon,\ 0<\varepsilon<1 . \end{aligned} \tag{3.9}$$

From (3.7), (3.8) and (3.9) it follows that

$$\sum_n n^{-2}(1-e^{-n(t-s)}) = \sum_n f(n) \leqq c|t-s|^{\varepsilon}, \qquad 0 \leqq s < t \leqq T\,(<\infty). \tag{3.10}$$

If we fix an ε, $1/2 < \varepsilon < 1$, and put $\delta = 2\varepsilon$, then the inequality (3.10) turns into (3.6.2). Thus we have proved the theorem under the assumption $\tilde{X}_0 = 0$.

In the general case where $\tilde{X}_0 \neq 0$, the continuity of $\tilde{X} = T_t\tilde{X}_0 + \int_0^t T_{t-s}dB_s$ follows from that of $T_t\tilde{X}_0$, because the second term has just been proved to be continuous. (Q.E.D.)

Since the equation (3.2) has a unique solution for any initial data $\tilde{X}_0 \in \tilde{H}_{-1/2}$, it determines a Markov process on $\tilde{H}_{-1/2}$. Let $P(t, \tilde{X}, d\tilde{Y})$ be the transition probability measure of this process. Then we obtain the following theorem.

THEOREM 3.3. *The transition probability measure $P(t, \tilde{X}, d\tilde{Y})$ is equivalent to $\nu(d\tilde{Y})$ if $t>0$ and $\tilde{X} \in \tilde{H}_{-1/2}$.*

Proof. Since the solution $\tilde{X}_t$ of (3.2) with initial value $\tilde{X}$ is given by $\tilde{X}_t = T_t\tilde{X} + \int_0^t T_{t-s}dB_s$, the transition probability measure $P(t, \tilde{X}, d\tilde{Y})$ is a Gaussian measure with mean $T_t\tilde{X}$ and supported by $\tilde{H}_{-1/2}$. The variance of $(\eta_j, \tilde{X}_t)_{-1/2}$ is equal to

$$E[(\eta_j, \tilde{X}_t)^2_{-1/2}] = \frac{1}{2}j^2(1-e^{-2jt}) .$$

Therefore the covariance operator $V(t, \tilde{X})$ of $P(t, \tilde{X}, d\tilde{Y})$ is given by

$$V(t, \tilde{X}) = \frac{1}{2}(\hat{\omega}^{-2} - T_t^2\hat{\omega}^{-2}) = \frac{1}{2}\hat{\omega}^{-1}(I - T_t^2)\hat{\omega}^{-1} . \tag{3.11}$$

While, as was seen from the proof of Theorem 3.1, the covariance operator V_ν of the measure ν is of the form

$$V_\nu = \frac{1}{2}\hat{\omega}^{-2} . \tag{3.12}$$

These two operators are linked by $V(t, \tilde{X}) = \sqrt{V_\nu}(I - T_t^2)\sqrt{V_\nu}$. Since $T_t\tilde{X} \in \tilde{H}_{+\infty} = \bigcap_{\alpha>0}\tilde{H}_\alpha$ for $t>0$ and $V_\nu(\tilde{H}_{-1/2}) = \hat{\omega}^{-2}(\tilde{H}_{-1/2}) = \tilde{H}_{3/2}$, it holds that $T_t\tilde{X} \in V_\nu(\tilde{H}_{-1/2})$. Hence, we can apply the well-known theorem on Gaussian measures to $\nu(d\tilde{Y})$ and $P(t, \tilde{X}, d\tilde{Y})$ (see H. Kuo [17] or A. V. Skorohod [24]) to obtain that $P(t, \tilde{X}, d\tilde{Y})$ is equivalent to $\nu(d\tilde{Y})$. (Q.E.D.)

COROLLARY 3.2. *Let $P(t, \tilde{X}, \tilde{Y})$ be the Radon-Nikodym derivative of $P(t, \tilde{X}, d\tilde{Y})$ with respect to $\nu(d\tilde{Y})$. Then $P(t, \tilde{X}, \tilde{Y})$ is expressed in the form*

$$\begin{aligned} &P(t, \tilde{X}, \tilde{Y}) \\ &= \prod_{j=1}^{\infty}(1-e^{-2jt})^{-1/2}\exp\{-(j(x_j^2+y_j^2)e^{-2jt} - 2jx_jy_je^{-jt})/(1-e^{-2jt})\} \end{aligned} \tag{3.13}$$

where $\tilde{X} = \sum x_j\xi_j$ and $\tilde{Y} = \sum y_j\xi_j$.

Proof. In case $\tilde{X} = 0$, the formula (3.13) is immediately obtained by the use of Theorem 3.3 of H. Kuo [17] p. 123 or Theorem 4 of A. V. Skorohod

[24] p. 95 (note that $\sqrt{V_\nu}=(1/\sqrt{2})\hat{\omega}^{-1}$ is a Hilbert-Schmidt operator). In the general case, applying Theorem 2 of A. V. Skorohod [17] p. 83 to $P(t,\tilde{X},d\tilde{Y})$ and $P(t,0,d\tilde{Y})$, we get

$$\frac{dP(t,\tilde{X},\cdot)}{dP(t,0,\cdot)}=\exp\left\{(V^{-1}(t,\tilde{X})T_t\tilde{X},\tilde{Y})_{-1/2}-\frac{1}{2}(V^{-1}(t,\tilde{X})T_t\tilde{X},T_t\tilde{X})_{-1/2}\right\}$$
$$=\prod_j\exp\{(2je^{-jt}x_jy_j-je^{-2jt}x_j^2)/(1-e^{-2jt})\}\,.$$

The formula (3.13) follows from this by the use of the formula

$$P(t,\tilde{X},\tilde{Y})=\frac{dP(t,\tilde{X},.)}{d\nu(\cdot)}=\frac{dP(t,\tilde{X},\cdot)}{dP(t,0,\cdot)}\times\frac{dP(t,0,\cdot)}{d\nu(\cdot)}\,.$$

(Q.E.D.)

With the explicit expression of $P(t,\tilde{X},\tilde{Y})$ we are able to speak of its further properties.

THEOREM 3.4. *The function* $P(t,\tilde{X},\tilde{Y})$ *has the following properties*:

(i) $P(t,\tilde{X},\tilde{Y})$ *is a continuous function on* $(0,\infty)\times\tilde{H}_{-1/2}\times\tilde{H}_{-1/2}$.

(ii) $P(t,\tilde{X},\tilde{Y})=P(t,\tilde{Y},\tilde{X})$, *that is* $P(t,\tilde{X},\tilde{Y})$ *is symmetric.*

(iii) *For fixed* $t>0$ *and* $\tilde{X}\in\tilde{H}_{-1/2}$, $P(t,\tilde{X},\tilde{Y})$ *is an* L^p $(d\nu)$*-function of* $\tilde{Y}$ *for every* $p>0$. *But* $P(t,\tilde{X},\tilde{Y})$ *is not bounded.*

(iv) $P(t,\tilde{X},\tilde{Y})$ *is the fundamental solution of the following equation*

$$\frac{\partial P(t,\tilde{X},\tilde{Y})}{\partial t}=\frac{1}{2}\Delta_V P(t,\tilde{X},\tilde{Y})-\int_0^\pi\hat{\omega}\tilde{Y}(\sigma)\frac{\delta P(t,\tilde{X},\tilde{Y})}{\delta(\tilde{Y}\sigma)}d\sigma\,,\tag{3.14}$$

$$P(t,\tilde{X},\tilde{Y})\longrightarrow\delta_{\tilde{X}}(\tilde{Y})\qquad(t\downarrow 0)\,,\tag{3.15}$$

where Δ_V *denotes the Volterra Laplacian and* $\delta P/\delta\tilde{Y}(\sigma)$ *denotes the functional derivative* (*see* I. Kubo [16] § 7 *and Remark* 3.1 *given below*), *and where the precise meaning of* (3.15) *is that*

$$\int_{\tilde{H}_{-1/2}}P(t,\tilde{X},\tilde{Y})f(\tilde{Y})d\nu(\tilde{Y})\longrightarrow f(\tilde{X})\qquad as\ t\downarrow 0$$

for any continuous bounded function $f(\tilde{Y})$ *of* $\tilde{H}_{-1/2}$.

Before we come to the proof we need some interpretation of the Volterra Laplacian and a lemma.

Remark 3.1. The function $P(t,\tilde{X},\tilde{Y})$ can naturally be extended to be a continuous function on $(0,\infty)\times\tilde{H}_{-1}\times\tilde{H}_{-1}$, and then $P(t,\tilde{X},\tilde{Y})$ can be expanded in the same form as (3.13) on the extended domain $(0,\infty)\times\tilde{H}_{-1}\times\tilde{H}_{-1}$. Putting $E=\tilde{H}_1$ and $E^*=\tilde{H}_{-1}$ in I. Kubo's notations, we know that Δ_V is well-defined in the sense of § 7 of I. Kubo [16]. The functional derivative $\delta P/\delta\tilde{Y}(\sigma)$ can be understood to be an element of $\tilde{H}^*_{-2}$ in the same way. Since $\hat{\omega}Y\in\tilde{H}_{-2}$, the dual product $\int\hat{\omega}\tilde{Y}(\sigma)(\delta P/\delta\tilde{Y}(\sigma))d\sigma$ is well-defined. In our cases, the Volterra Laplacian $\Delta_V P$ can be expressed in the form

$$\Delta_V P(t,\tilde{X},\tilde{Y}) \doteq \int_0^\pi P^{(2)}(t,\tilde{X},\tilde{Y};\sigma,\sigma)d\sigma$$

where $P^{(2)}(t,\tilde{X},\tilde{Y};\sigma,\sigma')$ is the second functional derivative of $P(t,\tilde{X},\tilde{Y})$.

LEMMA 3.1. *Let $\{X_n\}$, $n=1,2,\cdots$, be a sequence of random variables such that $X_n\to X$ a.e. and $\sup_n E[|X_n|^p]<\infty$. Then $X_n\to X$ in $L^{p'}$ for all p', $0<p'<p$.*

We are now ready to give a proof of the Theorem 3.4.

Proof of Theorem 3.4. Put

$$\phi_j(t,x,y)=(1-e^{-2jt})^{-1/2}\exp\{j(x^2+y^2)e^{-2jt}-2jxye^{-jt})/(1-e^{-2jt})\}$$

and

$$P_n(t,\tilde{X},\tilde{Y})=\prod_{j=1}^n\phi_j(t,x_j,y_j),\qquad \tilde{X}=\sum x_j\xi_j \text{ and } \tilde{Y}=\sum y_j\xi_j\,.$$

Then $P_n(t,\tilde{X},\tilde{Y})\to P(t,\tilde{X},\tilde{Y})$ (as $n\to\infty$) uniformly on $T\times A\times B\subset(0,\infty)\times\tilde{H}_{-1/2}\times\tilde{H}_{-1/2}$, where T is a compact subset of $(0,\infty)$ and where A and B are respectively bounded subsets of $\tilde{H}_{-1/2}$. This proves (i). The assertion (ii) is obvious from (3.13).

We then come to the proof of (iii). First, observe the moment $\int|P_n(t,\tilde{X},\tilde{Y})|^p d\nu(\tilde{Y})$,

$$\begin{aligned}\int|P_n(t,\tilde{X},\tilde{Y})|^p d\nu(\tilde{Y}) &= \prod_{j=1}^n\int_{-\infty}^{\infty}|\phi_j(t,x_j,y)|^p d\nu_j(y)\\ &=\prod_{j=1}^n\Big[(1-e^{-2jt})^{-(p-1)/2}(1+(p-1)e^{-2jt})^{-1/2}\\ &\quad\times\exp\{[-jp(1-p+(p-1)e^{-2jt})e^{-2jt}x_j^2]/[(1-e^{-2jt})(1+(p-1)e^{-2jt})]\}\Big]\,.\end{aligned}$$

It is easy to see that this product converges (as $n\to\infty$) if $\sum x_j^2/j^\alpha<\infty$ for some $\alpha>0$, and in fact this condition is always satisfied for $\tilde{X}\in\tilde{H}_{-1/2}$. Therefore $\left\{\int|P_n|^p d\nu(\tilde{Y}),\ n=1,2,\cdots,\right\}$ is bounded. Applying Lemma 3.1 to $\{P_n\}$, $n=1,2,\cdots$, and P, we prove that $P(t,\tilde{X},\cdot)\in L^p$.

Let $\tilde{X}$ be an element of $\tilde{H}_{-1/2}$ such that $\sum jx_j^2=\infty$. The maximum of $P_n(t, \tilde{X}, \tilde{Y})$ is $\prod_{j=1}^{n}(1-e^{-2jt})^{-1/2}\exp\{ix_j^2\}$ which is attained by $y_j = e^{jt}x_j$, $j = 1, \cdots, n$. Put $\tilde{Y}_n = \sum_{j=1}^{n} e^{jt}x_j\xi_j$. Then the sequence

$$P(t, \tilde{X}, \tilde{Y}_n) = \prod_{j=1}^{n}(1-e^{-2jt})^{-1/2}\exp\{jx_j^2\}\prod_{j=n+1}^{\infty}(1-e^{-2jt}), \quad n = 1, 2, \cdots,$$

diverges as $n \to \infty$. Thus we have proved the unboundedness of $P(t, \tilde{X}, \tilde{Y})$.

Finally we will prove the assertion (iv). It is easy to see that the function $\sum_{j=1}^{n} z_j(\partial\phi_j(t, x_j, y_j)/\partial y_j)$ of $\tilde{Z}$ and $\tilde{Y}$ (where $t > 0$ and $\tilde{X} \in H_{-1/2}$ are fixed) converges uniformly on every bounded set $B \times B' \subset \tilde{H}_{-1/2} \times \tilde{H}_{-1/2}$. From this it follows that $P(t, \tilde{X}, \tilde{Y})$ is Fréchet-differentiable as a function of $\tilde{Y}$. Similarly, it is easily verified that $P(t, \tilde{X}, \tilde{Y})$ is k-times Fréchet-differentiable, where k may be taken arbitrarily large.

