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PERTURBED BILLIARD SYSTEMS II BERNOULLI PROPERTIES

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§1. Introduction

One of the authors has shown the ergodicity of the perturbed billiard system which can describe the motion of a particle in a potential field of a special type [5], [6]. Since then, some development has been made, and we are now able to show the Bernoulli property of the system in this article. We hope, the result gives a new progress in statistical mechanics. Our method of the proof is inspired by the idea of D. S. Ornstein and B. Weiss [9], which has been used by G. Gallavotti and D. S. Ornstein [3] for a Sinai billiard system.

A perturbed billiard transformation will be prescribed in § 3. Roughly speaking, it is an automorphism T_* of two dimensional measure space (M, ν) which can be expressed as the product of T_1 and T, where T_1 is a ν-preserving C^2-diffeomorphism of M and where T is a Sinai billiard transformation. Such an automorphism T_* appears in a dynamical system of a particle moving in a potential field which is a composition of several finite range potentials (see [5], [6]). In order to discuss such a perturbed billiard system we need three assumptions (H–1)∼(H–3), which specify the diffeomorphism T_1. Under these assumptions, the perturbation of T by T_1 is not so much. Details of them will be found in § 3.

Our main results are the following:

THEOREM 2. *Under the assumptions* (H–1)∼(H–3), *partitions* $\alpha^{(c)}$ *and* $\alpha^{(e)}$ *are weak Bernoulli generators for* T_*. *Thus* T_* *is isomorphic to a Bernoulli shift.*

Here $\alpha^{(c)}$ and $\alpha^{(e)}$ are partitions of M whose elements are connected

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components such that the restrictions of T_*^{-1} and T_* on them are continuous, respectively.

THEOREM 3. *Under the assumptions* (H–1)$\sim$(H–3), *every countable partition* α *is weakly Bernoullian for* T_*, *whenever* $\log d(x;\partial\alpha)$ *is integrable.*

Here $d(x;\partial\alpha)$ is the distance between a point x and the union $\partial\alpha$ of the boundaries of all elements of α.

THEOREM 5. *Under the assumptions* (H–1)$\sim$(H–3) *and* (f–1)$\sim$(f–3), *if* $\{S_t^f\}$ *is a K-system and* α *is a finite partition such that* $\log \tilde{d}(w;\partial\alpha)$ *is integrable, then* α *is very weakly Bernoullian for* $\{S_t^f\}$ $(t \neq 0)$. *Furthermore* $\{S_t^f\}$ *is a Bernoulli flow.*

As stated in Corollary 5.3, $\{S_t^f\}$ is a K-system if it does not have any point spectrum. With this result, we have a stronger assertion Corollary 6.2. Here $\{S_t^f\}$ is a flow of Kakutani-Ambrose type whose basic transformation and ceiling function are T_* and $f(x)$, respectively. There we assume the conditions (f–1)$\sim$(f–3), which are prescribed in § 5, so as $f(x)$ to be regular. Actually, if $f(x)$ is positive and smooth on M, then they are obviously satisfied. Our formulation is complicated, but necessary in order to apply to the case of dynamical system on a potential field as described above.

In section 2 some lemmas to make easier checking weak Bernoulli property will be given. In section 3 some fundamental results of the perturbed billiard transformation T_*, which have been shown in [5], will be summarized. In section 4 the proofs of Theorem 2 and Theorem 3 will be shown appealing lemmas in § 2. The most complicated parts of the proofs are in the estimations of densities of measures related to transversal fibres of T_*. In section 5 we will discuss on the construction of transversal fibres and the K-properties of the flow $\{S_t^f\}$. In section 6 the proof of Theorem 5 will be shown by using properties of the transversal fibres.

Lastly we remark that the same results can be obtained for more general T_1 as discussed in [6], since the properties stated in § 3 are also true for the general case.

§ 2. Weak and very weak Bernoulli partitions

Let (M, ν) be a Lebesgue space with total mass $\nu(M) = 1$, and let T

be a bimeasurable measure preserving transformation on M. D. S. Ornstein gave the following definitions:

DEFINITION 2.1. A countable partition α of M is said to be *weakly Bernoullian* for T, if for any $\varepsilon > 0$ there exists $N \geq 0$ such that for all $N'' \geq N' \geq N$, all $n \geq 0$, and ε-a.e. B in $\bigvee_{i=N'}^{N''} T^{-i}\alpha$

$$\sum_{A \in \bigvee_{i=0}^{n} T^{i}\alpha} |\nu(A) - \nu(A \mid B)| < \varepsilon . \tag{2.1}$$

Here "ε-a.e. B in ξ" means "except element B of the partition ξ which is included in a set of measure ε". For two countable partitions $\alpha = \{A_i\}$ and $\beta = \{B_i\}$ of M, define the usual metric by

$$d(\alpha, \beta) \equiv \sum_i \nu(A_i \ominus B_i) ,$$

where $A \ominus B$ denotes the symmetric difference of the sets A and B. For given two sequences of partitions $\{\alpha_j\}_1^n$ and $\{\beta_j\}_1^n$ write

$$\{\alpha_j\}_1^n \sim \{\beta_j\}_1^n$$

if for all k_j, $1 \leq j \leq n$,

$$\mu_X\Big(\bigcap_1^n A_{k_j}^{(j)}\Big) = \mu_Y\Big(\bigcap_1^n B_{k_j}^{(j)}\Big) ,$$

where $\alpha_j = \{A_1^{(j)}, \cdots, A_{a(j)}^{(j)}\}$ are partitions on (X, μ_X) and $\beta_j = \{B_1^{(j)}, \cdots, B_{b(j)}^{(j)}\}$ are partitions on (Y, μ_Y). Further define the metric $\bar{d}$ by

$$\bar{d}(\{\alpha_j\}_1^n, \{\beta_j\}_1^n) = \inf_{\{\bar{\alpha}_j\}, \{\bar{\beta}_j\}} \frac{1}{n} \sum_1^n d(\bar{\alpha}_j, \bar{\beta}_j) ,$$

where $\{\bar{\alpha}_j\}$ and $\{\bar{\beta}_j\}$ run over all pairs of partitions on the same space such that $\{\alpha_j\}_1^n \sim \{\bar{\alpha}_j\}_1^n$ and $\{\beta_j\}_1^n \sim \{\bar{\beta}_j\}_1^n$. Let α be a partition and E be a subset of M. Then the normalized measure $\nu_E(A) = \nu(A \cap E)/\nu(E)$ will be associated to $\alpha|_E$.

DEFINITION 2.2. A finite partition α is said to be *very weakly Bernoullian* for T, if for every $\varepsilon > 0$ there exists $N \geq 0$ such that for all $N'' \geq N' \geq N$, all $n \geq 0$ and ε-a.e. B in $\bigvee_{i=N'}^{N''} T^{-i}\alpha$,

$$\bar{d}(\{T^i\alpha\}_1^n, \{T^i\alpha \mid B\}_1^n) \leq \varepsilon . \tag{2.2}$$

D. S. Ornstein and others [2], [4], [7], [8] have shown the following theorem:

THEOREM A. *If one of the following conditions is satisfied, then T is isomorphic to a Bernoulli shift.*

(i) *There is a weak Bernoulli generator for T.*

(ii) *There is a sequence of weak Bernoulli partitions α_n for T such that $\bigvee_{i=-\infty}^{\infty} T^i\alpha_n \uparrow \epsilon$**) *as $n \to \infty$.*

(iii) *There is a sequence of very weak Bernoulli partitions α_n for T such that $\bigvee_{i=-\infty}^{\infty} T^i\alpha_n \uparrow \epsilon$ as $n \to \infty$.*

In order to apply this theorem to a perturbed billiard system, it is convenient to prepare the following lemmas.

LEMMA 2.3. (i) *If for any $\varepsilon > 0$ and $\delta > 0$, there exist a natural number N and a finite family $\mathscr{F}$ of disjoint subsets of M with $\sum_{F\in\mathscr{F}} \nu(F) \geq 1-\delta$ such that for all $N'' \geq N' \geq N$, all $n \geq 0$ and ε-a.e. B in $\bigvee_{i=N'}^{N''} T^{-i}\alpha$,*

$$\sum_{A\in\bigvee_{i=0}^{n} T^i\alpha} |\nu(A|F) - \nu(A|F\cap B)| < \delta \qquad \text{for any } F \text{ in } \mathscr{F}, \tag{2.3}$$

$$\sum_{F\in\mathscr{F}} |\nu(F) - \nu(F|B)| < \delta, \tag{2.4}$$

then the partition α is weakly Bernoullian for T.

(ii) *In (i), the condition (2.4) is unnecessary if T is a K-system and the entropy of α is finite.*

(iii) *In (i), (2.3) is fulfilled if for all A in $\bigvee_{i=0}^{n} T^i\alpha$, all F in $\mathscr{F}$ and ε-a.e. B in $\bigvee_{i=N'}^{N''} T^{-i}\alpha$,*

$$\left|\frac{\nu(A\cap B\cap F)\nu(F)}{\nu(A\cap F)\nu(B\cap F)} - 1\right| < \delta. \tag{2.5}$$

Proof. Put $F_0 = M - \bigcup_{F\in\mathscr{F}} F$, then one has

$$\begin{aligned}\nu(F_0|B) &\leq 1 - \sum_F \nu(F|B) \leq 1 - (1-\delta)\sum_F \nu(F)\\ &\leq 1-(1-\delta)^2\end{aligned}$$

by (2.4) and by $\sum \nu(F) \geq 1-\delta$. From (2.3) the estimate

$$\begin{aligned}&\sum_F \sum_A |\nu(A\cap F) - \nu(A\cap F|B)|\\ &\quad\leq \sum_{A,F} |\nu(A|F)\nu(F) - \nu(A|B\cap F)\nu(F|B)|\end{aligned}$$

*) The symbol ϵ denotes the partition into the individual points.

$$\leq \sum_{A,F} |\nu(A|F) - \nu(A|B \cap F)|\,\nu(F) + \sum_{A,F} \nu(A|B \cap F)\,|\nu(F) - \nu(F|B)|$$
$$\leq 2\delta$$

is obtained. Hence for 2ε-a.e. B in $\bigvee_{i=N'}^{N''} T^{-i}\alpha$,

$$\sum_A |\nu(A) - \nu(A|B)|$$
$$\leq \sum_{F,A} |\nu(A \cap F) - \nu(A \cap F|B)| + \sum_A \{\nu(A \cap F_0) + \nu(A \cap F_0|B)\}$$
$$\leq 2\delta + \delta + 1 - (1-\delta)^2 \leq 5\delta.$$

Thus (i) is proved. If T is a K-system, then for the given $\mathscr{F}$ there exists N such that for all $N'' \geq N' \geq N$, all F in $\mathscr{F}$ and ε-a.e. B in $\bigvee_{N'}^{N''} T^{-i}\alpha$,

$$|\nu(F) - \nu(F|B)| \leq \delta\nu(F)\,.$$

It is easily seen that (2.5) implies (2.3). Q.E.D.

A mapping ϕ from X to Y is called *ε-measure preserving* if there exists a subset E of X with $\mu_X(E) \leq \varepsilon$ such that for all $A \subset X - E$

$$\left|\frac{\mu_Y(\phi A)}{\mu_X(A)} - 1\right| < \varepsilon\,.$$

Let $e(n)$ be the function on ordinal numbers defined by $e(0) = 0$, $e(n) = 1$ for $n \neq 0$. For a given partition $\alpha = \{A_j\}$, the name function of α is defined by $\ell(x) \equiv j$ if x is in A_j. The following lemma is due to D. S. Ornstein and B. Weiss [9].

LEMMA 2.4. *Let $\{\alpha_i\}_1^n$ be partitions of X with name functions $\ell_i(x)$, and $\{\beta_i\}_1^n$ be partitions of Y with name functions $m_i(y)$. If there is an ε-measure preserving mapping ϕ from X to Y such that*

$$\frac{1}{n}\sum_1^n e(\ell_i(x) - m_i(\phi x)) \leq \varepsilon$$

holds for x in $X - E$ with $\mu_X(E) \leq \varepsilon$, then

$$\bar{d}(\{\alpha_i\}_1^n, \{\beta_i\}_1^n) \leq 16\varepsilon\,.$$

The lemma is easily proved, but it is useful to check (2.2) for a suitable partition α (cf. § 6).

§ 3. Perturbed billiard systems

In the previous article by one of the authors [5], a perturbed billiard system was defined as follows. Let $\bar{Q}_\iota$, $\iota = 1, 2, \cdots, I$ be disjoint strictly convex domains in a 2-dimensional torus T whose boundaries ∂Q_ι are closed curves of C^3-class. Put $Q = T - \bigcup_\iota \overline{Q}_\iota$ and $\partial Q = \bigcup_\iota \partial Q_\iota$ and put $M_0 = \{(q, p); q \in Q, p = (\cos\omega, \sin\omega), 0 \leq \omega < 2\pi\}$. The flow $\{S_t\}$ on M_0 which describes the motion of a particle moving around in Q with unit speed and with elastic collision at ∂Q is called a Sinai billiard system in Q [11], [12]; the particle moves along straight lines in the interior of Q with speed one, and is reflected at ∂Q according to the law "the angle of reflection is equal to the angle of incidence". Denote by M the set of all unit incident vectors at ∂Q. Then every element $x = (q, p)$ of M can be represented by coordinates (ι, r, φ), where ι is the number of ∂Q_ι containing q, r is the arclength between q and a fixed origin in ∂Q_ι measured along ∂Q_ι clockwise, and φ is the angle between p and the inward normal of ∂Q_ι at q. For x in M_0, put

$$(3.1) \qquad \begin{aligned} v(x) &= \inf\{t \geq 0; S_t x \text{ collides with } \partial Q\} \\ \tau(x) &= \sup\{t < 0; S_t x \text{ collides with } \partial Q\}\,. \end{aligned}$$

Then almost every point x in M_0 (with respect to the measure $dqd\omega$) is parametrized by (ι, r, φ, v), where $v = v(x)$ and (ι, r, φ) represents the point $S_v x$ in M. One can define a transformation T of M, which is called *a Sinai billiard transformation*, by

$$(3.2) \qquad Tx = S_{\tau(x)-0}x \qquad \text{for } x \text{ in } M\,.$$

Then $\{S_{-t}\}$ is *a Kakutani-Ambrose flow* with the basic space M, the basic transformation T and the ceiling function $-\tau(x)$. The invariant measure μ of $\{S_t\}$ determined by Liouville's theorem is expressed in the form

$$(3.3) \qquad d\mu = -\mu_0 \cos\varphi d\varphi dr dv d\iota$$

and the corresponding invariant measure of T is expressed in the form

$$(3.4) \qquad d\nu = -\nu_0 \cos\varphi d\varphi dr d\iota$$

with $\mu_0 = (2\pi|Q|)^{-1}$ and $\nu_0 = (2|\partial Q|)^{-1}$, where $|Q|$ is the volume of Q and $|\partial Q|$ is the total arclength of the boundary ∂Q.

DEFINITION 3.1. A transformation T_* of M is called *a perturbed billiard transformation* if T_* is expressed in the form

$$(3.5) \qquad T_* = T_1 T\,,$$

where T is the Sinai billiard transformation and T_1 is a C^2-diffeomorphism of M which preserves the measure ν and each $M^{(\iota)} \equiv \{(\iota, r, \varphi); (\iota, r) \in \partial Q_\iota\}$.

In [5], a special class of perturbed billiard transformations has been investigated*):

(H–1) $T_1(\iota, r, \varphi) = (\iota, r - H(\iota, \varphi), \varphi)$, where $H(\iota, \varphi)$ is a function of C^2-class and satisfies $H(\iota, (\pi/2)) = H(\iota, (3/2)\pi) = 0$,

(H–2) every $\overline{Q}_\iota$ are disjoint strictly convex domains whose boundaries are curves of C^3-class,

(H–3) $\min_{\iota,\varphi} \{h(\iota, \varphi) + [\max_r k(\iota, r) + (\min_{\iota,r,\varphi'} |\tau(\iota, r, \varphi')|)^{-1}]^{-1}\} > 0$, where $h(\iota, \varphi) \equiv dH(\iota, \varphi)/d\varphi$ and $k(\iota, r)$ is the curvature of ∂Q_ι at (ι, r).

Under the above three assumptions the ergodicity and the K-property of the perturbed billiard transformation T_* were shown in [5]. In order to describe the results, it is necessary to introduce notation and terminology. A connected curve $\gamma; \varphi = \psi(r)$ in $M^{(\iota)}$ is called *K-increasing* if for $r \neq r'$

$$k_{\min} \leq \frac{\psi(r) - \psi(r')}{r - r'} \leq K_{\max}(\iota) \tag{3.6}$$

holds, where

$$k_{\min} \equiv \min_{\iota, r} k(\iota, r) \quad \text{and} \quad K_{\max}(\iota) \equiv \max_{r} k(\iota, r) + \left(\min_{r,\varphi} |\tau(T_*^{-1}(\iota, r, \varphi))|\right)^{-1}.$$

A connected curve $\gamma; \varphi = \psi(r)$ in $M^{(\iota)}$ is called *K-decreasing*, if for $r \neq r'$

$$K_{\min} \leq -\frac{\psi(r) - \psi(r')}{r - r'} \leq K_{\max} \tag{3.7}$$

holds, where $K_{\min} \equiv [\max_{\iota,\varphi} h(\iota, \varphi) + k_{\min}^{-1}]^{-1}$ and $K_{\max} \equiv \max_\iota [\min_\varphi h(\iota, \varphi) + K_{\max}(\iota)^{-1}]^{-1}$. Put $S \equiv \{(\iota, r, \varphi) \in M; \varphi = \pi/2 \text{ or } 3\pi/2\}$. Then $T_*^{-1}S$ (resp. T_*S) is called *the curves of discontinuity* of T_* (resp. T_*^{-1}). The image $T_*^{-1}S$ (resp. T_*S) consists of a countable number of K-increasing (resp. K-decreasing) curves. The curves of $T_*^{-1}S$ (resp. T_*S) decompose M into connected components and define the partition $\alpha^{(e)}$ (resp. $\alpha^{(c)}$) into the components $\{X_j^{(e)}\}$ (resp. $\{X_j^{(c)}\}$). Then T_* (resp. T_*^{-1}) is continuous in the interior of each component and belongs to C^2-class. If T_* (resp. T_*^{-1}) is continuous on a connected K-decreasing (resp. K-increasing) curve γ, then so is the image of γ.