Using the equation

$$\frac{\partial\phi_j}{\partial t} = \frac{1}{2}\frac{\partial\phi_j}{\partial y^2} - jy\frac{\partial\phi_j}{\partial y}$$

we get

$$\frac{\partial\phi_j}{\partial t}(t, x, y) = \phi_j(t, x, y) \times \psi_j(t, x, y)$$

where

$$\begin{aligned}\psi_j(t, x, y) = {} & 2j^2e^{-2jt}(ye^{-jt} - x)^2/(1-e^{-2jt})^2 - je^{-2jt}/(1-e^{-2jt}) \\ & + 2j^2e^{-jt}(y^2e^{-jt} - xy)/(1-e^{-2jt}).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\frac{\partial P_n(t, \tilde{X}, \tilde{Y})}{\partial t} &= \sum_{j=1}^{n}\left\{\frac{\partial\phi_j(t, x_j, y_j)}{\partial t}\prod_{k\neq j}^{n}\phi_k(t, x_k, y_k)\right\} \\ &= \left\{\sum_{j=1}^{n}\psi_j(t, x_j, y_j)\right\}P_n(t, \tilde{X}, \tilde{Y}).\end{aligned}$$

It is clear that the sequences $\{\sum_{j=1}^{n}\psi_j\}$ and $\{P_n\}$ converge uniformly on any t-interval (a, b), $0 < a < b < \infty$. We therefore prove that there exists the derivative $\partial P/\partial t$ and it satisfies

$$\frac{\partial P}{\partial t} = \lim_{n\to\infty}\frac{\partial P_n}{\partial t} = \left(\sum_{j=1}^{\infty}\psi_j\right)P(t, \tilde{X}, \tilde{Y}). \tag{3.16}$$

The functional derivatives of P is calculated as follows

$$P'(t, \tilde{X}, \tilde{Y}; \xi_j) = \{2j(x_je^{-jt} - y_je^{-2jt})/(1-e^{-2jt})\}P(t, \tilde{X}, \tilde{Y}),$$

$$P^{(2)}(t, X, Y; \xi_j, \xi_j) = \{[2j(x_j e^{-jt} - y_j e^{-2jt})/(1 - e^{-2jt})]^2 - 2je^{-2jt}/(1 - e^{-2jt})\}P(t, \tilde{X}, \tilde{Y}) .$$

Thus we have

$$\tfrac{1}{2}\Delta_V P(t, \tilde{X}, \tilde{Y}) = \left[\sum_{j=1}^{\infty} \{2j^2 e^{-2jt}(x_j - y_j e^{-jt})^2/(1 - e^{-2jt})^2 - je^{-2jt}/(1 - e^{-2jt})\}\right] P(t, \tilde{X}, \tilde{Y}) .$$

On the other hand, it holds that

$$\begin{aligned} P'(t, \tilde{X}, \tilde{Y}; \sigma) &= P'(t, \tilde{X}, \tilde{Y}; \delta_\sigma) \\ &= \sum_j \xi_j(\sigma) P'(t, \tilde{X}, \tilde{Y}; \xi_j) \\ &= \left[\sum_j \{2j(x_j e^{-jt} - y_j e^{-2jt})/(1 - e^{-2jt})\}\xi_j(\sigma)\right] P(t, \tilde{X}, \tilde{Y}) . \end{aligned}$$

From the formula $\hat{\omega}\tilde{Y}(\sigma) = \sum_j jy_j \xi_j(\sigma)$, it follows that

$$\int_0^\pi \hat{\omega}\tilde{Y}(\sigma) P'(t, \tilde{X}, \tilde{Y}; \sigma) d\sigma = \left\{\sum_j 2j^2 y_j (x_j e^{-jt} - y_j e^{-2tj})/(1 - e^{-2jt})\right\} P(t, \tilde{X}, \tilde{Y}) .$$

Finally we obtain

$$\frac{1}{2}\Delta_V P(t, \tilde{X}, \tilde{Y}) - \int \hat{\omega}\tilde{Y}(\sigma) \frac{\delta P}{\delta \tilde{Y}(\sigma)} d\sigma = \left[\sum_j \psi_j(t, \tilde{X}, \tilde{Y})\right] P(t, \tilde{X}, \tilde{Y}) . \tag{3.17}$$

From (3.16) and (3.17) it follows that $P(t, \tilde{X}, \tilde{Y})$ satisfies the equation (3.14).

From the definition of $P(t, \tilde{X}, \tilde{Y})$, it follows that

$$\int P(t, \tilde{X}, \tilde{Y}) f(\tilde{Y}) d\nu(\tilde{Y}) = \int f(\tilde{Y}) P(t, \tilde{X}, d\tilde{Y}) = E[f(\tilde{X}_t)], \qquad \tilde{X}_0 = \tilde{X} .$$

Since $\tilde{X}_t$ is continuous on $\tilde{H}_{-1/2}$ and since $f(\tilde{Y})$ is a bounded continuous function on $\tilde{H}_{-1/2}$, we have

$$\lim_{t \downarrow 0} E[f(\tilde{X}_t)] = E[f(\tilde{X}_0)] = f(\tilde{X}) .$$

Thus the condition (3.15) is satisfied, which proves the theorem. (Q.E.D.)

Remark 3.2. Theorem 3.4 holds true even if the function $P(t, \tilde{X}, \tilde{Y})$ is considered as a function on $(0, \infty) \times \tilde{H}_{-\alpha} \times \tilde{H}_{-\alpha}$ for any α, $1/2 \leqq \alpha < \infty$.

Let $\{U_t\}$, $t \geqq 0$, be the semi-group on $L^\infty(\tilde{H}_{-1/2}, \nu)$ determined by the transition probability measure $P(t, \tilde{X}, d\tilde{Y})$, that is

$$(U_t f)(\tilde{X}) = \int_{\tilde{H}_{-1/2}} P(t, \tilde{X}, d\tilde{Y}) f(\tilde{Y}) = \int_{\tilde{H}_{-1/2}} P(t, \tilde{X}, \tilde{Y}) f(\tilde{Y}) d\nu(\tilde{Y})$$
$$= E_{\tilde{X}}[f(\tilde{X}_t)], \qquad f \in L^\infty(\tilde{H}_{-1/2}, \nu),$$

and let L be the infinitesimal generator of the semi-group $\{U_t\}$. We denote by $C_0^2(\boldsymbol{R}^k)$ the class of twice continuously differentiable functions with compact support in $\boldsymbol{R}^k$. Then we obtain the following theorem.

THEOREM 3.5. *A function $F(\tilde{Y})$, given by*

$$F(\tilde{Y}) = f(v_1, \cdots, v_k), \quad v_j = \langle \xi_j, \tilde{Y} \rangle, \quad j = 1, 2, \cdots, k, \quad f \in C_0^2(\boldsymbol{R}^k),$$

belongs to the domain of L and satisfies

$$(LF)(\tilde{Y}) = \left[\frac{1}{2} \sum_{j=1}^{k} \frac{\partial^2 f}{\partial v_j^2} - \sum_{j=1}^{k} j v_j \frac{\partial f}{\partial v_j} \right]_{v_j = \langle \xi_j, \tilde{Y} \rangle}. \tag{3.18}$$

Proof. We need only to prove (3.18). A process X_t^k given by

$$X_t^k = (x_t^1, \cdots, x_t^k), \qquad x_t^j = \langle \xi_j, \tilde{X}_t \rangle$$

is a k-dimensional Ornstein-Uhlenbeck process, and the components of X_t^k satisfy

$$dx_t^j = -j x_t^j dt + dB_t(\xi), \qquad j = 1, 2, \cdots, k. \tag{3.19}$$

Now, the conclusion of the theorem follows. (Q.E.D.)

COROLLARY 3.2. *Let $D(L)$ denote the domain of the operator L. Then $D(L) \cap L^2(d\nu)$ is dense in $L^2(d\nu)$,*

Proof. The set

$$C_0^2 = \{F \in L^2(d\nu); F(\tilde{Y}) = f(v_1, \cdots, v_k), \ v_j = \langle \xi_j, \tilde{Y} \rangle,$$
$$j = 1, 2, \cdots, k, \ f \in C_0^2(\boldsymbol{R}^k), \ k = 1, 2, \cdots\},$$

is dense in $L_2(d\nu)$. Since $C_0^2 \subset D(L)$ by Theorem 3.5, $D(L) \cap L^2(d\nu)$ is dense in $L^2(d\nu)$. (Q.E.D).

By Corollary 3.2, L can be considered as an operator on $L^2(\tilde{H}_{-1/2}, d\nu)$. Let the closed extension on L be denoted by the same symbol L.

Noting that ν is the $\tilde{H}_{1/2}$-standard Gaussian measure on $\tilde{H}_{-1/2}$, $L^2(\tilde{H}_{-1/2}, d\nu)$ is decomposed in the form

$$L^2(\tilde{H}_{-1/2}, d\nu) = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n \qquad \text{(Wiener's direct sum decomposition)},$$

where $\mathscr{H}_n$ is the closed subspace spanned by $\{\prod_j H_{n_j}(\sqrt{j}\langle\xi_j, \tilde{Y}\rangle), \sum_j n_j = n\}$ (H_n Hermite polynomial of degree n) (see § 4 or T. Hida [14]). We note here that

$$\left\{\Phi_{\{n_j\}} = \left(\prod_j n_j!2^{n_j}\right)^{-1/2} \prod_j H_{n_j}(\sqrt{j}\langle\xi_j, \tilde{Y}\rangle),\ n_j = 0, 1, \cdots,\right.$$
$$\left. j = 1, 2, \cdots, \sum n_j < \infty\right\}$$

is an orthonormal base for $L^2(d\nu)$.

THEOREM 3.6. (i) *The functions* $\{\Phi_{\{n_j\}}(\tilde{Y}), n_j = 0, 1, \cdots, \sum n_j < \infty\}$, *satisfy the following formula*

$$L\Phi_{\{n_j\}}(\tilde{Y}) = -(\sum_j jn_j)\Phi_{\{n_j\}}(\tilde{Y})\,. \tag{3.20}$$

The operator L, acting on $L^2(d\nu)$, is non-positive definite and self-adjoint.

(ii) *The function $P(t, \tilde{X}, \tilde{Y})$ satisfies the equation*

$$\frac{dP}{dt}(t, \tilde{X}, \tilde{Y}) = LP(t, \tilde{X}, \tilde{Y}), \quad t > 0, \quad \textit{for fixed } \tilde{X}. \tag{3.21}$$

Proof. When $F(\tilde{Y})$ is expressed in the form

$$F(\tilde{Y}) = f(v_1, \cdots, v_k), \quad v_j = \langle\xi_j, \tilde{Y}\rangle, \quad j = 1, \cdots, k\,,$$

with a polynomial f in v_j's, $F(\tilde{Y})$ satisfies

$$(LF)(\tilde{Y}) = \left[\frac{1}{2}\sum_{j=1}^{k}\frac{\partial^2 f}{\partial v_j^2} - \sum_{j=1}^{k}\frac{\partial f}{\partial v_j}\right]_{v_j=\langle\xi_j, \tilde{Y}\rangle}\,. \tag{3.18}$$

If, in particular, $f(v_1, \cdots, v_k) = \prod_{j=1}^k H_{n_j}(\sqrt{j}\,v_j)$, then we have

$$L\prod_{j=1}^{k} H_{n_j}(\sqrt{j}\langle\xi_j, \tilde{Y}\rangle) = \left[\frac{1}{2}\sum\frac{\partial^2 f}{\partial v_j^2} - \sum jv_j\frac{\partial f}{\partial v_j}\right]_{v_j=\langle\xi_j, \tilde{Y}\rangle}$$
$$= \sum_j\left[\left\{\frac{1}{2}j\frac{\partial^2 H_{n_j}}{\partial x_j^2} - jx_j\frac{\partial H_{n_j}}{\partial x_j}\right\}\prod_{i\neq j} H_{n_i}\right]_{x_j=\sqrt{j}\langle\xi_j, \tilde{Y}\rangle}\,.$$

Using a formula

$$\frac{d^2H_n}{dx^2} - 2x\frac{dH_n}{dx} + 2nH_n = 0$$

for the Hermite polynomial, we obtain

$$L\prod_{j=1}^{k} H_{n_j}(\sqrt{j}\langle\xi_j, \tilde{Y}\rangle) = -\left(\sum_j jn_j\right)\prod_i H_{n_i}(\sqrt{i}\langle\xi_i, \tilde{Y}\rangle)\,,$$

which proves (3.20). Since $\{\Phi_{\{n_j\}}(\tilde{Y})\}$ is an orthonormal base for $L^2(d\nu)$, the formula (3.20) proves that L is non-positive definite and self-adjoint.