For a point $x = (\iota, r, \varphi)$ in M, put $x_i = (\iota_i, r_i, \varphi_i) \equiv T_*^{-i}(\iota, r, \varphi)$, $(\iota_i, r_i', \varphi_i') \equiv T_1^{-1}x_i$, $k_i \equiv k(\iota_i, r_i)$, $k_i' \equiv k(\iota_i, r_i')$, $h_i \equiv h(\iota_i, \varphi_i)$ and $\tau_i \equiv \tau(\iota_i, r_i, \varphi_i)$. Define

*) A more general case was discussed in [6].

functions $b_n(x; t)$, $-\infty < n < \infty$, of (x, t) in $M \times (-\infty, \infty)$ by

$$b_1(x; t) \equiv \frac{(\cos\varphi + k'\tau_1)(h+t) + \tau_1}{(k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1)(h+t) + \cos\varphi_1 + k_1\tau_1}$$

$$b_{n+1}(x; t) \equiv b_1(T_*^{-n}x; b_n(x; t)) \qquad n \geq 1$$

$$b_0(x; t) \equiv t$$

$$b_{-1}(x; t) \equiv -h - \frac{(\cos\varphi + k\tau)t - \tau}{(k\cos\varphi_{-1} + k'_{-1}\cos\varphi + kk'_{-1}\tau)t - (\cos\varphi_{-1} + k'_{-1}\tau)}$$

$$b_{-n-1}(x; t) \equiv b_{-1}(T_*^n x; b_{-n}(x; t)) \qquad n \geq 1 .$$

Suppose that γ and $T_*^{-n}\gamma$ are given by the equations $r = u(\varphi)$ and $r_n = u_n(\varphi_n)$, respectively, with $T_*^{-n}(\iota, u(\varphi), \varphi) = (\iota_n, u_n(\varphi_n), \varphi_n)$. Then the formula

$$\frac{du_n(\varphi_n)}{d\varphi_n} = b_n\left(\iota, r, \varphi; \frac{du(\varphi)}{d\varphi}\right)$$

holds for all n. Further one can see that for (x, t) in $M \times [1/K_{\min}, \infty)$ $b_n(T_*^n x; t)$ converges to a positive function $1/\chi^{(e)}(x)$ and that $b_{-n}(T_*^{-n}x; -t)$ converges to a negative function $1/\chi^{(c)}(x)$. The function $\chi^{(c)}$ (resp. $\chi^{(e)}$) is continuous at x not in $\bigcup_{n=0}^{\infty} T_*^n S$ (resp. $\bigcup_{n=0}^{\infty} T_*^{-n} S$) and satisfies

$$K_{\min} \leq -\chi^{(c)}(x) \leq K_{\max} \text{ (resp. } k_{\min} \leq \chi^{(e)}(x) \leq K_{\max}(\iota)) .$$

THEOREM 1. (i) *$\alpha^{(c)}$ and $\alpha^{(e)} = T_*^{-1}\alpha^{(c)}$ are generators for T_* with the same finite entropy.*

(ii) *Almost every element of $\zeta^{(c)} \equiv \bigvee_{i=0}^{\infty} T_*^i \alpha^{(c)}$ is a connected K-decreasing curve whose gradient*) at x is equal to $\chi^{(c)}(x)$. Alternatively, almost every element of $\zeta^{(e)} \equiv \bigvee_{i=0}^{\infty} T_*^{-i}\alpha^{(e)}$ is a connected K-increasing curve whose gradient at x is equal to $\chi^{(e)}(x)$.*

(iii) *T_* is a K-system. Actually, the partition $\zeta^{(c)}$ and the partition $\zeta^{(e)}$ satisfy the following conditions:*

$$T_*^{-1}\zeta^{(c)} > \zeta^{(c)} , \qquad T_*\zeta^{(e)} > \zeta^{(e)} ,$$

$$\bigvee_i T_*^i \zeta^{(c)} = \bigvee_i T_*^i \zeta^{(e)} = \epsilon$$

$$\bigwedge_i T_*^i \zeta^{(c)} = \bigwedge_i T_*^i \zeta^{(e)} = \text{the trivial partition} .$$

By the theorem, in order to show the Bernoulli property of T_*, it is enough to give a family $\mathscr{F}$ which satisfies the condition (2.5) in Lemma 2.3. For this purpose, it is necessary to investigate the structure of the

*) When a curve γ is given by the equation $\varphi = \psi(r)$, the gradient of γ at $x = (r, \varphi)$ is $d\psi/dr$.

measure μ in connection with the partitions $\zeta^{(c)}$ and $\zeta^{(e)}$. Denote by $\gamma^{(c)}(x)$ the curve which is the element of $\zeta^{(c)}$ involving x. Alternatively, denote by $\gamma^{(e)}(x)$ the curve which is the element of $\zeta^{(e)}$ including x. For two decreasing curves γ and γ', define *the canonical mapping* $\Psi^{(e)}_{\gamma',\gamma}$ by

$$\text{(3.8)}\qquad \begin{aligned} &\Psi^{(e)}_{\gamma',\gamma}\colon \gamma \to \gamma' \\ &\Psi^{(e)}_{\gamma',\gamma}\colon x \mapsto \gamma^{(e)}(x) \cap \gamma' . \end{aligned}$$

Let σ_γ and $\sigma_{\gamma'}$ be the measures on γ and γ' respectively defined as follows; for $\bar{\gamma}$ in γ and $\bar{\gamma}'$ in γ

$$\sigma_\gamma(\bar{\gamma}) = \int_{\bar{\gamma}} |d\varphi| \quad \text{and} \quad \sigma_{\gamma'}(\bar{\gamma}') = \int_{\bar{\gamma}'} |d\varphi| .$$

Define the measure $\Psi^{(e)}_{\gamma,\gamma'}\sigma_{\gamma'}$ by

$$\Psi^{(e)}_{\gamma,\gamma'}\sigma_{\gamma'}(\bar{\gamma}) \equiv \sigma_{\gamma'}(\Psi^{(e)}_{\gamma',\gamma}\bar{\gamma}) .$$

Then the Radon-Nikodym density relative to $d\sigma_\gamma$ is given by

$$\text{(3.9)}\qquad \frac{d\Psi^{(e)}_{\gamma,\gamma'}\sigma_{\gamma'}}{d\sigma_\gamma} = g^{(e)}_{\gamma,\gamma'} \equiv \prod_{i=-\infty}^{0} \frac{\Lambda^*(x_i, T_*^{-i}\gamma)}{\Lambda^*(x_i', T_*^{-i}\gamma')}$$

with x in γ and $x' = \Psi^{(e)}_{\gamma',\gamma}x$, where

$$\text{(3.10)}\qquad \Lambda^*(x, \gamma) = \frac{\{k_1 \cos\varphi + k' \cos\varphi_1 + k_1 k' \tau_1\} b_1(x; du/d\varphi) - k'\tau_1 - \cos\varphi}{\cos\varphi} .$$

Similarly, $\Psi^{(c)}_{\gamma',\gamma}$, $\sigma_{\gamma'}$, σ_γ are defined for increasing curves γ', γ and one has

$$\text{(3.9)}'\qquad \frac{d\Psi^{(c)}_{\gamma,\gamma'}\sigma_{\gamma'}}{d\sigma_\gamma} = g^{(c)}_{\gamma,\gamma'} = \prod_{i=0}^{\infty} \frac{\Lambda(x_i, T_*^{-i}\gamma)}{\Lambda(x_i', T_*^{-i}\gamma')}$$

with x in γ and $x' = \Psi^{(c)}_{\gamma',\gamma}x$, where

$$\text{(3.10)}'\qquad \Lambda(x, \gamma) = -\frac{\{k_1 \cos\varphi + k' \cos\varphi_1 + k_1 k' \tau_1\}\{du/d\varphi + h\} + k_1\tau_1 + \cos\varphi_1}{\cos\varphi_1}$$

By Lemmas 6.1, 6.1′ and 7.1 in [5], for any $\delta > 0$ there exist an even natural number $\ell_0 = \ell_0(\delta, 1, 1/4)$ and a positive function $\varepsilon_0 = \varepsilon_0(x, \delta, 1)$ which guarantee the following property: For an x not in $\bigcup_{i=-\ell_0}^{\ell_0} T_*^i S$, let G be a K-quadrilateral*) (a domain which is enclosed by four curves such that a pair of opposite curves $\gamma_b(G)$, $\gamma_d(G)$ are K-increasing and the other pair of opposite curves $\gamma_a(G)$, $\gamma_c(G)$ are K-decreasing) in the ε_0-neighbourhood

*) The notation for G and some properties of G are explained in [5].

$U_{\epsilon_0}(x)$ of x. Suppose that $\delta_0 \equiv \theta(\gamma_a(G)) = \theta(\gamma_b(G))$*) and that $T_*^{-\ell_0}G$, $T_*^{\ell_0}G$ are K-quadrilaterals. Then there exist subsets $G^{(c,\delta)}$ and $G^{(e,\delta)}$ which satisfy the four conditions;

(C–1) for all x in $G^{(c,\delta)}$ (resp. $G^{(e,\delta)}$), $\gamma^{(c)}(x) \cap G^{(c,\delta)}$ (resp. $\gamma^{(e)}(x) \cap G^{(e,\delta)}$) is a connected segment which joins $\gamma_b(G)$ and $\gamma_d(G)$ (resp. $\gamma_a(G)$ and $\gamma_c(G)$),

(C–2) $\nu(G^{(c,\delta)}) \geq (1-\delta)\nu(G)$ and $\nu(G^{(e,\delta)}) \geq (1-\delta)\nu(G)$,

(C–3) for any K-increasing (resp. K-decreasing) curve γ and γ' in G, the canonical mapping $\Psi^{(c)}_{\gamma',\gamma}$ (resp. $\Psi^{(e)}_{\gamma',\gamma}$) is absolutely continuous on $\gamma \cap G^{(c,\delta)}$ (resp. $\gamma \cap G^{(e,\delta)}$) with respect to σ_γ and $\sigma_{\gamma'}$,

(C–4) for any $m \geq 0$, $T_*^{-m}G^{(c,\delta)} \cap V_m(\delta_0) = \phi$**) (resp. $T_*^{m}G^{(e,\delta)} \cap V_m(\delta_0) = \phi$).