Let $P_n(t, \tilde{X}, \tilde{Y})$ and $\psi_k(t, x, y)$ be functions given in the proof of Theorem 3.4. Because $P_n(t, \tilde{X}, \tilde{Y})$ is a tame function, it is easily seen that

$$\frac{\partial P_n}{\partial t}(t, \tilde{X}, \tilde{Y}) = LP_n(t, \tilde{X}, \tilde{Y}) = \Big(\sum_{k=1}^{n} \psi_k(t, x_k, y_k)\Big)P_n(t, \tilde{X}, \tilde{Y}) . \tag{3.22}$$

Recall that

$$P_n(t, \tilde{X}, \tilde{Y}) \longrightarrow P(t, \tilde{X}, \tilde{Y}) \text{ in } L^2(d\nu) . \tag{3.23}$$

We can further prove that

$$\frac{\partial P_n}{\partial t}(t, \tilde{X}, \tilde{Y}) \longrightarrow \frac{\partial P}{\partial t}(t, \tilde{X}, \tilde{Y}) \qquad \text{in } L^2(d\nu) \tag{3.24}$$

and

$$LP_n(t, \tilde{X}, \tilde{Y}) = \Big(\sum_{k=1}^{n} \psi_k\Big)P_n(t, \tilde{X}, \tilde{Y}) \longrightarrow \Big(\sum_{k=1}^{\infty} \psi_k\Big)P(t, \tilde{X}, \tilde{Y}) \quad \text{in } L^2(d\nu) . \tag{3.25}$$

Since the operator L is closable, the formulae (3.24) and (3.25) prove that $P(t, \tilde{X}, \tilde{Y}) \in D(L)$, $t > 0$, and that

$$LP(t, \tilde{X}, \tilde{Y}) = \lim_{n\to\infty} LP_n(t, \tilde{X}, \tilde{Y}) = \Big(\sum_{k=1}^{\infty} \psi_k\Big)P(t, \tilde{X}, \tilde{Y}) . \tag{3.26}$$

It is easily verified that dP/dt the derivative of $P(t, \tilde{X}, \cdot)$ in $L^2(d\nu)$-norm is equal to the point-wise partial derivative $\partial P/\partial t$. Therefore the equality (3.21) follows from (3.22), (3.24) and (3.26). (Q.E.D.)

We will now consider a generalized equation of (3.2)

$$d\tilde{X}_t = -\hat{\omega}\tilde{X}_t dt + BdB_t , \tag{3.27}$$

where B is a bounded linear operator defined on $\tilde{H}$. We can carry on the same analysis on (3.27) as we have done on (3.2), to obtain the following results:

1. There exists a unique invariant probability measure ν_B supported by $\tilde{H}_{-1/2}$ and its characteristic function $C_{\nu_B}(\xi)$ is given by

$$C_{\nu_B}(\xi) = \int_{\tilde{H}_{-1/2}} e^{-i\langle\xi, \tilde{X}\rangle} d\nu_B(\tilde{X}) = \exp\Big\{-\frac{1}{4}\|B^*\hat{\omega}^{1/2}\xi\|^2\Big\}, \qquad \xi \in \tilde{H}_{1/2} .$$

2. There is a dichotomy that the measures ν_B and ν are either mutually

equivalent or singular. They are equivalent if and only if $\hat{\omega}^{1/2}BB^*\hat{\omega}^{-1/2}-I$ is of Hilbert-Schmidt type.

3. The solution of (3.27) is continuous in $\tilde{H}_{-1/2}$ norm.

4. In order that the transition probability measure $P_B(t, \tilde{X}, d\tilde{Y})$ is equivalent to $\nu_B(d\tilde{Y})$, it is necessary and sufficient that $T_t\tilde{X}\in S_1(\tilde{H}_{-1/2})$ and that $S_1^{-1/2}T_tS_1T_tS_1^{-1/2}$ is of Hilbert-Schmidt type, where $S_1=(1/2)\hat{\omega}^{-1/2}BB^*\hat{\omega}^{-1/2}$.

5. A necessary and sufficient condition for $P_B(t, \tilde{X}, \tilde{Y})$ to be ν_B-symmetric is that the operator BB^* commutes with $\hat{\omega}$.

Remark 3.3. Our methods are applicable to some generalized cases. Let H be any real separable Hilbert space and let A be a positive unbounded operator on H such that A^{-1} is of Hilbert-Schmidt type. Then, for the equation on H of the form

$$dX_t = -AX_tdt + dB_t\ ,$$

we can do the same discussions as we have done for the equation (3.2).

§ 4. Multiple Wiener integrals on a Hilbert space

In this section we will first consider the Wiener's direct sum decompositions of $L^2(\mathscr{E}^*, \mu)$ and $L^2(\mathscr{E}^*\to K)$, and then we will proceed to investigate their integral representations. (See § 1 for the notations.)

1. Functionals of multi-parameter white noise.

Introduce the space of real valued functionals of white noise

$$\begin{aligned}(L^2)_D &= L^2(\mathscr{E}^*) = L^2(\mathscr{E}^*, \mathscr{B}, \mu)\\ &= \left\{\phi\,;\int_{\mathscr{E}^*}|\phi(\omega)|^2d\mu(\omega)<\infty\right\}.\end{aligned}$$

Then we are able to obtain the Wiener's direct sum decomposition of $(L^2)_D$ as follows.

For any $\phi\in(L^2)_D$ define $\mathscr{T}\phi$ by

$$(\mathscr{T}\phi)(\eta)=\int_{\mathscr{E}^*}e^{i\langle\eta,\omega\rangle}\phi(\omega)d\mu(\omega), \qquad \eta\in\mathscr{E}\ . \tag{4.1}$$

Putting

$$\phi(\omega)=\exp\left\{2t\frac{\langle\tilde{\eta},\omega\rangle}{\sqrt{2}}-t^2\right\}=\sum_k\frac{t^k}{k!}H_k(\langle\tilde{\eta},\omega\rangle/\sqrt{2}), \qquad \|\tilde{\eta}\|=1\ ,$$

where $H_k(\ )$ is the Hermite polynomial of degree k, we obtain

$$(\mathscr{T}\phi)(\eta) = C_\mu(\eta) \sum_{k=1}^{\infty} \frac{(i\sqrt{2}\,t)^k}{k!} (\eta, \tilde{\eta})^k .$$

Therefore we have proved

$$(4.2) \qquad (\mathscr{T} H_k(\langle\tilde{\eta}, \omega\rangle/\sqrt{2}))(\eta) = C_\mu(\eta)(\sqrt{2}\,i)^k(\tilde{\eta}, \eta)^k, \qquad \|\tilde{\eta}\| = 1 .$$

We then show a generalization of (4.2). Put $\phi(\omega) = \prod_{j=1}^{m} H_{K_j}(\langle\eta_j, \omega\rangle/\sqrt{2})$, $\sum K_j = n$, where $\{\eta_j\}$, $\eta_j \in \mathscr{E}$, is an orthonormal system in $\mathscr{H} = L^2(D\times T)$. Then we can easily prove that

$$\begin{aligned}(\mathscr{T}\phi)(\eta) &= C_\mu(\eta)(\sqrt{2}\,i)^n \prod_{j=1}^{m} (\eta, \eta_j)^{k_j} \\ &= C_\mu(\eta) i^n \int_{(D\times T)^n} \cdots \int F(x_1, \cdots, x_n)\eta^{n\otimes}(x_1, \cdots, x_n) dx_1 \cdots dx_n \\ &= C_\mu(\eta) i^n \int_{(D\times T)^n} \cdots \int \tilde{F}(x_1, \cdots, x_n)\eta^{n\otimes}(x_1, \cdots, x_n) dx_1 \cdots dx_n\end{aligned}$$

where $F(x_1, \cdots, x_n) = 2^{n/2}[\eta_1(x_1) \cdots \eta_1(x_{k_1})] \cdots [\cdots \eta_m(x_n)]$, and $\tilde{F}(x_1, \cdots, x_n)$ = symmetrization of $F(x_1, \cdots, x_n) = 1/n! \sum_\pi F(x_{\pi(1)}, \cdots, x_{\pi(n)})$, π; permutation of $(1, 2, \cdots, n)$, and $\eta^{n\otimes}$ = n-times tensor product of η.
The following formulae can easily be proved:

$$\|F\|_{L^2} = 2^{n/2}, \qquad \|\tilde{F}\|_{L^2} = 2^{n/2}(n!)^{-1/2}\left(\prod_{j=1}^{m} k_j!\right)^{1/2} .$$

On the other hand it is obvious that

$$\|\phi\|_{(L^2)_D} = \left(\prod_{j=1}^{m} k_j!\right)^{1/2} 2^{n/2} ,$$

and so we have established

$$(4.3) \qquad \|\phi\|_{(L^2)_D} = (n!)^{1/2}\|\tilde{F}\|_{L^2((D\times T)^n)} .$$

Let $\mathscr{H}_n$ be the subspace of $(L^2)_D$ spanded by the Hermite polynomials of degree n. Then we can prove

$$(L^2)_D = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n \qquad \text{(Wiener's direct sum decomposition)}.$$

In fact the correspondence between $\phi = \prod H_{k_j}(\langle\eta_j, \omega\rangle/\sqrt{2})$ and $\tilde{F} \in \hat{L}^2((D\times T)^n)$ can be extended to the one-to-one mapping from $\mathscr{H}_n$ to $\hat{L}^2((D\times T)^n)$, $n = 0, 1, 2, \cdots$, where $\hat{L}^2((D\times T)^n) = \{F; F \in L^2((D\times T)^n)$ and F is symmetric$\}$. We denote this transformation from $\mathscr{H}_n$ to $\hat{L}^2((D\times T)^n)$ by τ. Thus we have obtained the following diagram.

$$(L^2)_D = \sum \oplus \mathscr{H}_n \cong \sum \oplus \sqrt{n!}\, \hat{L}^2((D \times T)^n), \qquad \text{under } \tau:$$
$$\tau:\ \phi \longrightarrow \tau\phi \in \hat{L}^2((D \times T)^n), \qquad \phi \in \mathscr{H}_n,$$
$$(4.4) \quad (\mathscr{T}\phi)(\eta) = i^n C(\eta) \int_{(D\times T)^n} \cdots \int \tau\phi(x_1, \cdots, x_n)\eta^{n\otimes}(x_1, \cdots, x_n)dx_1 \cdots dx_n,$$
$$\|\phi\|_{(L^2)_D} = \sqrt{n!}\, \|\tau\phi\|_{L^2((D\times T)^n)} .$$

DEFINITION 4.1. $\tau\phi$ is called the kernel of the integral representation of ϕ.

EXAMPLE 4.1.

$$\tau(\prod_j H_{k_j}(\langle \eta_j, \omega\rangle/\sqrt{2})) = \text{the symmetrization of } 2^{n/2} \prod_j \otimes \eta_j^{k_j\otimes} ,$$

where $\{\eta_j\}$ is an orthonormal system in $L^2(D \times T)$.

2. Multiple Wiener integrals and iterated stochastic integrals I.

We are going to define multiple Wiener integrals $I(F)$, $F \in L^2((D \times T)^n)$. Let $\{\xi_\alpha\}$ be an orthonormal base in H and let $\{\zeta_\beta\}$ be an orthonormal base in $L^2(T)$. Then $\{\eta_j\}$, where η_j is of the form $\xi_\alpha \otimes \zeta_\beta$, is an orthonormal base in $\mathscr{H} = L^2(D \times T)$, and $\{\eta_{j_1\cdots j_n} = \eta_{j_1} \otimes \cdots \otimes \eta_{j_n}\}$ is an orthonormal base in $L^2((D \times T)^n)$. Therefore, if we define the multiple Wiener integrals for $\eta_{j_1\cdots j_n}$, we can extend this definition to all functions of $L^2((D \times T)^n)$. Let B_t be the cylindrical Brownian motion on $H = L^2(D)$ introduced in § 1. Put $F(x_1, \cdots, x_n) = \eta_{j_1} \otimes \cdots \otimes \eta_{j_n}(x_1, \cdots, x_n)$, then the multiple Wiener integral $I(F)$ of F is given by the next formula

$$I(F) = I_n(F) = \int \cdots \int \zeta_{\beta_1}(u_1) \cdots \zeta_{\beta_n}(u_n) dB_{u_1}(\xi_{\alpha_1}) \cdots dB_{u_n}(\xi_{\alpha_n}) ,$$

where $B_{u_k}(\xi_{\alpha_k})$, $k = 1, 2, \cdots, n$ are respectively 1-dimensional Brownian motions, so the right hand side of the formula is well-defined as the usual (finite dimensional) multiple Wiener integral of degree n. By simple calculation we know that $I(F)$ is of the form

$$I(F) = 2^{-n/2} \prod_j H_{k_j}(\langle \eta_j, \omega\rangle/\sqrt{2}), \qquad \Sigma k_j = n ,$$

where k_j is the multiplicity of η_j in $\eta_{j_1\cdots j_n}$.

DEFINITION 4.2. For F, $F \in L^2((D \times T)^n)$, $I(F)$ is called the multiple Wiener integral of F. The set of all $I(F)$, $F \in L^2((D \times T)^n)$, is called the space of multiple Wiener integrals of degree n.

From the definition of $I(F)$, we can easily prove that

$$(4.5) \qquad I(F) = I(\tilde{F}) .$$

THEOREM 4.1. *It holds that*

$$\{I(F)\colon F\in L^2((D\times T)^n)\}=\{I(\hat{F})\colon \hat{F}\in \hat{L}^2((D\times T)^n)\}=\mathscr{H}_n\,.$$

And the mapping $I|_{\hat{L}^2((D\times T)^n)}$, *the restriction of* I, *is the inverse of* τ.

Proof. From the formulae

$$I(\eta_{j_1\cdots j_n})=2^{-n/2}\prod_j H_{k_j}(\langle\eta_j,\omega\rangle/\sqrt{2}),\qquad \Sigma k_j=n\,,$$

and

$$\tau(\prod_j H_{k_j}(\langle\eta_j,\omega\rangle/\sqrt{2}))=\text{symmetrization of }\prod\otimes\eta_j^{k_j\otimes}\,,$$

it follows that $\phi=I(\tau\phi)$ for $\phi=\prod H_{k_j}(\langle\eta_j,\omega\rangle/\sqrt{2})$ (note (4.4)). This equality holds true for all $\phi\in\mathscr{H}_n$ by the linearity of τ and I. (Q.E.D.)

We have obtained the following diagram:

$$\begin{aligned}&\tau\colon \phi\longrightarrow\tau\phi\in\hat{L}^2((D\times T)^n),\quad \phi\in\mathscr{H}_n,\quad\text{bijection},\\ &I\colon F\longrightarrow I(F)\in\mathscr{H}_n,\qquad F\in L^2((D\times T)^n),\\ &\phi=I(\tau\phi),\\ &\|\phi\|_{(L^2)_D}=(n!)^{1/2}\|\tau\phi\|_{L^2((D\times T)^n)}\,.\end{aligned}\tag{4.6}$$

We will now give the definition of iterated stochastic integrals. Let $F(x_1,\cdots,x_n)$ be an element of $L^2((D\times T)^n)$. At first we assume that F is a simple function. Then, for fixed $(x_2,\cdots,x_n)$ and t_1 $F(\ )$ is identified with an element of $L^2(D)=H$. So we can consider the stochastic integral (defined in § 1)

$$\int_{-\infty}^{t_2}\langle F(x_1,x_2,\cdots,x_n),dB_{t_1}\rangle=\int_{-\infty}^{t_2}\langle F,dB_{t_1}(x_1)\rangle=\hat{I}_1(F)(x_1,\cdots,x_n)\tag{4.7}$$

for all $(x_2,\cdots,x_n)$, and we have

$$\int_{\mathscr{E}^*}|\hat{I}_1(F)(x_2,\cdots,x_n)|^2d\mu(\omega)=\int_{-\infty}^{t_2}\|F\|_H^2dt_1\leqq\|F\|^2_{L^2(D\times T)}\tag{4.8}$$

For fixed $(x_3,\cdots,x_n)$, $\hat{I}_1(F)$ is a function of x_2 and ω, and it is $\mathscr{B}_{t_2}$-adapted. Considering these facts, we know that the following formula holds

$$\begin{aligned}&\|\hat{I}_1(F)(\cdot,x_3,\cdots,x_n)\|^2_{L^2(T\times\mathscr{E}^*\to H)}\\ &\quad=\int_{D\times T}\|\hat{I}_1(F)\|^2_{L^2(\mathscr{E}^*)}dx_2\leqq\int_{D\times T}\|F\|^2_{L^2(D\times T)}dx_2<\infty\,,\end{aligned}$$

for all $(x_3,\cdots,x_n)$. Therefore the stochastic integral

$$\hat{I}_2(F)(x_3, \cdots, x_n) = \int_{-\infty}^{t_3} \langle \hat{I}_2(F), dB_{t_2} \rangle = \int_{-\infty}^{t_3} \langle \hat{I}_2(F), dB_{t_2}(\boldsymbol{x}_2) \rangle$$

is well-defined. Thus we are given the iterated stochastic integrals $\hat{I}_3(F)$, $\hat{I}_4(F), \cdots, \hat{I}_n(F) = \hat{I}(F)$, and we can easily verify the following inequality

$$(4.9) \qquad \| \hat{I}_k(F)(x_{k+1}, \cdots, x_n; \omega) \|^2_{L^2((D\times T)^{n-k}\times \mathscr{E}^*)} \leqq \| F \|^2_{L^2((D\times T)^n)}, \qquad k = 1, 2, \cdots, n,$$

where we have introduced the notation $\hat{I}$ in order to discriminate the iterated stochastic integral from the multiple Wiener integral $I(F)$.

The set $\{F$: $F \in L^2((D \times T)^n)$ and F is a linear combination of simple functions$\}$ is dense in $L^2((D \times T)^n)$. From this and (4.9), the mapping $\hat{I}_k$ can be extended to be a bounded linear mapping from $L^2((D \times T)^n)$ to $L^2((D \times T)^{n-k} \times \mathscr{E}^*)$. Therefore the iterated stochastic integral $\hat{I}(F) = \hat{I}_n(F)$ has been defined for all $F \in L^2((D \times T)^n)$.

The following notations will be used to denote $\hat{I}_k(F)$:

$$(4.10) \qquad \begin{aligned} \hat{I}_k(F) &= \int^{t_{k+1}} \int^{t_k} \cdots \int^{t_2} \langle F, dB_{t_1} \cdots dB_{t_k} \rangle \\ &= \int^{t_{k+1}} \left\langle \int^{t_k} \left\langle \cdots \int^{t_2} \langle F, dB_{t_1} \rangle, dB_{t_2} \right\rangle, \cdots, dB_{t_k} \right\rangle \\ &= \int^{t_{k+1}} \left\langle \int^{t_k} \left\langle \cdots \int^{t_2} \langle F, dB_{t_1}(\boldsymbol{x}_1) \rangle, \cdots, dB_{t_k}(\boldsymbol{x}_k) \right\rangle \right\rangle. \end{aligned}$$

THEOREM 4.2. *If $F \in \hat{L}^2((D \times T)^n)$, then the next formula holds*

$$(4.11) \qquad I(F) = n! \hat{I}(F).$$

Proof. Let $F = \eta_{i_1} \otimes \cdots \otimes \eta_{j_n}$, $\eta_{j_k} = \xi_{j_k} \otimes \zeta_{j_k}$, and put

$$\tilde{F} = \frac{1}{n!} \sum_\pi F^\pi \qquad \text{(symmetrization of } F\text{)},$$

where $F^\pi(x_1, \cdots, x_n) = F(x_{\pi(1)}, \cdots, x_{\pi(n)})$. From the definition of $I(F)$, we obtain

$$\begin{aligned} I(\tilde{F}) &= I(F) \\ &= \int \cdots \int \zeta_{j_1}(u_1) \cdots \zeta_{j_n}(u_n) dB_{u_1}(\xi_{j_1}) \cdots dB_{u_n}(\xi_{j_n}) \\ &= \sum_\pi \int_{\Delta\pi} \cdots \int \zeta_{i_1}(u_1) \cdots \zeta_{j_n}(u_n) dB_{u_1}(\xi_{j_1}) \cdots dB_{u_n}^{r}(\xi_{j_n}) \\ \Delta\pi &= \{(u_1, \cdots, u_n);\ u_{\pi(1)} < u_{\pi(2)} < \cdots < u_{\pi(n)}\} \\ &= \sum_\pi \int_{u_1 < \cdots < u_n} \cdots \int \zeta_{j_{\pi(1)}}(u_1) \cdots \zeta_{j_{\pi(n)}}(u_n) dB_{u_1}(\xi_{j_{\pi(1)}}) \cdots dB_{u_n}(\xi_{j_{\pi(n)}}) \end{aligned}$$

$$= \sum_{\pi} \hat{I}(F^{\pi^{-1}}) = \hat{I}\Big(\sum_{\pi} F^{\pi^{-1}}\Big) = \hat{I}(n!\tilde{F}) = n!\hat{I}(\tilde{F}) .$$

Since $\{\tilde{F};\ F$ is in the form of $\eta_{j_1} \otimes \cdots \otimes \eta_{j_n}\}$ is dense in $\hat{L}^2((D \times T)^n)$, the theorem has been proved.

COROLLARY 4.1. *$\phi \in \mathscr{H}_n$ has the following representation*

$$\begin{aligned} \phi &= I(\tau\phi) = n!\,\hat{I}(\tau\phi) \\ &= n! \int_{u_1 < \cdots < u_n} \cdots \int \langle \tau\phi(x_1, \cdots, x_n), dB_{u_1} \cdots dB_{u_n} \rangle . \end{aligned} \tag{4.12}$$

Remark 4.1. The equality (4.12) explains the reason why we call $\tau\phi$ the kernel of the integral representation of ϕ.

3. Hilbert space valued functionals of white noise.

Let K be a real separable Hilbert space and put

$$L^2(\mathscr{E}^* \longrightarrow K) = \Big\{\Phi;\ \mathscr{E}^* \longrightarrow K, \int_{\mathscr{E}^*} \|\Phi(\omega)\|_K^2 d\mu(\omega) < \infty\Big\} .$$

First we shall consider the Wiener's direct sum decomposition of $L^2(\mathscr{E}^* \to K)$. Let ψ be an element of K. Then $(\phi, \psi)_K$ is an element of $(L^2)_D$, so we can apply the mapping $\mathscr{T}$ (which is already defined by (4.1)) to $(\phi, \psi)_K$:

$$(\mathscr{T}\Phi)(\psi, \eta) = (\mathscr{T}(\Phi, \psi)_K(\eta) = \int_{\mathscr{E}^*} e^{i\langle \eta, \omega \rangle}(\Phi, \psi)_K d\mu(\omega) . \tag{4.13}$$

By (4.4), $(\Phi, \psi)_K$ has the integral kernel $\tau(\Phi, \psi)_K$ which belongs to $\hat{L}^2((D\times T)^n)$ and satisfies

$$\|(\Phi, \psi)_K\|_{(L^2)_D} = (n!)^{1/2} \|\tau(\Phi, \psi)_K\|_{L^2((D\times T)^n)} . \tag{4.14}$$

Put

$$\mathscr{H}_n(K) = \{\Phi \in L^2(\mathscr{E}^* \to K);\ (\Phi, \psi)_K \in \mathscr{H}_n \quad \text{for any } \psi \in K\} . \tag{4.15}$$

Then the mapping $(\Phi, \psi) \to \tau(\Phi, \psi)_K$, where $\Phi \in \mathscr{H}_n(K)$ and $\psi \in K$, gives a bounded bilinear operator from $\mathscr{H}_n(K) \times K$ to $\hat{L}^2((D \times T)^n)$. The mapping τ can be regarded as an operator from $\mathscr{H}_n(K)$ to $\mathscr{L}(K \to \hat{L}^2((D\times T)^n))$ such that

$$\tau\Phi:\ \psi \longrightarrow (\tau\Phi)(\psi) = \tau(\Phi, \psi)_K \in \hat{L}^2((D \times T)^n), \qquad \psi \in K , \tag{4.16}$$

where $\mathscr{L}(K_1 \to K_2)$ is the linear space of all bounded linear operators from K_1 to K_2.

THEOREM 4.3. *For $\Phi \in \mathscr{H}_n(K)$ the operator $\tau\Phi$: $K \to \hat{L}^2((D \times T)^n)$, is of Hilbert-Schmidt type and satisfies*

$$\|\Phi\|_{L^2(\mathscr{E}^* \to K)} = (n!)^{1/2} \|\tau\Phi\|_{H-S} , \tag{4.17}$$

where $\| \quad \|_{H-S}$ stands for the Hilbert-Schmidt norm.

Proof. It is easy to see that $\tau\Phi$ is a bounded linear operator. The Hilbert-Schmidt norm of $\tau\Phi$ is calculated as follows. Let $\{e_j\}$ be an orthonormal base in K. Then it holds that

$$\|\tau\Phi\|_{H-S}^2 = \sum_j \|\tau(\Phi, e_j)_K\|_{L^2((D\times T)^n)}^2 = \sum_j \frac{1}{n!} \|(\Phi, e_j)_K\|_{(L^2)_D}^2 = \frac{1}{n!} \|\Phi\|_{L^2(\mathscr{E}^* \to K)}^2$$

(Q.E.D.)

We denote by σ_2 (H_1, H_2) the Hilbert space consisting of all Hilbert-Schmidt operators from H_1 to H_2.

COROLLARY 4.2. *The mapping τ can be regarded as a linear mapping from $\mathscr{H}_n(K)$ to $\sigma_2(K, \hat{L}^2((D\times T)^n))$ and then τ is bijective. The operator norm $\|\tau\|$ of τ equals $(n!)^{-1/2}$.*

DEFINITION 4.3. The mapping τ: $\mathscr{H}_n(K) \to \sigma_2$ $(K, \hat{L}^2((D \times T)^n))$ is called the first representation of $\mathscr{H}_n(K)$ and $\tau\Phi$ is called the first representation of Φ.

Since $\tau\Phi$ is a Hilbert-Schmidt operator from K to $\hat{L}^2((D \times T)^n)$, $(\tau\Phi)^*$, the adjoint operator of $\tau\Phi$, is a Hilbert-Schmidt operator from $\hat{L}^2((D\times T)^n)$ to K and $\|(\tau\Phi)^*\|_{H-S} = \|\tau\Phi\|_{H-S}$. We denote the isomorphism form $\mathscr{H}_n(K)$ to $\sigma_2(\hat{L}^2((D\times T)^n), K)$ by τ^* (that is $\tau^*\Phi = (\tau\Phi)^*$).

DEFINITION 4.4. The operator τ^*: $\mathscr{H}_n(K) \to \sigma_2(\hat{L}^2((D \times T)^n), K)$ is called the second representation of $\mathscr{H}_n(K)$ and $\tau^*\Phi$ is called the second representation of Φ.

COROLLARY 4.3. *The mapping τ^* is bijective, and satisfies*

$$\|\Phi\|_{L^2(\mathscr{E}^* \to K)} = \sqrt{n!}\, \|\tau^*\Phi\|_{H-S} \tag{4.18}$$

THEOREM 4.4. *It holds that*

$$L^2(\mathscr{E}^* \longrightarrow K) = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n(K). \qquad (\textit{direct sum}) \tag{4.19}$$

Proof. Let $\{e_j\}$ be an orthonormal base in K. Then any element Φ of $L^2(\mathscr{E}^* \to K)$ is expanded as $\Phi(\omega) = \sum_{j=1}^{\infty} \phi_j(\omega)e_j$, where $\phi_j(\omega) \in (L^2)_D$. By

(4.4), $\phi_j(\omega)$ is decomposed as $\phi_j(\omega) = \sum \phi_{n,j}(\omega)$, $\phi_{n,j}(\omega) \in \mathscr{H}_n$, to have $\Phi(\omega) = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \phi_{n,j}(\omega) e_j$. Put $K_{n,j} = \{\phi(\omega)e_j : \phi(\omega) \in \mathscr{H}_n\}$, then $L^2(\mathscr{E}^* \to K) = \sum_{n,j} \oplus K_{n,j}$. Putting $\tilde{K}_n = \sum_j \oplus K_{n,j}$, we obtain $L^2(\mathscr{E}^* \to K) = \sum \oplus \tilde{K}_n$. Since the equality $\tilde{K}_n = \mathscr{H}_n(K)$ is obvious, the proof of the theorem has been complete. (Q.E.D.)