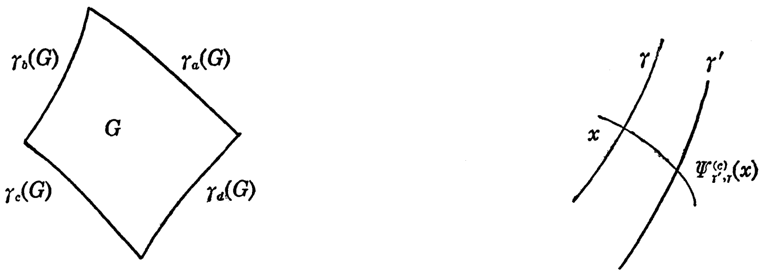

Fig. 1

Define the measure ρ_γ on a K-decreasing (or K-increasing) curve γ by

$$\rho_\gamma(\bar{\gamma}) \equiv \int_{\bar{\gamma}} dr \qquad \text{for } \bar{\gamma} \subset \gamma\,.$$

Let γ be a K-decreasing curve in G which joins $\gamma_a(G)$ and $\gamma_c(G)$, and let $\tilde{\gamma}_0$ be a K-increasing curve which is an extension of $\tilde{\gamma}$ and is given by the equation $r = \tilde{u}(\varphi)$, $\pi/2 \leq \varphi \leq (3/2)\pi$. For given $\dot{\varphi}$ and $\dot{\psi}$, let $\tilde{\gamma}^{\dot{\varphi},\dot{\psi}}$ be the curve defined by $r = \tilde{u}(\varphi) - \tilde{u}(\dot{\varphi}) - u^{\dot{\varphi}}(\dot{\psi})$, where $r = u^{\dot{\varphi}}(\dot{\psi})$ is the equation of $\gamma^{(c)}(\iota, \tilde{u}(\dot{\varphi}), \dot{\varphi})$. Then by Lemma 8.3 in [5], the measure ν is expressed in the form

$$\nu(B \cap G^{(c,\delta)}) = \int d\sigma_{\tilde{\gamma}}(\varphi) \int_{\gamma^{(c)}(\iota,\tilde{u}(\varphi),\varphi) \cap B \cap G^{(c,\delta)}} g_0(\varphi, \psi)\, d\sigma_{\gamma^{(c)}}(\psi)\,, \tag{3.11}$$

where $g_0(\varphi, \psi)$ is defined by

*) For monotone connected curve γ, $\theta(\gamma)$ denotes the total variation of γ in φ-direction; $\theta(\gamma) \equiv \sigma_\gamma(\gamma) = \int_\gamma d\varphi$.

**) $V_m(\delta_0) \equiv \{x\,;\, -\cos\varphi(x) \leq (1+\eta_1)^{-m/32}\delta_0\}$ with $\eta \equiv k_{\min}|\tau|_{\min}$ and $\eta_1 \equiv \min\{\eta, (1-\eta)^{-1}\cdot K_{\min}/K_{\max}\}$.

(3.12) $$g_0(\varphi, \psi) = \frac{\nu_0 \cos \varphi}{\chi^{(c)}(\iota, u^{\varphi}(\psi), \psi)} \prod_{i=0}^{\infty} \frac{\Lambda(T_*^{-i}x; T_*^{-i}\tilde{\gamma}_0)}{\Lambda(T_*^{-i}\hat{x}; T_*^{-i}\tilde{\gamma}^{\varphi,\psi})}$$

with $x = (\iota, \tilde{u}(\varphi), \varphi)$ and $\hat{x} = (\iota, u^{\varphi}(\psi), \psi)$.

Let $\hat{\gamma}$ be a K-decreasing curve in G which joins $\gamma_b(G)$ and $\gamma_d(G)$, and be defined by the equation $r = \hat{u}(\varphi)$. Then by (3.9), one can see that

(3.13) $$\begin{aligned}&\nu(B \cap G^{(c,\delta)} \cap G^{(e,\delta)}) \\ &= \int_{\hat{\gamma}\cap G^{(e,\delta)}} d\sigma_{\hat{\gamma}}(\varphi) \int_{\hat{\gamma}\cap \Psi^{(e)}_{\hat{\gamma},\gamma'}(B\cap G^{(c,\delta)}\cap\gamma')} g_0(\varphi, \Psi^{(e)}_{\hat{\gamma},\gamma'}\hat{\psi}) g^{(e)}_{\hat{\gamma},\gamma'}(\hat{\psi}) d\sigma_{\hat{\gamma}}(\hat{\psi})\end{aligned}$$

where $\gamma' = \gamma^{(c)}(\iota, \tilde{u}(\varphi), \varphi)$, $\Psi^{(e)}_{\gamma',\gamma}(\iota, \hat{u}(\hat{\psi}), \hat{\psi}) = (\iota, u^{\varphi}(\tilde{\varphi}), \tilde{\varphi})$ and $\Psi^{(e)}\hat{\psi} \equiv \tilde{\varphi}$. One can easily see that for any fixed $m \geq \ell_0$, there exists a positive number $\varepsilon_2 = \varepsilon_2(x, m) < \varepsilon_0$ such that $T_*^j U_{\varepsilon_2}(x) \cap V_0(2(1+\eta_1)^m \varepsilon_2) = \phi$ for $|j| \leq m$.

LEMMA 3.1. *Suppose that $G \subset U_{\varepsilon_2}(x)$ be as above and $T_*^{-m}\hat{\gamma}$ is a K-decreasing curve, then*

$$\begin{aligned}&\exp[-c_{31}(1+\eta_1)^{-m/2}] \\ &\leq \frac{-\nu_0}{\nu(B\cap G^{(c,\delta)}\cap G^{(e,\delta)})} \int_{\hat{\gamma}\cap G^{(e,\delta)}} \cos\varphi d\sigma_{\hat{\gamma}}(\varphi) \int_{\Psi^{(e)}_{\gamma,\gamma'}(B\cap G^{(c,\delta)}\cap\gamma')} d\rho_{\gamma}(\hat{\psi}) \\ &\leq \exp[c_{31}(1+\eta_1)^{-m/2}] .\end{aligned}$$

In particular, if A is a $\zeta^{(c)}|_G$-measurable subset of G and B is $\zeta^{(e)}|_G$-measurable subset of G, then

$$\begin{aligned}&\exp[-c_{31}(1+\eta_1)^{-m/2}] \\ &\leq \frac{\nu_0}{\nu(A\cap B\cap G^{(c,\delta)}\cap G^{(e,\delta)})} \int_{\hat{\gamma}\cap B\cap G^{(e,\delta)}} \cos\varphi d\sigma_{\hat{\gamma}}(\varphi) \int_{\hat{\gamma}\cap A\cap G^{(c,\delta)}} d\rho_{\hat{\gamma}}(\hat{\psi}) \\ &\leq \exp[c_{31}(1+\eta_1)^{-m/2}] .\end{aligned}$$

In order to prove this lemma, we will prepare two lemmas.

LEMMA 3.2. *Let G be as in Lemma 3.1. Let $\hat{\gamma}$ and $\hat{\hat{\gamma}}$ be K-decreasing curves in G which join $\gamma_b(G)$ and $\gamma_d(G)$. If $T_*^{-m}\hat{\gamma}$ and $T_*^{-m}\hat{\hat{\gamma}}$ are K-decreasing, then for $\hat{x}$ in $\hat{\gamma} \cap G^{(e,\delta)}$*

$$\exp[-c_{32}(1+\eta_1)^{-m/2}] \leq g^{(e)}_{\hat{\gamma},\hat{\hat{\gamma}}}(\hat{x}) \leq \exp[c_{32}(1+\eta_1)^{-m/2}]$$

holds with a positive constant c_{32}.

Proof. Put $\gamma \equiv \gamma^{(e)}(\hat{x}) \cap G$ and $\gamma_j = T_*^{-j}\gamma$. Put $\hat{x}_j = T_*^{-j}\hat{x}$ and $\hat{\hat{x}}_j = T_*^{-j}\hat{\hat{x}}$ with $\hat{\hat{x}} = \Psi^{(e)}_{\hat{\gamma},\hat{\hat{\gamma}}}(\hat{x})$. Since G is in $U_{\varepsilon_2}(x)$ with an x, $\min\{-\cos\varphi(y); y \in \gamma_j\} \geq 2\varepsilon_2(1+\eta_1)^m$ holds for $-m \leq j \leq 0$ and $\theta(\gamma) \leq 2\varepsilon_2$ holds. Put $c_2 =$

$K_{\max}/K_{\min}$. Applying Lemma 5.3 and Lemma 5.4 (i) in [5], one has the estimation

$$\log [\Lambda^*(\hat{\hat{x}}_j, \hat{\hat{\gamma}}_j)/\Lambda^*(\hat{x}_j, \hat{\gamma}_j)] \leq (c_{22} + c_{21} + \log c_2)(1 + \eta_1)^{j-m}$$

for $-m \leq j \leq 0$. Since $\hat{x}$ is in $G^{(e,\delta)}$, it holds that $\min\{-\cos\varphi(y); y \in \gamma_j\} \geq \delta_0(1 + \eta_1)^{j/32}$ for $j \leq 0$ and that $\theta(\gamma) \leq (1 + c_2)\delta_0$. Therefore again one has

$$\log [\Lambda^*(\hat{\hat{x}}_j, \hat{\hat{\gamma}}_j)/\Lambda^*(\hat{x}_j, \hat{\gamma}_j)] \leq 2(c_2 + 1)c_{22}(1 + \eta_1)^{31j/32} + (\log c_2)(1 + \eta_1)^j$$

for $j \leq -m$. These estimates imply Lemma 3.2 by (3.9). Q.E.D.

LEMMA 3.3. *Let G be as in Lemma 3.1 and let $T_*^m\tilde{\gamma}_0$ be K-increasing. Then*

$$\exp[-c_{33}(1 + \eta_1)^{-m/2}] \leq \frac{g_0(\varphi, \psi)}{\nu_0} \frac{\cos\varphi}{\cos\psi} \frac{\chi^{(c)}(\iota, u(\psi), \psi)}{\chi^{(c)}(\iota, \tilde{u}(\varphi), \varphi)} \leq \exp[c_{33}(1 + \eta_1)^{-m/2}]$$

with a positive constant c_{33}.

Proof. Put $x = (\iota, \tilde{u}(\varphi), \varphi)$ and $\hat{x} = (\iota, u^{\varphi}(\psi), \psi)$. Similarly in Lemma 3.2,

$$\sum_{j=0}^{\infty} \left|\log \frac{\Lambda(T_*^{-j}x, T_*^{-j}\tilde{\gamma}_0)}{\Lambda(T_*^{-j}\hat{x}, T_*^{-j}\tilde{\gamma}^{\varphi,\psi})}\right| \leq c'_{32}(1 + \eta_1)^{-m} .$$

By Lemma 5.3 in [5], for $(\iota, \hat{u}(\hat{\varphi}), \hat{\varphi}) = \Psi_{\hat{\gamma},\gamma^{(c)}(\iota,\tilde{u}(\varphi),\varphi)}(\iota, u^{\varphi}(\psi), \psi)$

$$\left|\log \frac{\chi^{(c)}(\iota, u^{\varphi}(\psi), \psi)}{d\hat{u}/d\hat{\psi}}\right| \leq (\pi c_{21} + c_3)(1 + \eta_1)^{-m} . \tag{3.14}$$

On the other hand, the estimate

$$\left|\log \frac{\cos\varphi}{\cos\psi}\right| \leq 2(1 + \eta_1)^{-m}$$

holds, since G is in $U_{\varepsilon_2}(x)$. Q.E.D.

Proof of Lemma 3.1. Since $|dr| = |d\hat{u}/d\hat{\psi}|\, d\hat{\psi}$ on $\hat{\gamma}$, Lemma 3.2 and Lemma 3.3 imply the first statement in Lemma 3.1. If A and B are as in Lemma 3.1, then

$$\Psi^{(e)}_{\hat{\gamma},\gamma'}(A \cap B \cap G^{(c,\delta)} \cap \gamma') = \begin{cases} B \cap G^{(e,\delta)} \cap \hat{\gamma} & \text{if } \gamma' \subset A \cap G^{(c,\delta)} \\ \phi & \text{if } \gamma' \cap A \cap G^{(c,\delta)} = \phi \end{cases}$$

where $\gamma' = \gamma^{(c)}(y)$ with some y. Further, $\gamma' \subset A \cap G^{(c,\delta)}$ if and only if $\tilde{\gamma} \cap \gamma'$ is in $\tilde{\gamma} \cap A \cap G^{(c,\delta)}$. Therefore one has the second statement. Q.E.D.

§4. The perturbed billiard transformation is isomorphic to a Bernoulli shift

Applying the lemmas in §2 and §3, the following theorems will be shown.

THEOREM 2. *Under the assumptions* (H–1)~(H–3), $\alpha^{(c)}$ *and* $\alpha^{(e)}$ *are weak Bernoulli generators for* T_*. *Thus* T_* *is isomorphic to a Bernoulli shift.*

Proof. By Theorem 1, it is sufficient for the proof to give a family $\mathscr{F}$ which satisfies (2.3) in Lemma 2.3. For given $\delta > 0$, let m_0 be a natural number such that $\exp c_{31}(1+\eta_1)^{-m_0/2} < 1+\delta$ and $m_0 \geq \ell_0 \equiv \ell_0(1, 1/4, \delta)$. For every x not in $\bigcup_{i=-m_0}^{m_0} T^i S$ and for any $\delta_0 > 0$, there exists a K-quadrilateral G in $U_{\varepsilon_2(x,m_0)}(x)$ such that $\theta(\gamma_a(G)) = \theta(\gamma_b(G)) < \delta_0$, G involves the point x and $T_*^{-m_0}G$, $T_*^{m_0}G$ are K-quadrilaterals. By the covering theorem of Vitali, there exists a finite family $\mathscr{G}$ of such G's which satisfies $G \cap G' = \phi$ for $G \neq G'$ in $\mathscr{G}$ and $\nu(M - \bigcup_{G\in\mathscr{G}} G) < \delta$. Then by Lemmas 6.1, 6.1′ and 7.1 in [5], there exist subsets $G^{(c,\delta)}$ and $G^{(e,\delta)}$ which satisfy (C–1), (C–2) and (C–3) in §3. Let A be an element of the partition $\bigvee_{i=0}^{n} T_*^i \alpha^{(c)}$ and let B be an element of $\bigvee_{i=N'}^{N''} T_*^{-i}\alpha^{(c)}$. Since A is $\zeta^{(c)}$-measurable and B is $\zeta^{(e)}$-measurable, Lemma 3.1 is applicable. Put $\tilde{\gamma} \equiv \gamma^{(c)}(x) \cap G$ and $\hat{\gamma} \equiv \gamma^{(e)}(x) \cap G$ for a fixed x in $G^{(c,\delta)} \cap G^{(e,\delta)}$. Then one has

$$(1+\delta)^{-1} \leq \frac{-\nu_0}{\nu(A \cap B \cap G^{(c,\delta)} \cap G^{(e,\delta)})} \int_{\tilde{\gamma}\cap B\cap G^{(e,\delta)}} \cos\varphi d\sigma(\varphi) \times \int_{\hat{\gamma}\cap A\cap G^{(c,\delta)}} d\rho(\gamma) \leq 1+\delta\,.$$

Since A and B are arbitrary, the above inequality holds even if one replaces A to $G^{(c,\delta)}$ (B to $G^{(e,\delta)}$). Hence the estimate

$$(1+\delta)^{-4} \leq \frac{\nu(A \cap B \cap G^{(c,\delta)} \cap G^{(e,\delta)})\nu(G^{(c,\delta)} \cap G^{(e,\delta)})}{\nu(A \cap G^{(c,\delta)} \cap G^{(e,\delta)})\nu(B \cap G^{(c,\delta)} \cap G^{(e,\delta)})} \leq (1+\delta)^4$$

is obtained. Therefore the family $\mathscr{F} \equiv \{G^{(c,\delta)} \cap G^{(e,\delta)}; G \in \mathscr{G}\}$ satisfies (2.3).
Q.E.D.

COROLLARY 4.1. *A Sinai billiard transformation is isomorphic to a Bernoulli shift. In particular, the natural generators* $\alpha^{(c)}$ *and* $\alpha^{(e)}$ *are weakly Bernoullian for* T.

Let $\alpha = \{X_j\}$ be a countable partition. Denote the boundary of X_j by ∂X_j. The union $\partial\alpha \equiv \bigcup_j \partial X_j$ is called the boundary of the partition α.

Let $d(x; \partial\alpha)$ be the distance between a point x in M and the boundary $\partial\alpha$.

LEMMA 4.2. (i) *If* $\log d(x; \partial\alpha)$ *is integrable, then the entropy of* α *is finite.*

(ii) *If the boundary* $\partial\alpha$ *consists of curves whose total arclength is finite, then* $\log d(x; \partial\alpha)$ *is integrable.*

Proof. Put $R = \sup_{x \in X_j} d(x; \partial X_j)$, then for $x \in X_j$

$$\nu(X_j) \geq -\nu_0 \int_0^R \int_{\pi/2}^{R+\pi/2} \cos\varphi d\varphi dr \geq \nu_0 R^3/4$$
$$\geq \nu_0 \{d(x; \partial X_j)\}^3/4 .$$

This inequality implies $-\sum \nu(X_j) \log \nu(X_j) < \infty$. The second assertion is obvious. Q.E.D.