DEFINITION 4.5. The decomposition $L^2(\mathscr{E}^* \to K) = \sum \oplus \mathscr{H}_n(K)$ is called the Wiener's direct sum decomposition of $L^2(\mathscr{E}^* \to K)$, and the second representation $\tau^*\Phi$, $\Phi \in \mathscr{H}_n(K)$, is called the kernel of Φ.

4. Multiple Wiener integrals and iterated stochastic integrals II.

We will now define the multiple Wiener integral of S, $S \in \sigma_2(L^2((D \times T)^n), K)$.

DEFINITION 4.6. For an element S in $\sigma_2(L^2((D \times T)^n), K)$, the multiple Wiener integral of S, call it $I(S) = \int \cdots \int S dB_{t_1} \cdots dB_{t_n}$, is an element of $L^2(\mathscr{E}^* \to K)$ determined by the formula

$$(I(S), \psi)_K = I(S^*\psi) \qquad \text{for any } \psi \in K, \tag{4.20}$$

where $I(S^*\psi)$ is the multiple Wiener integral of $S^*\psi$ in the sense of Definition 4.2. The set of all $I(S)$, $S \in \sigma_2(L^2((D \times T)^n), K)$, is called the space of multiple Wiener integrals of degree n.

Remark 4.2. If $\hat{S} \in \sigma_2(\hat{L}^2((D \times T)^n), K)$, then $\hat{S}^*\psi \in \hat{L}^2((D \times T)^n)$ for $\psi \in K$. Therefore an element in $L^2(\mathscr{E}^* \to K)$ is uniquely determined by

$$(I(\hat{S}), \psi)_K = I(\hat{S}^*\psi) \qquad \text{for any } \psi \in K .$$

$I(\hat{S}) = \int \cdots \int \hat{S} dB_{t_1} \cdots dB_{t_n}$ is called the multiple Wiener integral of $\hat{S}$.

THEOREM 4.5. (i) *For $S \in \sigma_2(L^2((D \times T)^n), K)$ put $\hat{S} = S|_{\hat{L}^2((D\times T)^n)}$ (the restriction of S to $\hat{L}^2((D \times T)^n)$). Then it holds that $I(S) = I(\hat{S})$.*

(ii) *The operator I: $\sigma_2(L^2((D \times T)^n), K) \to L^2(\mathscr{E}^* \to K)$ is linear and bounded. In addition, for $S \in \sigma_2(\hat{L}^2((D \times T)^n), K)$, we have*

$$\|I(S)\|_{L^2(\mathscr{E}^* \to K)} = (n!)^{1/2} \|S\|_{H-S}. \tag{4.21}$$

Proof. (i) We will first prove that

$$\hat{S}^*\psi = \widetilde{S^*\psi} \qquad \text{for any } \psi \in K, \tag{4.22}$$

where $\sim$ stands for the symmetrization. This equality comes from the next formulae, for $f \in \hat{L}^2((D \times T)^n)$

$$(\widetilde{S^*\psi}, f)_{L^2} = \frac{1}{n!} \sum_\pi ((S^*\psi)^\pi, f)_{L^2} = \frac{1}{n!} \sum_\pi (S^*\psi, f)_{L^2}$$
$$= (S^*\psi, f)_{L^2} = (\psi, Sf)_K$$

and

$$(\hat{S}^*\psi, f)_{L^2} = (\psi, \hat{S}f)_K = (\psi, Sf)_K .$$

From (4.5) and (4.22) we obtain

$$(I(S), \psi)_K = I(S^*\psi) = I(\widetilde{S^*\psi}) = I(\hat{S}^*\psi) = (I(\hat{S}), \psi)_K ,$$

which is to be proved.

(ii) Let $\{e_j\}$ be an orthonormal base in K. Then, from (4.6) we have

$$\int_{\mathscr{E}^*} \|I(S)\|^2 d\mu(\omega) = \int_{\mathscr{E}^*} \sum_j |(I(S), e_j)|^2 d\mu(\omega)$$
$$= \sum_j \int_{\mathscr{E}^*} |I(S^*e_j)|^2 d\mu(\omega) = \sum_j n! \| S^*e_j \|^2_{L^2}$$
$$= n! \| S^* \|^2_{H-S} = n! \| S \|^2_{H-S} .$$

The rest of Theorem is trivial. (Q.E.D.)

THEOREM 4.6. *Let $\Phi(\omega)$ be an element of $\mathscr{H}_n(K)$. Then Φ is represented as the multiple Wiener integral of $\tau^*\Phi$, that is*

$$\Phi(\omega) = I(\tau^*\Phi) = \int \cdots \int \tau^*\Phi dB_{t_1} \cdots dB_{t_n} . \tag{4.23}$$

Proof. Since the inner product (Φ, ψ), $\psi \in K$, belongs to $\mathscr{H}_n$, we obtain

$$(\Phi, \psi)_K = I(\tau(\Phi, \psi)_K) = I((\tau\Phi)\psi) .$$

On the other hand it holds that

$$(I(\tau^*\Phi), \psi)_K = I((\tau^*\Phi)^*\psi) = I(((\tau\Phi)^*)^*\psi) = I((\tau\Phi)\psi) .$$

Therefore we have proved the Theorem. (Q.E.D.)

Let B_t be the cylindrical Brownian motion on $H = L^2(D)$ introduced in § 1. Then the stochastic integral $\int_T S(t)dB_t$ is defined, where S is $\mathscr{B}_t$-adapted and $S \in L^2(T \times \mathscr{E}^* \to \sigma_2(H, K))$. The integral $\int_T S(t)dB_t$ is an element of $L^2(\mathscr{E}^* \to K)$. We will treat the iteration of stochastic integral of this type.

Let $S \in \sigma_2(H, K)$, then S is represented as

$$S\xi = \sum_j \lambda_j(\xi, \xi_j)e_j = \int_D F_S(x)\xi(x)dx \tag{4.24}$$

where $\{\xi_j\}$ is an orthonormal base in H, $\{e_j\}$ is an orthonormal system of K, $\sum \lambda_j^2 = \|S\|_{H-S}^2 < \infty$, and $F_S(x) = \sum_j \lambda_j \xi_j(x) e_j$ $(F_S: D \to K)$. Since F_S satisfies

$$\int_D \|F_S\|_K^2 dx = \sum_j \int \lambda_j^2 \xi_j^2(x) dx = \sum_j \lambda_j^2 = \|S\|_{H-S}^2 < \infty ,$$

F_S belongs to $L^2(D \to K)$.

PROPOSITION 4.1. *The above correspondence of S and F_S determines an isomorphism between $\sigma_2(H, K)$ and $L^2(D \to K)$.*

Proof. It is already proved that $F_S \in L^2(D \to K)$ and $\|F_S\| = \|S\|_{H-S}$ for any $S \in \sigma_2(H, K)$. Let $F \in L^2(D \to K)$ and put S_F the operator defined by

$$S_F \xi = \int_D F(x)\xi(x)dx, \qquad \xi \in H .$$

Then S_F: $H \to K$ is well-defined and is a bounded linear operator. The Hilbert-Schmidt norm of S_F is equal to

$$\begin{aligned}
\sum_j \|S_F \xi_j\|_K^2 &= \sum_j \sum_i |(S_F \xi_j, e_i)_K|^2 \\
&= \sum_{i,j} \left| \int_D (F(x), e_i)_K \cdot \xi_j(x) dx \right|^2 \\
&= \sum_i \int_D |(F(x), e_i)_K|^2 dx \\
&= \|F\|_{L^2(D \to K)}^2 .
\end{aligned}$$

Thus the proof is completed. (Q.E.D.)

In the same manner we can prove the next proposition.

PROPOSITION 4.2. *The following diagrams are true.*

$$\begin{aligned}
&\sigma_2(L^2(D^n), K) \cong L^2(D^n \to K) \\
&\sigma_2(\hat{L}^2(D^n), K) \cong \hat{L}^2(D^n \to K) \\
&\sigma_2(L^2((D \times T)^n), K) \cong L^2((D \times T)^n \to K)
\end{aligned}$$

$$\sigma_2(\hat{L}^2((D \times T)^n), K) \cong \hat{L}^2((D \times T)^n \to K)$$

where $\cong$ *denotes isomorphism.*

PROPOSITION 4.3. (i) *We have the following diagram*

$$L^2(\mathscr{E}^* \longrightarrow \sigma_2(L^2(D \times T), K)) \cong L^2(\mathscr{E}^* \longrightarrow L^2((D \times T) \longrightarrow K))$$
$$L^2(T \times \mathscr{E}^* \longrightarrow \sigma^2(L^2(D), K) \cong L^2(T \times \mathscr{E}^* \longrightarrow L^2(D \longrightarrow K))$$
$$\cong L^2(\mathscr{E}^* \times D \times T \longrightarrow K)$$

(ii) *If* $S \in L^2(T \times \mathscr{E}^* \to \sigma_2(L^2(D), K))$ *and* S *is* $\mathscr{B}_t$*-adapted, then the stochastic integral* $\int_T S(t, \omega)dB_t$ *is well-defined. Put* $\hat{I}(S) = \int_T S(t)dB_t$, *then*

$$E[\|\hat{I}(S)\|_K^2] = \int E[\|S\|_{H-S}^2]dt = \int_{D \times T} E[\|F_S\|_K^2]dx ,$$

where F_S *is the element of* $L_2(\mathscr{E}^* \times D \times T \to K)$ *corresponding to* S *by* (4.24).

Proof. (i) is easily proved by the use of Proposition 4.2. (ii) is obvious from the definition of the stochastic integral and Proposition 4.1. (Q.E.D.)

Remark 4.3. We identify the elements of $L^2(T \times \mathscr{E}^* \to \sigma_2(L^2(D), K))$ with those of $L^2(\mathscr{E}^* \times D \times T \to K)$ and we use the notation $\int_T S(t, \omega)dB_t$ or $\int_T S(t, \omega; x)dB_t(x)$ to indicate the stochastic integral $\hat{I}(S)$. We also use the notation $S(x_1, \cdots, x_n)$ instead of $F_S(x_1, \cdots, x_n)$.

We are now ready to define the iterated stochastic integrals. Let $S \in \sigma_2(L^2((D \times T)^n), K)$, then S is considered as an element of $L^2((D \times T)^n \to K)$ by Proposition 4.2. We assume here that S is a simple functional. Once $(x_2, \cdots, x_n)$ is fixed, we can regard S as an element of $L^2(D \times T \to K)$. While S is also regarded as an element of $\sigma_2(L^2(D \times T), K)$ or $L^2(T \to \sigma_2(L^2(D), K))$ by Proposition 4.3 (i), and this function is denoted by $S^{x_2 \cdots x_n}(t_1)$. Thus we know that the stochastic integral

$$\hat{I}_1(S)(x_2, \cdots, x_n) = \int_{-\infty}^{t_2} S^{x_2 \cdots x_n}(t_1)dB_{t_1}$$

is well-defined in the sense of Definition 1.4 and we have

$$E\|\hat{I}_1(S)(x_2, \cdots, x_n)\|_K^2 = \int^{t_2}\left\{\int_D \|S(x_1, \cdots, x_n)\|_K^2 dx_1\right\}dt_1 \leqq \int_{D \times T} \|S\|_K^2 dx_1 .$$

From this inequality we obtain

$$\int_{(D\times T)^{n-1}}\cdots\int E\|\hat{I}_1(S)(x_2,\cdots,x_n)\|_K^2dx_2\cdots dx_n\leqq\int_{(D\times T)^n}\int\|S\|_K^2dx_1\cdots dx_n\,,$$
$$\hat{I}_1(S)\in L^2(\mathscr{E}^*\longrightarrow L^2((D\times T)^{n-1}\longrightarrow K))\cong L^2(\mathscr{E}^*\longrightarrow\sigma_2(L^2((D\times T)^{n-1},K))\,.$$

Using Proposition 4.3, we can define the stochastic integral

$$\hat{I}_2(S)(x_3,\cdots,x_n)=\int^{t_3}\hat{I}_1(S)(x_2,\cdots,x_n)dB_{t_2}(x_2)\,,$$

which has the same properties as $\hat{I}_1(S)(x_2,\cdots,x_n)$ described above. Repeating this procedure n times, we finally arrive at the iterated stochastic integral

$$\hat{I}(S)=\int\left\{\int^{t_n}\left\{\cdots\left\{\int^{t_2}S^{x_2\cdots x_n}(t_1)dB_{t_1}\right\}dB_{t_2}\cdots\right\}dB_{t_{n-1}}\right\}dB_{t_n}\,,\tag{4.25}$$

and $\hat{I}(S)$ satisfies

$$\|\hat{I}(S)\|^2_{L^2(\mathscr{E}^*\to K)}\leqq\int_{(D\times T)^n}\cdots\int\|S\|_K^2dx_1\cdots dx_n=\|S\|^2_{H-S}\,.\tag{4.26}$$

Since the set $\{S\colon S\in L^2((D\times T)^n\to K),\ S$ is a linear combination of simple functions$\}$ is dense in $L^2((D\times T)^n\to K)$, the mapping $\hat{I}\colon S\to\hat{I}(S)$, can be extended to be a bounded linear mapping from $L^2((D\times T)^n\to K)$ ($\cong\sigma_2(L^2((D\times T)^n),K)$) to $L^2(\mathscr{E}^*\to K)$.