THEOREM 3. *Under the assumptions* (H-1)~(H-3), *every countable partition* α *is weakly Bernoullian for* T_* *whenever* $\log d(x; \partial\alpha)$ *is integrable.*

Proof. For a fixed x and $i > 0$, the distance between $T_*^{-i}x$ and $T_*^{-i}\gamma^{(c)}(x) \cap \partial\alpha$ measured along $\gamma^{(c)}(T_*^{-i}x)$ is greater than $d(T_*^{-i}x; \partial\alpha)$, if $T_*^{-i}\gamma^{(c)}(x)$ intersects $\partial\alpha$. Hence

$$d(T_*^{-i}x; \partial\alpha) \leq c_1\theta(T_*^{-i}\gamma^{(c)}(x)) \leq \pi c_1(1 + \eta_1)^{-i}$$

holds with $c_1 \equiv (1 + K_{\min}^{-2})^{1/2}$, if $T_*^{-i}\gamma^{(c)}(x)$ intersects $\partial\alpha$. Since $\log d(x; \partial\alpha)$ is integrable, for almost every x, $1/i \log d(T_*^{-i}x; \partial\alpha)$ converges to 0 as $i \to \infty$, by the Birkhoff ergodic theorem. Thus for almost every x, the boundary $\partial\alpha$ is not intersected by $T_*^{-i}\gamma^{(c)}(x)$ of infinitely many i's. Hence for almost every x, there exists a natural number $n^{(c)}(x)$ such that for all $i \geq n^{(c)}(x)$ $\gamma^{(c)}(x)$ is included in an element of $T_*^i\alpha$. Further since $\log d(x; \partial\alpha)$ is integrable, the partition of $\gamma^{(c)}(T_*^{-i}x)$ into the connected components of the sets $\{\gamma^{(c)}(T_*^{-i}x) \cap X_j\}_{j=1}^{\infty}$ is a countable partition. Put

$$\zeta_\alpha^{(c)} \equiv \bigvee_{i=0}^{\infty} T_*^{-i}\alpha \quad \text{and} \quad \zeta_\alpha^{(e)} \equiv \bigvee_{i=1}^{\infty} T_*^{i}\alpha .$$

Then by the above discussions, the restriction of the partition $\zeta_\alpha^{(c)}$ to almost every element $\gamma^{(c)}$ of $\zeta^{(c)}$ is a countable partition, whose elements are countable unions of connected segments of $\gamma^{(c)}$. Let $C^{(c)}(x)$ be the connected component of x in the element of $\zeta_\alpha^{(c)} \vee \zeta^{(c)}$ which contains x. The partition $\zeta_\alpha^{(e)}$ and $C^{(e)}(x)$ are similarly defined. Denote by $\varphi(x)$ the φ-coordinate of $x = (\iota, r, \varphi)$. Then for almost every x

$$\bar{\theta}(C^{(c)}(x); x) \equiv \sup_{y \in C^{(c)}(x)} \varphi(y) - \varphi(x) , \qquad \underline{\theta}(C^{(c)}(x); x) \equiv \varphi(x) - \inf_{y \in C^{(c)}(x)} \varphi(y)$$
$$\bar{\theta}(C^{(e)}(x); x) \equiv \sup_{y \in C^{(e)}(x)} \varphi(y) - \varphi(x) , \qquad \underline{\theta}(C^{(e)}(x); x) \equiv \varphi(x) - \inf_{y \in C^{(e)}(x)} \varphi(y)$$

are all positive.

For $\delta > 0$, let ℓ_0, m_0 and ε_2 be as in the proof of Theorem 2. Let t be a number which satisfies $\nu(E) > 1 - \delta$ with $E = \{x; \bar{\theta}(C^{(c)}(x); x) > t,\ \underline{\theta}(C^{(c)}(x); x) > t,\ \bar{\theta}(C^{(e)}(x); x) > t$ and $\underline{\theta}(C^{(e)}(x); x) > t\}$. By the similar way to the proof of Theorem 2, there exists a finite family $\mathscr{G}_1$ of K-quadrilaterals such that

$$\begin{aligned}
&\theta(\gamma_a(G)) = \theta(\gamma_b(G)) < t/(1 + K_{\max}/K_{\min}) , \\
&\nu(E \cap G) > (1 - \delta)\nu(G) , \\
&\nu\Big(E - \bigcup_{G \in \mathscr{G}_1} G\Big) < \delta , \\
&G \cap G' = \phi \qquad \text{if } G \neq G' ,
\end{aligned}$$

and that $T_*^{-m_0}G$, $T_*^{m_0}G$ are K-quadrilaterals and there exist subsets $G^{(c,\delta)}$ and $G^{(e,\delta)}$ of G which satisfy the conditions (C–1), (C–2), (C–3), and (C–4) in § 3. For G in $\mathscr{G}_1$, put

$$\begin{aligned}
\hat{G}^{(c)} &\equiv \{x \in G^{(c,\delta)}; C^{(c)}(x) \text{ intersects both } \gamma_b(G) \text{ and } \gamma_d(G)\} , \\
\hat{G}^{(e)} &\equiv \{x \in G^{(e,\delta)}; C^{(e)}(x) \text{ intersects both } \gamma_a(G) \text{ and } \gamma_c(G)\} .
\end{aligned}$$

Then $\hat{G}^{(c)}$ is a $\zeta^{(c)}|_G$-measurable subset and includes $E \cap G^{(c,\delta)}$. Alternatively, $\hat{G}^{(e)}$ is $\zeta^{(e)}|_G$-measurable subset and includes $E \cap G^{(e,\delta)}$. Since for any element A of $\bigvee_0^n T_*^i\alpha$, $A \cap \hat{G}^{(c)}$ is $\zeta^{(c)}|_G$-measurable and since for any element B of $\bigvee_{i=N'}^{N''} T_*^{-i}\alpha$, $B \cap \hat{G}^{(e)}$ is $\zeta^{(e)}|_G$-measurable, Lemma 3.1 is applicable to the subsets A and B. Thus one has the estimates

$$\begin{aligned}
&\left| \frac{\nu(A \cap B \cap \hat{G}^{(c)} \cap \hat{G}^{(e)})\nu(\hat{G}^{(c)} \cap \hat{G}^{(e)})}{\nu(A \cap \hat{G}^{(c)} \cap \hat{G}^{(e)})\nu(B \cap \hat{G}^{(c)} \cap \hat{G}^{(e)})} - 1 \right| \leq (1 + \delta)^4 - 1 , \\
&\nu(\hat{G}^{(c)} \cap \hat{G}^{(e)}) \geq \nu(E \cap G^{(c,\delta)} \cap G^{(e,\delta)}) \geq (1 - 3\delta)\nu(G) ,
\end{aligned}$$

and

$$\nu\Big(\bigcup_{G \in \mathscr{G}_1} (\hat{G}^{(c)} \cap \hat{G}^{(e)})\Big) \geq (1 - 2\delta)(1 - 3\delta) .$$

Hence the conditions in Lemma 2.3 are fulfilled. Q.E.D.

§ 5. K-properties of the flow $\{S_t^f\}$

Let $f(x)$ be a positive function on M and let $\{S_{-t}^f\}$ be a Kakutani-Ambrose flow with the basic space M, the basic transformation T_* and the ceiling function $f(x)$; that is, $\{S_t^f\}$ is defined on the space $W \equiv \{(x, v); 0 \le v < f(x), x \in M\}$ by

$$(5.1)\qquad S_t^f(x, v) \equiv \begin{cases} (T_*^{-k}x, v - t + \sum_{j=1}^k f(T_*^{-j}x)) \\ \qquad \text{if } 0 \le v - t + \sum_{j=1}^k f(T_*^{-j}x) < f(T_*^{-k}x),\ k \ge 1\,, \\ (x, v - t) \qquad \text{if } 0 \le v - t < f(x)\,, \\ (T_*^{-k}x, v - t - \sum_{j=k+1}^0 f(T_*^{-j}x)) \\ \qquad \text{if } 0 \le v - t - \sum_{j=k+1}^0 f(T_*^{-j}x) < f(T_*^{-k}x),\ k \le -1\,. \end{cases}$$

Associate the invariant probability measure μ_f with $\{S_t^f\}$: $d\mu_f = c_f dv d\nu$. Suppose that the assumptions (H–1)～(H–3) are satisfied. Then $\{S_t^f\}$ is ergodic, since T_* is ergodic. Moreover suppose the following three assumptions (f–1)～(f–3):

(f–1) $f(x)$ is strictly positive and continuously differentiable on each element $X_j^{(e)}$ of $\alpha^{(e)}$,

(f–2) there exists a constant K such that

$$\left\{\left|\frac{\partial f(\iota, r, \varphi)}{\partial r}\right| + \left|\frac{\partial f(\iota, r, \varphi)}{\partial \varphi}\right|\right\}\frac{\cos\varphi - 1}{\tau} \le K$$

with $(\iota_{-1}, r_{-1}, \varphi_{-1}) = T_*(\iota, r, \varphi)$,

(f–3) $f(x)\log|\tau(x)|$ is integrable.

For x not in $\bigcup_{i=0}^\infty T_*^i S$, put

$$(5.2)\qquad \begin{aligned} f^{(+)}(x) &\equiv \sum_{i=1}^\infty \left\{\frac{1}{\chi^{(c)}(x_i)}\frac{\partial f(x_i)}{\partial r} + \frac{\partial f(x_i)}{\partial \varphi}\right\}\prod_{j=0}^{i-1}[\Lambda(x_j, \gamma^{(c)}(x_j))]^{-1}\,, \\ \hat{f}^{(+)}(x) &\equiv \sum_{i=1}^\infty \left\{\left|\frac{1}{\chi^{(c)}(x_i)}\frac{\partial f(x_i)}{\partial r}\right| + \left|\frac{\partial f(x_i)}{\partial \varphi}\right|\right\}\prod_{j=0}^{i-1}|\Lambda(x_j, \gamma^{(c)}(x_j))|^{-1} \end{aligned}$$

with $x_j \equiv T_*^{-j}x$. For x not in $\bigcup_{i=0}^\infty T_*^{-i}S$, put

$$(5.3)\qquad \begin{aligned} f^{(-)}(x) &\equiv \sum_{i=-\infty}^{-1}\left\{\frac{1}{\chi^{(e)}(x_i)}\frac{\partial f}{\partial r}(x_i) + \frac{\partial f}{\partial \varphi}(x_i)\right\}\prod_{j=i+1}^{0}[\Lambda^*(x_j, \gamma^{(e)}(x_j))]^{-1} \\ &\quad + \frac{1}{\chi^{(e)}(x)}\frac{\partial f}{\partial r}(x) + \frac{\partial f}{\partial \varphi}(x)\,, \\ \hat{f}^{(-)}(x) &\equiv \sum_{i=-\infty}^{-1}\left\{\left|\frac{1}{\chi^{(e)}(x_i)}\frac{\partial f}{\partial r}(x_i)\right| + \left|\frac{\partial f}{\partial \varphi}(x_i)\right|\right\}\prod_{j=i+1}^{0}|\Lambda^*(x_j, \gamma^{(e)}(x_j))|^{-1} \\ &\quad + \left|\frac{1}{\chi^{(e)}(x)}\frac{\partial f}{\partial r}(x)\right| + \left|\frac{\partial f}{\partial r}(x)\right|\,. \end{aligned}$$

Then by assumptions (f–1), (f–2) and by Lemma 3.2, Lemma 3.3, the series in (5.2) and (5.3) converge and $f^{(+)}(x)$ (resp. $f^{(-)}(x)$) is continuous at x not in $\bigcup_{i=0}^{\infty} T_*^i S$ (resp. $\bigcup_{i=0}^{\infty} T_*^{-i} S$).

For $w = (\tilde{x}, \tilde{v})$ in W, define curves $\bar{\gamma}^{(+)}(w)$ and $\bar{\gamma}^{(-)}(w)$ passing through $(\tilde{x}, \tilde{v})$ by the following way. Let $r = u^{(c)}(\varphi; \tilde{x})$ be the equation of the curve $\gamma^{(c)}(\tilde{x})$ and let $r = u^{(e)}(\varphi; \tilde{x})$ be the equation of the curve $\gamma^{(e)}(\tilde{x})$. Let $\bar{\gamma}^{(+)}(w)$ and $\bar{\gamma}^{(-)}(w)$ be the curves defined respectively by the equations

$$(5.4)\qquad \begin{cases} \iota = \tilde{\iota} \\ r = u^{(c)}(\varphi; \tilde{x}) \\ v = \tilde{v} - \int_{\tilde{\varphi}}^{\varphi} f^{(-)}(\tilde{\iota}, u^{(c)}(\varphi; \tilde{x}), \varphi) d\varphi \end{cases} \quad \text{and} \quad \begin{cases} \iota = \tilde{\iota} \\ r = u^{(e)}(\varphi; \tilde{x}) \\ v = \tilde{v} + \int_{\tilde{\varphi}}^{\varphi} f^{(+)}(\tilde{\iota}, u^{(e)}(\varphi; \tilde{x}), \varphi) d\varphi \end{cases}$$

for $0 \leq v < f(\iota, r, \varphi)$ with $\tilde{x} = (\tilde{\iota}, \tilde{r}, \tilde{\varphi})$. Then, obviously, $\bar{\gamma}^{(+)}(w)$ and $\bar{\gamma}^{(-)}(w)$ are locally transversal fibres; that is,

(i) for w' in $\bar{\gamma}^{(+)}(w)$ (resp. $\bar{\gamma}^{(-)}(w)$), $\bar{\gamma}^{(+)}(w') = \bar{\gamma}^{(+)}(w)$ (resp. $\bar{\gamma}^{(-)}(w') = \bar{\gamma}^{(-)}(w)$),

(ii) $S_t^f \bar{\gamma}^{(+)}(w)$ coincides with $\bar{\gamma}^{(+)}(S_t^f w)$ and $S_t^f \bar{\gamma}^{(-)}(w)$ coincides with $\bar{\gamma}^{(-)}(S_t^f w)$ in a neighbourhood of $S_t^f w$.

Therefore $\tilde{\Gamma}^{(+)}(w) \equiv \bigcup_t S_t^f \bar{\gamma}^{(+)}(S_{-t}^f w)$ and $\tilde{\Gamma}^{(-)}(w) \equiv \bigcup_t S_t^f \bar{\gamma}^{(-)}(S_{-t}^f w)$ consist of countably many connected curves in W. Further $\tilde{\Gamma}^{(+)}(w)$ and $\tilde{\Gamma}^{(-)}(w)$ are transversal fibres; that is,

(i) $\tilde{\Gamma}^{(+)}(w') = \tilde{\Gamma}^{(+)}(w)$ for $w' \in \tilde{\Gamma}^{(+)}(w)$,
$\tilde{\Gamma}^{(-)}(w') = \tilde{\Gamma}^{(-)}(w)$ for $w' \in \tilde{\Gamma}^{(-)}(w)$,

(ii) $S_t^f \tilde{\Gamma}^{(+)}(w) = \tilde{\Gamma}^{(+)}(S_t^f w)$ and $S_t^f \tilde{\Gamma}^{(-)}(w) = \tilde{\Gamma}^{(-)}(S_t^f w)$.

For each x in M, identify two points $(x, f(x))$ and $(T_* x, 0)$. Under the identification, let $\tilde{\gamma}^{(+)}(w)$ be the connected component of w in $\tilde{\Gamma}^{(+)}(w)$. Then $\{\tilde{\gamma}^{(+)}(w); w \in W\}$ gives a partition $\tilde{\zeta}^{(+)}$ of W. Similarly, $\tilde{\gamma}^{(-)}(w)$ and $\tilde{\zeta}^{(-)}$ are given by $\{\tilde{\Gamma}^{(-)}(w)\}$.

A curve $\tilde{\gamma}$ in W which is given by the equations $\iota = \tilde{\iota}$, $r = u(\varphi)$ and $v = t(\varphi)$ is said to be K-increasing (resp. K-decreasing), if the curve γ in $M^{(\tilde{\iota})}$ defined by $r = u(\varphi)$ is K-increasing (resp. K-decreasing) and $t(\varphi)$ is locally Lipschitzc ontinuous. For a given K-increasing curve $\tilde{\gamma}$ in W, define a measure $\sigma_{\tilde{\gamma}}$ by

$$(5.5)\qquad \sigma_{\tilde{\gamma}}(A) = \int_A |d\varphi|$$

for A in $\tilde{\gamma}$. Put for a subset R of $(-\infty, \infty)$

$$A^{(+)}[\tilde{\gamma}; t] \equiv \bigcup_{w \in S_{-t}\tilde{\gamma}} \tilde{\gamma}^{(+)}(w), \quad A^{(+)}[\tilde{\gamma}; R] \equiv \bigcup_{t \in R} A^{(+)}[\tilde{\gamma}; t]. \tag{5.6}$$

Let Π be the natural projection from W to M; $\Pi(\tilde{x}, \tilde{v}) = \tilde{x}$. Then for sufficiently small R the measure $\mu = \mu_f$ satisfies

$$\mu(B \cap A^{(+)}[\tilde{\gamma}; R]) = c_f \int_R \nu(\Pi(A^{(+)}[\tilde{\gamma}; t] \cap B))dt. \tag{5.7}$$

Similarly, subsets $A^{(-)}[\tilde{\gamma}; t]$ and $A^{(-)}[\tilde{\gamma}; R]$ are defined. The local fibres $\{\tilde{\gamma}^{(+)}\}$ and $\{\tilde{\gamma}^{(-)}\}$ are called *mutually integrable* (with each other), if for almost every $w = (\tilde{x}, \tilde{v})$ in W and for almost every y in $\Pi(A^{(+)}[\tilde{\gamma}^{(-)}(w); 0]) \cap \Pi(A^{(-)}[\tilde{\gamma}^{(+)}(w); 0])$, the relation

$$A^{(+)}[\tilde{\gamma}^{(-)}(w); 0] \cap \Pi^{-1}(y) = A^{(-)}[\tilde{\gamma}^{(+)}(w); 0] \cap \Pi^{-1}(y)$$

holds.

THEOREM 4. *Under the assumptions* (H-1)$\sim$(H-3) *and* (f-1)$\sim$(f-3),

(i) $S_t^f\tilde{\zeta}^{(+)} > \tilde{\zeta}^{(+)}$, $S_t^f\tilde{\zeta}^{(-)} < \tilde{\zeta}^{(-)}$, $t > 0$,
$\bigvee_t S_t^f\tilde{\zeta}^{(+)} = \bigvee_t S_t^f\tilde{\zeta}^{(-)} = \epsilon$,
$\bigwedge_t S_t^f\tilde{\zeta}^{(+)} = \bigwedge_t S_t^f\tilde{\zeta}^{(-)} = \pi(\{S_t^f\})$,

(ii) *the conditional measure* $\mu(\cdot|\tilde{\gamma}^{(+)})$ *(resp.* $\mu(\cdot|\tilde{\gamma}^{(-)})$ *is equivalent to* $\sigma_{\tilde{\gamma}^{(+)}}$ *(resp.* $\sigma_{\tilde{\gamma}^{(-)}}$),

(iii) $h(S_t^f) = h(S_t^f\tilde{\zeta}^{(+)}|\tilde{\zeta}^{(+)}) = h(S_{-t}^f\tilde{\zeta}^{(-)}|\tilde{\zeta}^{(-)}) = th(T_*)\Big/\int f(x)d\nu$,

(iv) *if* $\{\tilde{\gamma}^{(+)}\}$ *and* $\{\tilde{\gamma}^{(-)}\}$ *are not mutually integrable, then* $\pi(\{S_t^f\})$ *is the trivial partition, and hence* $\{S_t^f\}$ *is a K-system,*