DEFINITION 4.7. The K-valued random variable $\hat{I}(S)$, which has just been obtained above, is called the iterated stochastic integral of S.

$\hat{I}(S)$ is also denoted by

$$\int_{t_1\leqq\cdots\leqq t_n}\cdots\int S(x_1,\cdots,x_n)dB_{t_1}(x_1)\cdots dB_{t_n}(x_n)$$

or simply by $\int_{t_1\leqq\cdots\leqq t_n}\cdots\int SdB_{t_1}\cdots dB_{t_n}$.

Remark 4.4. If $\hat{S}\in\sigma_2(\hat{L}^2((D\times T)^n),K)\cong\hat{L}^2((D\times T)^n\to K)$, then $\hat{S}$ can be regarded as an element of $L^2((D\times T)^n\to K)$. So the iterated stochastic integral $\hat{I}(\hat{S})$ is well-defined.

PROPOSITION 4.4. *Let* $S\in\sigma_2(L^2((D\times T)^n),K)$ *and put* $\hat{S}=S|_{\hat{L}^2((D\times T)^n)}$. *Then* $\hat{S}\in\sigma_2(\hat{L}^2((D\times T)^n),K)$ *and* $F_{\hat{S}}=\tilde{F}_S$(*=the symmetrization of* F_S).

Proof. For any $f\in\hat{L}^2((D\times T)^n)$ it holds that

$$Sf=\int\cdots\int F_S(x_1,\cdots,x_n)f(x_1,\cdots,x_n)dx_1\cdots dx_n$$

$$= \int \cdots \int F_S^{\pi}(x_1, \cdots, x_n) f(x_1, \cdots, x_n) dx_1 \cdots dx_n$$

$$= \int \cdots \int \tilde{F}_S(x_1, \cdots, x_n) f(x_1, \cdots, x_n) dx_1 \cdots dx_n .$$

Therefore we obtain $F_{\hat{S}} = \tilde{F}_S$. (Q.E.D.)

We will finally discuss the connection between $I(S)$ and $\hat{I}(S)$.

THEOREM 4.7. *It holds that*

$$I(S) = n!\,\hat{I}(S) \qquad \text{for any } S \in \sigma_2((D \times T)^n), K). \tag{4.27}$$

Proof. Let $S \in \sigma_2(L^2((D \times T)^n), K)$, then by Proposition 4.2 there is a function $F_S \in L^2((D \times T)^n \to K)$ such that

$$Sf = \int \cdots \int F_S(x_1, \cdots, x_n) f(x_1, \cdots, x_n) dx_1 \cdots dx_n,\ f \in L^2((D \times T)^n) .$$

Define the operator $S^{x_2 \cdots x_n} \in \sigma_2(L^2(D \times T), K)$ by

$$S^{x_2 \cdots x_n} \eta = \int F_S(x_1, x_2, \cdots, x_n) \eta(x_1) dx_1 .$$

Then $S^{x_2 \cdots x_n}$ is well-defined for almost all $(x_2, \cdots, x_n)$, and the next equality holds

$$\begin{aligned} ((S^{x_2 \cdots x_n})^* \psi, \eta)_{L^2(D \times T)} &= (\psi, S^{x_2 \cdots x_n} \eta)_K \\ &= (\psi, \int F_S(x_1, x_2, \cdots, x_n) \eta(x_1) dx_1)_K \\ &= \int (\psi, F_S(x_1, x_2, \cdots, x_n))_K \eta(x_1) dx_1 \end{aligned}$$

for $\psi \in K$ and $\eta \in L^2(D \times T)$. We can regard $((S^{x_2 \cdots x_n})^* \psi)(x_1)$ as an element of $L^2((D \times T)^n)$, and then we have

$$\begin{aligned} &(((S^{x_2 \cdots x_n})^* \psi)(x_1), \eta \otimes g)_{L^2((D \times T)^n)} \\ &= \int_{(D \times T)^{n-1}} \cdots \int ((S^{x_2 \cdots x_n})^* \psi, \eta)_{L^2(D \times T)} g(x_2, \cdots, x_n) dx_2 \cdots dx_n \\ &= \int_{(D \times T)^n} \cdots \int (\psi, F_S)_K \eta(x_1) g(x_2, \cdots, x_n) dx_1 \cdots dx_n \end{aligned} \tag{4.28}$$

for $g \in L^2((D \times T)^{n-1})$. On the other hand it holds that

$$\begin{aligned} &(S^* \psi, \eta \otimes g)_{L^2((D \times T)^n)} = (\psi, S^*(\eta \otimes g))_K \\ &= \left(\psi, \int \cdots \int F_S(x_1, \cdots, x_n) \eta(x_1) g(x_2, \cdots, x_n) dx_1 \cdots dx_n \right)_K \end{aligned} \tag{4.29}$$

$$= \int \cdots \int (\psi, F_S)\eta(x_1)g(x_2, \cdots, x_n)dx_1 \cdots dx_n .$$

From (4.28) and (4.29) we obtain

$$((S^{x_2 \cdots x_n})^* \psi)(x_1) = (S^* \psi)(x_1, \cdots, x_n) \qquad \text{in } L^2((D \times T)^n) ,$$

and so we know that for almost all $(x_2, \cdots, x_n)$

$$(4.30) \qquad (S^{x_2 \cdots x_n})^* \psi = (S^* \psi)(\cdot, x_2, \cdots, x_n)$$

in $L^2(D \times T)$ sense. From this equality it follows that

$$\begin{aligned} &(\psi, \hat{I}_1(S)(x_2, \cdots, x_n))_K \\ &= \int^{t_2} \langle ((S^{x_2 \cdots x_n})^* \psi)(x_1), \ dB_{t_1}(x_1) \rangle \\ &= \int^{t_2} \langle (S^* \psi)(x_1, \cdots, x_n), dB_{t_1}(x_1) \rangle \end{aligned}$$

for almost all $(x_2, \cdots, x_n)$.

Now we shall prove inductively that

$$(4.31) \qquad (\psi, \hat{I}_k(S))_K = \int^{t_{k+1}} \Big\langle \int \cdots \int^{t_2} \langle S^* \psi, dB_{t_1} \rangle, \cdots, dB_{t_k} \Big\rangle$$

for almost all $(x_{k+1}, \cdots, x_n)$, $k = 1, 2, \cdots, n$. Assume that (4.31) is proved for $k-1$. Then, by the definition of $\hat{I}_{k-1}(S)$, for any $\eta \in L^2(D \times T)$ we have

$$(4.32) \qquad \begin{aligned} &(\hat{I}_{k-1}(S)^* \psi, \eta)_{L^2(D \times T)} = (\psi, \hat{I}_{k-1}(S)\eta)_K \\ &= \int_{D \times T} (\psi, \hat{I}_{k-1}(S)(x_k, \cdots, x_n))_K \eta(x_k) dx_k \\ &= ((\psi, \hat{I}_{k-1}(S))_K, \eta)_{L^2(D \times T)} . \end{aligned}$$

Using the assumption of induction and (4.32), we know that

$$\hat{I}_{k-1}(S)^* \psi = \int^{t_k} \Big\langle \cdots \int^{t_2} \langle S^* \psi, dB_{t_1} \rangle, \cdots, dB_{t_{k-1}} \Big\rangle ,$$

so we obtain

$$\begin{aligned} (\psi, \hat{I}_k(S))_K &= \int^{t_{k+1}} \langle \hat{I}_{k-1}(S)^* \psi, dB_{t_k} \rangle \\ &= \int^{t_{k+1}} \Big\langle \Big\{ \int^{t_k} \Big\langle \cdots \int^{t_2} \langle S^* \psi, dB_{t_1} \rangle, \cdots, dB_{t_{k-1}} \Big\rangle \Big\}, dB_{t_k} \Big\rangle , \end{aligned}$$

which is to be proved.

Putting $k = n$ in the formula (4.31), we obtain

$$(\psi, \hat{I}(S))_K = \hat{I}(S^*\psi) .$$

Let $S \in \sigma_2(\hat{L}^2((D \times T)^n), K)$, then $S^*\psi \in \hat{L}^2((D \times T)^n)$. By Theorem 4.2 we have

$$(\psi, I(S))_K = I(S^*\psi) = n!\hat{I}(S^*\psi) = n!(\psi, \hat{I}(S))_K$$

for any $\psi \in K$. Thus we have proved (4.27). (Q.E.D.)

Summing up what have been discussed, we have obtained the following diagram:

(4.33)
$$L^2(\mathscr{E}^* \longrightarrow K) = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n(K),$$
$$\mathscr{H}_n(K) \cong \sqrt{n!}\,\sigma_2(\hat{L}^2((D \times T)^n), K) \cong \sqrt{n!}\,\hat{L}^2((D \times T)^n \longrightarrow K),$$
$$\tau^*: \Phi \longrightarrow \tau^*\Phi \in \sigma_2(\hat{L}^2((D\times T)^n), K) \cong \hat{L}^2((D\times T)^n \longrightarrow K),$$
$$\Phi \in \mathscr{H}_n(K), \text{ bijection },$$
$$I: F \longrightarrow I(F) = n!\hat{I}(F) = \int_{t_1 \leqq \cdots \leqq t_n} \int F dB_{t_1} \cdots dB_{t_n},$$
$$F \in \hat{L}^2((D\times T)^n \longrightarrow K),$$
$$I \cdot \tau^* = \text{identity} .$$

Before closing this section, we mention an interesting result in connection with the stochastic differential equation (3.2) in § 3. Since the unique solution $\tilde{X}_t$ of (3.2) is an element of $L^2(\mathscr{E}^* \to \tilde{H}_{-1/2})$, $\tilde{X}_t$ has a kernel of integral representation.

THEOREM 4.8. *The kernel of the integral representation of* $\tilde{X}_t$ *with* $\hat{X}_0 = 0$ *is given by*

$$\chi_{[0,t]}(u) \sum_{j=1}^{\infty} e^{-j(t-u)} \xi_j(\sigma) \xi_j .$$

Proof. This result follows from the above discussions without any difficulties. (Q.E.D.)

§ 5. Stochastic differential equations with multiplicative operator

This section is devoted to a development of our theory. Actually we shall discuss the so-called bilinear stochastic differential equation on the Hilbert space $H = L^2([0, \pi])$ given by

(5.1)
$$dX_t = -\hat{\omega} X_t dt + X_t \cdot dB_t ,$$

where $\hat{\omega}$ $(= -\sqrt{-\Delta})$ was given in § 3 and $X_t\cdot$ denote the multiplicative operator, i.e. $(X_t\cdot\xi)(\sigma)=X_t(\sigma)\xi(\sigma)$, $\sigma\in[0,\pi]$.

Although our techniques developed in § 3 are not available to the equation of this type, our results established in § 4 do work in the investigation of the equation above. The reason could be seen in the discussions what follow.

Before we come to details, we have to overcome a difficulty. Namely, there is no solution to (5.1) living even in $H_{-\infty}$ formed by the Hilbert scale derived from $\hat{\omega}$. To avoid this difficulty, we are led to consider a modified equation (5.4) under the assumption (5.5) (note that if $\Gamma(\sigma,\sigma')=\delta(\sigma-\sigma')$, then (5.4) turns into (5.1)), which will be prescribed later.

We first investigate some properties of the multiplicative operator. Let A be a positive self-adjoint operator on H determined by

$$A\xi_0=\xi_0$$
$$A\xi_j=\hat{\omega}\xi_j=j\xi_j,\qquad j=1,2,\cdots,$$

where $\{\xi_j\}$, $j=0,1,2,\cdots$, is the orthonormal base for H given in § 3 (i.e. $\xi_0=\pi^{-1/2}$ and $\xi_j=(2/\pi)^{1/2}\cos j\sigma$). Since A^{-1} is of Hilbert-Schmidt type, a Hilbert scale $\{H_\alpha\}$, $-\infty<\alpha<\infty$, is generated by A, where H_α is a Hilbert space with an inner product $(\xi,\eta)_\alpha=(A^\alpha\xi,A^\alpha\eta)_H$.

Put $T_t=e^{-t\hat{\omega}}$, $t\geqq 0$, (T_t is the same operator as given in § 3, but the domain is H, not $\tilde{H}$). Then we have

PROPOSITION 5.1. (i) *The multiplicative operator $X\cdot$, $X\in H$, is not bounded on H. If $t>0$, then the closed extension of $T_tX\cdot$ is a Hilbert-Schmidt operator on H.*

(ii) *The multiplicative operator $X\cdot$, $X\in H$, is considered as a bounded operator from H into $H_{-\alpha}$, $\alpha>1/2$. Moreover it holds that $X\cdot\in\sigma_2(H,H_{-\alpha})$ and that*

$$\|X\cdot\|^2_{\sigma_2(H,H_{-\alpha})}\leqq\text{const.}\left(1+\sum_{k=1}^{\infty}k^{-2\alpha}\right)\|X\|^2\,.$$

(iii) *If $X\in H_{-\alpha}$ for some $\alpha>0$ but $X\notin H$, then $T_tX\cdot\notin\sigma_2(H,H_{-\beta})$ however large β may be chosen.*

Remark 5.1. For $X\in H$ $T_tX\cdot$ is of Hilbert-Schmidt type by Proposition 5.1 (i). But the integration $\int_0^t\|T_{t-S}X\cdot\|^2_{\sigma_2(H)}ds$ does not always converge. For example, if $X\neq 0$ and $a_n\geqq 0$ in the expansion $X=\sum a_n\xi_n$, then $\int_0^t\|T_{t-s}X\cdot\|^2_{\sigma_2(H)}ds$ diverges.

Proof of Proposition 5.1. (i) Take an element X in $H=L^2([0,\pi])$ such that $X\notin L^4([0,\pi])$. Then $X\cdot X=X^2\notin H$, i.e. $X\cdot$ is not a bounded operator on H. Having the X expressed in the form $X=\sum a_n\xi_n$, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\|T_t(X\xi_m)\|^2&=\sum_m\sum_k(T_t(X\xi_m),\xi_k)^2\\ &=\sum_m\sum_k\Big(\sum_n a_ne^{-kt}(\xi_n\xi_m,\xi_k)\Big)^2\\ &\leqq 2\pi^{-2}\sum_k e^{-2kt}\sum_m(a_{m+k}^2+a_{|m-k|}^2)\\ &\leqq 6\pi^{-2}\|X\|^2\sum e^{-2kt}<\infty\qquad(\text{if } t>0)\,,\end{aligned}\tag{5.2}$$

where we have used the following formula

$$\sum_n a_n(\xi_n\xi_m,\xi_k)=\begin{cases}\dfrac{1}{\pi}(a_{m+k}+a_{|m-k|}), & \text{if } m,\ k\geqq 1\,,\\ \dfrac{1}{\sqrt{2}\,\pi}(a_{m+k}+a_{|m-k|}), & \text{if } mk=0,\quad m+k\geqq 1\,,\\ \dfrac{1}{\pi}a_0, & \text{if } m=k=0\,.\end{cases}\tag{5.3}$$

The inequality (5.2) proves that the closed extension of $T_tX\cdot$ is an element of $\sigma_2(H)$.

(ii) With the expression $X=\sum a_n\xi_n$, we obtain

$$\begin{aligned}\|X\cdot\|^2_{\sigma_2(H,H_{-\alpha})}&=\sum_m\|X\cdot\xi_m\|^2_{-\alpha}=\sum_{k,m}(X\cdot\xi_m,\xi_k)^2_{-\alpha}\\ &=\sum_{k,m}\big(\sum a_n(\xi_n\xi_m,\xi_k)_{-\alpha}\big)^2\\ &=\sum_m\Big(\sum_n a_n(\xi_n\xi_m,\xi_0)\Big)^2+\sum_{k\geqq1}\sum_m k^{-2\alpha}\Big(\sum_n a_n(\xi_n\xi_m,\xi_k)\Big)^2\\ &\leqq 4\pi^{-2}\sum_m a_m^2+\pi^{-2}\sum_{k\geqq1}k^{-2\alpha}\sum_m(a_{m+k}+a_{|m-k|})^2\\ &\leqq 6\pi^{-2}\|X\|^2(1+\sum_{k\geqq1}k^{-2\alpha})\,.\end{aligned}$$

(iii) Suppose that $\sum a_n^2=\infty$, $\sum n^{-2\alpha}a_n^2<\infty$ and that $a_n\geqq 0$ in the expansion $X=\sum a_n\xi_n$. If $T_tX\cdot$ were well-defined and $T_tX\cdot\in\sigma_2(H,H_{-\beta})$, then the following formula should hold

$$\begin{aligned}\|T_tX\cdot\|^2_{\sigma_2(H,H_{-\beta})}&=\sum_m\|T_tX\xi_m\|^2_{-\beta}\\ &=\sum_m\Big\{(T_tX\xi_m,\xi_0)^2+\sum_{k\geqq1}(T_tX\xi_m,k^\beta\xi_k)^2_{-\beta}\Big\}\\ &=\sum_m\Big\{\Big(\sum_n a_n(\xi_n\xi_m,\xi_0)\Big)^2+\sum_{k\geqq1}k^{-2\beta}e^{-2kt}\Big(\sum_n a_n(\xi_n\xi_m,\xi_0)\Big)^2\Big\}\end{aligned}$$

$$\geqq (1/4\pi^2)\Big\{\sum_m a_m^2 + \sum_{k\geqq 1}\Big(k^{-2\beta}e^{-2kt}\sum_m (a_{m+k}^2 + a_{|m-k|}^2)\Big)\Big\}$$
$$\geqq (1/4\pi^2)\Big(1 + \sum_{k\geq 1} k^{-2\beta}e^{-2kt}\Big)\Big(\sum_m a_m^2\Big) = \infty\ .$$

Therefore we conclude that $T_tX\cdot \notin \sigma_2(H, H_{-\beta})$. (Q.E.D.)

If the equation (5.1) has a solution in H, then the solution should be given by $X_t = T_tX_0 + \int_0^t T_{t-s}X_s\cdot dB_s$. As was mentioned in the Remark 5.1, the integral $\int_0^t T_{t-s}X_s\cdot dB_s$ does not always converge in the space H. Suppose next that a solution X_t were obtained in $H_{-\infty}$. Then, by Proposition 5.1 (iii), the integral $\int_0^t T_{t-s}X_s\cdot dB_s$ would not be defined in $H_{-\infty}$. Thus we have seen that the equation (5.1) has no solution in $H_{-\infty}$. Accordingly, we will consider a modified equation of the form

(5.4) $$dX_t = -\hat{\omega}X_tdt + f(X_t)\cdot dB_t\ ,$$

where f is a mapping from $H_{-\alpha}$ to H ($\alpha > 1/2$).

THEOREM 5.1. *If the mapping* f: $H_{-\alpha}\to H$, $\alpha > 1/2$, *is Lipschitz continuous, then the equation* (5.4) *has a unique solution in* $H_{-\alpha}$ *for a given initial data* $X_0 \in H_{-\alpha}$.

Proof. From Proposition 5.1 (ii) and the Lipschitz continuity of f, it follows that

$$\|f(X) - f(Y)\|^2_{\sigma_2(H,H_{-\alpha})} \leqq \text{const.}\Big(1 + \sum_{k=1}^{\infty} k^{-2\alpha}\Big)\|f(X) - f(Y)\|^2$$
$$\leqq \text{const.}\ \|X - Y\|^2_{-\alpha}\ .$$

Therefore $f(X)\cdot$ is a Lipschitz continuous mapping from $H_{-\alpha}$ to $\sigma_2(H, H_{-\alpha})$. Using Theorem 2.3, we know the existence of a solution of (5.4) as well as uniqueness. (Q.E.D.)

From now on we treat only such a special case as f is linear and of the form

(5.5) $$f(X)(\sigma) = \int_0^\pi \Gamma(\sigma, \sigma')X(\sigma')d\sigma', \qquad X \in H_{-1}\ ,$$

where $\Gamma(\sigma, \sigma')$ is an element of $H \otimes H_1$. Now, by Theorem 2.1 and Theorem 2.2, the equation (5.4) is equivalent to

(5.6) $$X_t = T_t X_0 + \int_0^t T_{t-s}(f(X_s)\cdot)dB_s\,.$$

We are now ready to apply the results obtained in § 4 in terms of the integral representation of (5.6).

LEMMA 5.1. *Let $S \in L^2(T \times \mathscr{E}^* \to \sigma_2(H, H_{-1}))$ and assume that $S(t, \omega)$ satisfies the following conditions:*

(i) $S(t, \omega)$ is $\mathscr{B}_t$-adapted.

(ii) $S(t, \cdot) \in L^2(\mathscr{E}^* \to \sigma_2(H, H_{-1}))$ *and the Wiener's direct sum decomposition of $S(t)$ is given by*

$$S(t, \omega) = \sum S_n(t),\ S_n(t) \in \mathscr{H}_n(\sigma_2(H, H_{-1}))\,,$$
$$S_n(t) \cong S_n(t; x_1, \cdots, x_n; x),\ \boldsymbol{x}_j = (x_j, t_j) \in D \times T,\ x \in D\,,$$

where $S_n(t; \cdot, \cdots, \cdot; \cdot) \in L^2(\widehat{(D \times T)^n} \times D \to H_{-1})$, $D = [0, \pi]$ and $T = [0, \infty)$.

Then the $\mathscr{H}_{n+1}(H_{-1})$-component of $\int_0^t S(s)dB_s$ is equal to $\int_0^t S_n(s)dB_s$ and its kernel is given by

(5.7) $$\frac{1}{n+1}\chi_{[0,t]}(t_{n+1})S_n(t_{n+1}; x_1, \cdots, x; \boldsymbol{x}_{n+1}),\ t_1 \leqq t_2 \leqq \cdots \leqq t_{n+1}\,.$$

Proof. Using the results in § 4, we get

$$\begin{aligned}\int_0^t S_n(s)dB_s &= \int_0^t n!\,\hat{I}(S_n(s))dB_s \\ &= n!\int_{0\leqq t_1\leqq\cdots\leqq t_{n+1}\leqq t}\int S_n(t_{n+1}; x_1, \cdots, x_n; \boldsymbol{x}_{n+1})dB_{t_1}(x_1)\cdots dB_{t_{n+1}}(x_{n+1}) \\ &= (n+1)!\,\hat{I}\Big(\frac{1}{n+1}\chi_{[0,t]}S_n(t_{n+1}; x_1, \cdots, x_n; \boldsymbol{x}_{n+1})\Big)\,.\end{aligned}$$

(Q.E.D.)

LEMMA 5.2. *Let $Z \in L^2(\mathscr{E}^* \to H)$ and let $Z_n(x_1, \cdots, x_n; x)$, $n = 0, 1, 2, \cdots$, be the kernels of the integral representation of Z. Then the kernels $F_n(x_1, \cdots, x_n; x)$, $n = 0, 1, 2, \cdots$, of the integral representation of the multiplicative operator $Z\cdot \in L^2(\mathscr{E}^* \to \sigma_2(H, H_{-1}))$ are of the form*

(5.8) $$F_n(x_1, \cdots, x_n; x) = \sum_{j=0}^{\infty} Z_n(x_1, \cdots, x_n; x)\xi_j(x)\xi_j,\ n = 0, 1, 2, \cdots\,.$$

Proof. Put $(Z)_n = \mathscr{H}_n(H)$-element of Z. Then $(Z)_n$ satisfies

$$((Z)_n\xi, \xi_j) = \int_0^\pi (Z)_n(\sigma)\xi(\sigma)\xi_j(\sigma)d\sigma = ((Z)_n, \xi\xi_j)$$

$$= (I(Z_n), \xi\xi_j) = I(Z_n^* \xi\xi_j)$$
$$= I\Big(\int_0^\pi Z_n(x_1, \cdots, x_n; \sigma)\xi(\sigma)\xi_j(\sigma)d\sigma\Big) .$$

Therefore we have

$$(Z)_n\xi = I\Big(\sum_{j=0}^\infty \Big(\int_0^\pi Z_n(x_1, \cdots, x_n; \sigma)\xi(\sigma)\xi_j(\sigma)d\sigma\Big)\xi_j\Big) .$$

From this equality we obtain

$$(5.9)\qquad \begin{aligned}(Z\xi)_n = (Z)_n\xi &= I\Big(\sum_{j=0}^\infty \Big(\int_0^\pi Z_n(x, \cdots, x_n; \sigma)\xi(\sigma)\xi_j(\sigma)d\sigma\Big)\xi_j\Big) \\ &= I\Big(\int_0^\pi \Big\{\sum_{j=0}^\infty Z_n(x_1, \cdots, x_n; \sigma)\xi_j(\sigma)\cdot\xi_j\Big\}\xi(\sigma)d\sigma\Big) .\end{aligned}$$

The equality (5.9) proves (5.8). (Q.E.D.)

LEMMA 5.3. *Let* $Z \in L^2(T \times \mathscr{E}^* \to H)$ *and let* $Z_n(t; x_1, \cdots, x_n; x)$, $n = 0, 1, 2, \cdots$, *be the kernels of* Z. *Then the kernels of the operator* $T_{t-s}(Z(s)\cdot)$ *are of the form*

$$(5.10)\qquad Z_n(s; x_1, \cdots, x_n; x)\Big\{\sum_{j=0}^\infty e^{-j(t-s)}\xi_j(x)\xi_j\Big\} .$$

Proof. From (5.9) it follows that

$$T_{t-s}(Z(s)\cdot\xi)_n$$
$$= I\Big(\int_0^\pi \Big\{\sum_{j=0}^\infty Z_n(s; x_1, \cdots, x_n; \sigma)\xi_j(\sigma)e^{-j(t-s)}\xi_j\Big\}\xi(\sigma)d\sigma\Big) .$$

This equality proves the lemma. (Q.E.D.)

LEMMA 5.4. *Let* $X_s \in L_2(\mathscr{E}^* \to H_{-1})$ *and let its kernels be* $\Phi_n(s; x_1, \cdots, x_n; \sigma')$, $n = 0, 1, 2, \cdots$. *Then the kernels of the integral representation of* $\int_0^\pi \Gamma(\sigma, \sigma')X_s(\sigma')d\sigma'$ *are expressed in the form*

$$(5.11)\qquad \int_0^\pi \Gamma(\sigma, \sigma')\Phi_n(s; x_1, \cdots, x_n; \sigma')d\sigma' ,$$

where the integration should be understood in the same sense as in (5.5).

Proof. This lemma is obvious from the definition of kernels.

(Q.E.D.)

COROLLARY 5.1. *Under the same assumptions as in Lemma* 5.4, *the kernels of the operator* $T_{t-s}(f(X_s)\cdot)$ *are given by*

$$\int \Gamma(x, \sigma')\Phi_n(s; x_1, \cdots, x_n; \sigma')d\sigma' \sum_{j=0}^{\infty} e^{-j(t-s)}\xi_j(x)\xi_j . \tag{5.12}$$

Proof. This comes from Lemma 5.3 and 5.4. (Q.E.D.)

We are now in a position to describe the equation (5.6) in terms of the integral representation.

THEOREM 5.2. (i) *Let X_t be the solution of* (5.6), *where* $f(X) = \int_0^\pi \Gamma(\sigma, \sigma')X(\sigma')d\sigma'$, *and let $\Phi_n(t; x_1, \cdots, x_n; \cdot)$ be the kernels of the integral representation of X_t. Then $\Phi_n, n = 0, 1, 2, \cdots$,* satisfies the following functional equation

$$\begin{aligned} &\Phi_{n+1}(t; x_1, \cdots, x_{n+1}; \cdot) \\ &\quad = \frac{1}{n+1}\chi_{[0,t]}^{(n+1)\otimes}(t_1, \cdots, t_{n+1})\left\{\sum_{j=0}^{\infty} e^{-j(t-t_{n+1})}\xi_j(x_{n+1})\xi_j\right\} \\ &\quad\quad \times \int_0^\pi \Gamma(x_{n+1}, \sigma')\Phi_n(t_{n+1}; x_1, \cdots, x_n; \sigma')d\sigma' , \\ &\quad\quad t_1 \leqq t_2 \leqq \cdots \leqq t_{n+1},\ n = 0, 1, 2, \cdots , \\ &\Phi_0(t; \cdot) = T_t X_0 . \end{aligned} \tag{5.13}$$

(ii) *Conversely, if a solution $\Phi_n(t; x_1, \cdots, x_n; \cdot)$, $n = 0, 1, 2, \cdots$, of the equation* (5.13) *satisfies the following condition*

$$\sum_{n=0}^{\infty} (n!)^2 \int_{t_1 \leqq t_2 \leqq \cdots \leqq t_n} \cdots \int \|\Phi_n(t)\|_{-1}^2 dx_1 \cdots dx_n < \infty , \tag{5.14}$$

then $\sum_{n=0}^{\infty} I(\Phi_n(t))$ is the solution of (5.6).

Proof. (i) follows from Lemma 5.1 and Corollary 5.1. The condition (5.14) assures that $\sum_{n=0}^{\infty} I(\Phi_n(t)) \in L^2(\mathscr{E}^* \to H_{-1})$. Hence (ii) follows from the uniqueness of the integral representation. (Q.E.D.)

THEOREM 5.3. *The equation* (5.13) *has a unique solution for a given initial data $X_0 \in H_{-1}$, and the solution $\Phi_n(t)$ itself satisfies*

$$(n!)^2 \int_{t_1 \leqq \cdots \leqq t_n} \cdots \int \|\Phi_n(t)\|_{-1}^2 dx_1 \cdots dx_n \leqq \frac{1}{n!} c_0 c^n t^n , \tag{5.15}$$

where c_0 and c are positive constants.

Remark 5.2. The condition (5.15) is equivalent to

$$\|I(\Phi_n(t))\|_{L^2(\mathscr{E}^* \to H_{-1})}^2 \leqq \frac{1}{n!} c_0 c^n t^n . \tag{5.16}$$

Proof of Theorem 5.3. When a point $X_0 \in H_{-1}$ is given, $\{\Phi_n(t); n = 0, 1, 2, \cdots\}$ is determined inductively by (5.13). The proof of (5.15) proceeds by induction. Since $\|T_t\| \leqq 1$, $\Phi_0(t)$ satisfies (5.15) with $c_0 = \|X_0\|_{-1}^2$. Let c_1 be a constant which satisfies

$$\left\|\int \Gamma(\cdot\,, \sigma')X(\sigma')d\sigma'\right\|^2 \leqq c_1\|X\|_{-1}^2 \qquad \text{for any } X \in H_{-1}\,.$$

Put $c = (1 + \sum_{j=1}^{\infty}(1/j^2))c_1$. To prove the induction step from n to $n+1$, we note that

$$\begin{aligned}
&((n+1)!)^2\int_{t_1\leqq\cdots\leqq t_{n+1}}\cdots\int \|\Phi_{n+1}(t)\|_{-1}^2 dx_1\cdots dx_{n+1}\\
&= (n!)^2\int_{0\leqq t_1\leqq\cdots\leqq t_{n+1}\leqq t}\int\left\{\left\|\sum_{j=0}^{\infty} e^{-j(t-t_{n+1})}\xi_j(x_{n+1})\xi_j\right\|_{-1}^2\right\}\\
&\qquad \times\left\{\int_0^{\pi}\Gamma(x_{n+1}, \sigma')\Phi_n(t_{n+1}; x_1, \cdots, x_n; \sigma')d\sigma'\right\}^2 dx_1\cdots dx_{n+1} \qquad (5.17)\\
&\leqq (n!)^2 c\int_{0\leqq t_1\leqq\cdots\leqq t_{n+1}\leqq t}\int\|\Phi_n(t_{n+1}; x_1, \cdots, x_n; \cdot)\|_{-1}^2 dx_1\cdots dx_n dt_{n+1}\\
&\leqq \frac{1}{n!}c_0c^{n+1}\int_0^t t_{n+1}^n dt_{n+1} = \frac{1}{(n+1)!}c_0c^{n+1}t^{n+1}\,.
\end{aligned}$$

Then, we have proved (5.15). (Q.E.D.)

COROLLARY 5.2. *The equation* (5.4) *has a unique solution* X_t *under the assumption that* $f(X) = \int_0^{\pi}\Gamma(\cdot\,, \sigma')X(\sigma')d\sigma'$, *and* X_t *satisfies*

$$\|X_t\|_{L^2(\mathcal{E}^*\to H_{-1})}^2 \leqq \|X_0\|_{-1}^2 e^{ct}\,. \tag{5.18}$$

Proof. From Theorem 5.2 and 5.3, it follows that $\sum_{n=0}^{\infty} I(\Phi_n(t))$ is the unique solution of (5.4). The inequality (5.18) is obvious from (5.16).

(Q.E.D.)

Remark 5.3. The results in Corollary 5.2 are in fact part of Theorem 5.1, because the estimation (5.18) can be obtained by successive approximation method.

Since $\Gamma(\sigma, \sigma')$ has been assumed to be an element of $H \otimes H_1$, $\Gamma(\sigma, \sigma')$ is expressed in the form of

$$\begin{aligned}
&\Gamma(\sigma, \sigma') = \sum_{k,i=0}^{\infty} a_{ki}\xi_k(\sigma)\xi_i(\sigma')\,,\\
&\|\Gamma\|_{H\otimes H_1}^2 = \sum_{k=0}^{\infty} a_{k0}^2 + \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} a_{ki}^2 i^2\,.
\end{aligned} \qquad (5.19)$$

Then we have the following theorem.

THEOREM 5.4. *When $\Gamma(\sigma, \sigma')$ is of the form* (5.19), *the unique solution of* (5.13) *is given by*

$$\Phi_0(t) = T_t X_0 = \sum_{j=0}^{\infty} e^{-jt} c_j \xi_j ,$$

$$\Phi_n(t; x_1, \cdots, x_n; \cdot)$$

$$(5.20) \qquad = \frac{1}{n!} \chi_{[0,t]}^{n\otimes}(t_1, \cdots, t_n) \left[\sum_{j=0}^{\infty} \{\xi_j(\boldsymbol{x}_n) \exp\{-j(t - t_n)\} \xi_j(\cdot)\} \right.$$

$$\times \left\{ \sum_{\substack{k_1, \cdots, k_n = 0 \\ i_1, \cdots, i_n = 0}}^{\infty} a_{k_n i_n} \cdots a_{k_1 i_1} \xi_{k_n}(\boldsymbol{x}_n) \exp\{-i_n(t_n - t_{n-1})\} \xi_{i_n}(\boldsymbol{x}_{n-1}) \right.$$

$$\left. \left. \times \cdots \times \xi_{k_2}(\boldsymbol{x}_2) \exp\{-i_2(t_2 - t_1)\} \xi_{i_2}(\boldsymbol{x}_1) \xi_{k_1}(\boldsymbol{x}_1) \exp(-i_1 t_1\} c_{i_1} \right\} \right] ,$$

$$t_1 \leqq t_2 \leqq \cdots \leqq t_n, \qquad n = 1, 2, \cdots ,$$

where $X_0 = \sum_{j=0}^{\infty} c_j \xi_j \in H_{-1}$ is a given initial value.

Proof. Using the expression (5.19), we are able to solve the equation (5.13) step by step in an explicit form. Carrying out this procedure, we can obtain the formula (5.19). (Q.E.D.)

For $\eta \in \mathscr{E}$ put

$$(5.21) \qquad U^{(n)}(t; \eta) = \int \cdots \int \Phi_n(t; x_1, \cdots, x_n; \cdot) \eta^{n\otimes}(x_1, \cdots, x_n) dx_1 \cdots dx_n ,$$

and put

$$(5.22) \qquad U(t; \eta) = \sum_{n=0}^{\infty} U^{(n)}(t; \eta) .$$

Then it is easily seen that the system $\{U(t; \eta); \eta \in \mathscr{E}\}$ determines $\{\Phi_n(t; x_1, \cdots, x_n), n = 0, 1, 2, \cdots\}$ completely. Without loss of generality, we may assume that $\{\eta: \eta = \xi \otimes \zeta \in \mathscr{E}\}$ is dense in $\mathscr{E}$. We therefore conclude that $\{U(t; \eta): \eta = \xi \otimes \zeta \in \mathscr{E}\}$ determines $\{\Phi_n(t); n = 0, 1, 2, \cdots\}$ completely.

THEOREM 5.5. *For $\eta = \xi \otimes \zeta \in \mathscr{E}$, $U(t; \eta)$ satisfies the following equation*

$$(5.23) \qquad \frac{dU(t; \eta)}{dt} = -\hat{\omega} U(t; \eta) + \zeta(t) G U(t; \eta), \quad t > 0, \quad U(0; \eta) = X_0 ,$$

where G is a linear operator on H_{-1} depending on Γ and ξ (the explicit form of G is given in the proof).

Proof. Using (5.20), we get

$$
\begin{aligned}
& U^{(n)}(t; \eta) \\
(5.24) \qquad & = n! \int_{t_1 \leqq \cdots \leqq t_n} \cdots \int \Phi_n(t; x_1, \cdots, x_n; \cdot)\eta^{n\otimes}(x_1, \cdots, x_n)dx_1 \cdots dx_n \\
& = \sum_{j=0}^{\infty} \xi_j [\sum a_{k_n i_n} \cdots a_{k_1 i_1} c_{i_1} b_{j k_n} b_{i_n k_{n-1}} \cdots b_{i_2 k_1} d^{(n)}_{(i_1, \cdots, i_n), j}(t)] ,
\end{aligned}
$$

where

$$
b_{ik} = \int_0^{\pi} \xi_i(\sigma)\xi_k(\sigma)\xi(\sigma)d\sigma, \qquad i, k = 0, 1, 2, \cdots ,
$$

and

$$
\begin{aligned}
d^{(n)}_{(i_1, \cdots, i_n), j}(t) = & \int_{0 \leqq t_1 \leqq \cdots \leqq t_n \leqq t} \int \exp\{-j(t - t_n) - \cdots - i_1 t_1\} \\
& \times \zeta(t_1) \cdots \zeta(t_n) dt_1 \cdots dt_n, \qquad n = 1, 2, \cdots .
\end{aligned}
$$

From

$$
d^{(n)}_{(i_1, \cdots, i_n), j}(t) = \int_0^t \exp\{-j(t - t_n)\}\zeta(t_n) d^{(n-1)}_{(i_1, \cdots, i_{n-1}), i_n}(t_n) dt_n ,
$$

it follows that

$$
\begin{aligned}
(5.25) \qquad & \frac{d}{dt} d^{(n)}_{(i_1, \cdots, i_n), j}(t) \\
& = \zeta(t) d^{(n-1)}_{(i_1, \cdots, i_{n-1}), i_n}(t) - j d^{(n-1)}_{(i_1, \cdots, i_{n-1}), j}(t), \qquad n = 1, 2, \cdots ,
\end{aligned}
$$

where $d^{(0)}(t) = 0$.
From (5.24) and (5.25) we have

$$
\begin{aligned}
& \frac{d}{dt} U^{(n)}(t; \eta) \\
(5.26) \qquad & = -\hat{\omega} U^{(n)}(t; \eta) \\
& \quad + \zeta(t) \sum_{j=0}^{\infty} \xi_j [\sum a_{k_n i_n} \cdots a_{k_1 i_1} c_{i_1} b_{j k_n} \cdots b_{i_2 k_1} d^{(n-1)}_{(i_1, \cdots, i_{n-1}), i_n}(t)] .
\end{aligned}
$$

Introduce an operator G on H_{-1}

$$
(5.27) \qquad G\left(\sum_{p=0}^{\infty} \alpha_p \xi_p\right) = \sum_{j=0}^{\infty} \left(\sum_{p,q} b_{jq} a_{qp} \alpha_p\right) \xi_j .
$$

Then the second term of (5.26) is equal to $\zeta(t) G U^{(n-1)}(t; \eta)$. Thus we have

$$
(5.28) \qquad
\begin{aligned}
& \frac{d}{dt} U^{(n)}(t; \eta) = -\hat{\omega} U^{(n)}(t; \eta) + \zeta(t) G U^{(n-1)}(t; \eta), \qquad n = 1, 2, \cdots , \\
& \frac{d}{dt} U^{(0)}(t; \eta) = -\hat{\omega} U^{(0)}(t; \eta) .
\end{aligned}
$$

Summing up (5.28) for $n=0, 1, 2, \cdots$, we obtain (5.23). The convergence of the series appeared above is guaranteed by Theorem 5.3. (Q.E.D.)

Remark 5.4. The operator G is expressible as

$$G = \tilde{G}(\xi)\Gamma$$

where $\tilde{G}(\xi)$ is a bounded linear operator on H, linearly depending on ξ, and where Γ is the bounded linear operator from H_{-1} to H given by (5.5)

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