(v) *if* $\{S_t^f\}$ *has no point spectrum, then* $\{S_t^f\}$ *is a K-system.*

Proof. By the above discussions and the definitions,

$$S_t^f\tilde{\zeta}^{(+)} > \tilde{\zeta}^{(+)},\ S_t^f\tilde{\zeta}^{(-)} < \tilde{\zeta}^{(-)}\ (t > 0) \quad \text{and } \bigvee_t S_t^f\tilde{\zeta}^{(+)} = \bigvee_t S_t^f\tilde{\zeta}^{(-)} = \epsilon$$

are obvious. Let β be the partition of W given by $\beta \equiv \Pi^{-1}\alpha^{(e)} = \{\Pi^{-1}X_j^{(e)}; X_j^{(e)} \in \alpha^{(e)}\}$. For any countable partition $\alpha = \{Y_j\}$ of W let $\tilde{d}\,(w; \partial\alpha)$ be the distance between w and the boundaries $\bigcup_j \partial Y_j \cup W_* \cup W^* \cup \Pi^{-1}(S)$ where $W_* \equiv \{(x, 0); x \in M\}$, $W^* \equiv \{(x, f(x)); x \in M\}$ and $\Pi^{-1}(S) \equiv \{(x, v); 0 \leq v < f(x), x \in S\}$. Then $\log \tilde{d}(w; \partial\beta)$ is integrable by virtue of (f-1)$\sim$(f-3). Since the flow $\{S_t^f\}$ is ergodic, except for a countable number of t's the transformation S_t^f is ergodic. Fix such a sufficiently small positive t and suppose that $\log \tilde{d}(w; \partial\alpha)$ is integrable. Then by the same way as in the proof of Theorem 3, one can see that for almost every element $\tilde{\gamma}^{(+)}$ of $\tilde{\zeta}^{(+)}$, the restriction of $\zeta_{\alpha}^{(+)} \equiv \bigvee_{k=0}^{\infty} S_{-kt}^{f}\alpha$ to $\tilde{\gamma}^{(+)}$ is a countable partition, each element of which is a union of a countable number of segments of $\tilde{\gamma}^{(+)}$. Hence one can see

$$\bigwedge_{n} \bigvee_{k=0}^{\infty} S_{(n-k)t}^{f}\alpha \leq \bigwedge_{s} S_{s}^{f}\tilde{\zeta}^{(+)} \equiv \tilde{\zeta}_{\infty}^{(+)} .$$

Since there exists a sequence of partitions $\{\alpha_n\}$ of W increasing to ϵ such that $\log \tilde{d}(w; \partial\alpha_n)$ are integrable, $\pi(S_t^f) \leq \tilde{\zeta}_{\infty}^{(+)}$. If $\alpha \geq \beta$, then $\zeta_{\alpha}^{(+)} \geq \Pi^{-1}\zeta^{(c)} \equiv \{\Pi^{-1}\gamma^{(c)};\ \gamma^{(c)} \in \zeta^{(c)}\}$, since $\alpha^{(c)}$ generates $\zeta^{(c)}$. For any $\varepsilon > 0$ and for almost every w, $\{S_{-kt}^{f}w; k \geq 0\}$ visits the set $Y_{\varepsilon} = \{(x, v) \in W; u - \varepsilon < v < u, (x, u) \in \bigcup_j \partial Y_j\}$ infinitely many times, since S_t^f is ergodic. Hence $\zeta_{\alpha}^{(+)} = \bigvee_{k=0}^{\infty} S_{-kt}^{f}\alpha \geq \tilde{\zeta}^{(+)}$ if $\alpha \geq \beta$ and if $\log \tilde{d}(w; \partial\alpha)$ is integrable. Hence one obtains

$$\pi(S_t^f) = \pi(\{S_s^f\}) = \bigwedge_{s} S_s^f\tilde{\zeta}^{(+)} .$$

Thus (i) is proved. The second assertion (ii) is obvious by definition and § 3. The third assertion (iii) comes from the theorem of Rohlin and Sinai [10] and a theorem of Abramov [1]. For almost every y in M and for a sufficiently small neighbourhood $U_{\varepsilon}(y)$, there exists a quartet $\{y, y_1, y_2, y_3\}$ in $U_{\varepsilon}(y)$ such that y_1 in $\gamma^{(c)}(y)$, y_2 in $\gamma^{(e)}(y_1)$, y_3 in $\gamma^{(c)}(y_2)$ and y in $\gamma^{(e)}(y_3)$. Then one can define a mapping $\Psi$of $\Pi^{-1}(y)$ by

$$\Psi w = \tilde{\gamma}^{(-)}(\tilde{\gamma}^{(+)}(\tilde{\gamma}^{(-)}(\tilde{\gamma}^{(+)}(w) \cap \Pi^{-1}(y_1)) \cap \Pi^{-1}(y_2)) \cap \Pi^{-1}(y_3)) \cap \Pi^{-1}(y)$$

for w in $\Pi^{-1}(y)$. Obviously, there exists a real number $a = a(y, y_1, y_2, y_3)$ such that

$$\Psi(y, u) = (y, u + a)$$

for (y, u) in the domain of Ψ. If $\{\tilde{\gamma}^{(+)}\}$ and $\{\tilde{\gamma}^{(-)}\}$ are not mutually integrable, then there exists a subset Y of positive measure such that for all $\delta > 0$ and all y in Y one can choose a quartet $\{y, y_1, y_2, y_3\}$ with $0 < |a(y, y_1, y_2, y_3)| < \delta$. Put $\tilde{\zeta}_{-\infty}^{(-)} \equiv \bigwedge S_t^f\tilde{\zeta}^{(-)}$ and let $h(w)$ be a $\tilde{\zeta}_{\infty}^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$-measurable bounded function. Since $\tilde{\zeta}_{\infty}^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$ is $\{S_t^f\}$-invariant, $h_b(w) = \dfrac{1}{b}\int_0^b h(S_t^f w)dt$ is again $\tilde{\zeta}_{\infty}^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$-measurable. Then $h_b(y, u)$ is continuous in u and $h_b(w)$ converges to $h(w)$ a.e. w as $b \to 0$. There exist measurable functions $h_b^{(+)}(w)$ and $h_b^{(-)}(w)$ such that $h_b(w) = h_b^{(+)}(w) = h_b^{(-)}(w)$ for a.e. w and that $h_b^{(+)}(w)$ is constant on $\tilde{\Gamma}^{(+)}$ and $h_b^{(-)}(w)$ is constant on $\tilde{\Gamma}^{(-)}$. Since canonical mappings $\Psi^{(c)}$ and $\Psi^{(e)}$ are absolutely continuous, one can choose

y, y_1, y_2, y_3 such that $h_b(y_i, u) = h_b^{(+)}(y_i, u) = h_b^{(-)}(y_i, u)$ for almost every u in $[0, f(y_i))$, $i = 0, 1, 2, 3$ with $y_0 = y$. Hence one can obtain

$$h_b(y, u) = h_b(y, u + a)$$

for almost every u in $[\delta, f(y)-\delta]$ with small $\delta > 0$. Since δ can be taken arbitrary small and $h_b(y, u)$ is continuous in u, $h_b(y, u)$ is constant in u. Hence $h_b(y, u)$ is constant in a subset with positive measure. Since b and h are arbitrary, one can see that the partition $\tilde{\zeta}_\infty^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$ contains an element of positive measure. Since $\{S_t^f\}$ is ergodic and $\tilde{\zeta}_\infty^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$ is invariant under $\{S_t^f\}$, the partition $\zeta_\infty^{(+)} \wedge \zeta_\infty^{(-)}$ is trivial. Thus (iv) was proved. Suppose that $\{\tilde{\gamma}^{(+)}\}$ and $\{\tilde{\gamma}^{(-)}\}$ are mutually integrable and that $\tilde{\zeta}_\infty^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$ is not trivial. Since $\bigwedge_k T_*^k \zeta^{(c)} \wedge \bigwedge_k T_*^k \zeta_e^{(\,)}$ is the trivial partition, the factor flow of $\{S_t^f\}$ with respect to $\tilde{\zeta}_\infty^{(+)} \wedge \tilde{\zeta}_{-\infty}^{(-)}$ is a circle flow. Hence $\{S_t^f\}$ has a point spectrum. Q.E.D.

It is very difficult to check that $\{\tilde{\gamma}^{(+)}\}$ and $\{\tilde{\gamma}^{(-)}\}$ are not mutually integrable for general cases.

LEMMA 5.1. *For a K-quadrilateral G, put*

$$v_f(G) \equiv \int_{\gamma_a} f^{(+)} d\varphi + \int_{\gamma_b} f^{(-)} d\varphi - \int_{\gamma_c} f^{(+)} d\varphi - \int_{\gamma_d} f^{(-)} d\varphi .$$

(i) *If $v_f(G) = 0$ for any G whose lateral sides are segments of $\{\gamma^{(c)}\}$ and $\{\gamma^{(e)}\}$, then $\{\tilde{\gamma}^{(+)}\}$ and $\{\tilde{\gamma}^{(-)}\}$ are mutually integrable.*

(ii) *If $v_f(G) > 0$ for any G in an open set whose lateral sides are segments of $\{\gamma^{(c)}\}$ and $\{\gamma^{(e)}\}$, then $\{\tilde{\gamma}^{(+)}\}$ and $\{\tilde{\gamma}^{(-)}\}$ are not mutually integrable.*

COROLLARY 5.2 ([12]). *A Sinai billiard system is a K-system.*

Proof. Since $f(x) = -\tau(x)$, it holds that

$$\frac{1}{\chi^{(c)}(x)} \frac{\partial f}{\partial r} + \frac{\partial f}{\partial \varphi} = \frac{1}{\chi^{(c)}(x)} \sin \varphi(x) - \frac{1}{\chi^{(c)}(x_1)} \sin \varphi(x_1) \frac{d\varphi_1}{d\varphi} ,$$

and hence $f^{(+)}(x) = \sin \varphi / \chi^{(c)}(x)$ (cf. [5]). Similarly one has $f^{(-)}(x) = -\sin \varphi / \chi^{(e)}(x)$. Hence

$$v_f(G) = -\int_{\gamma_a} \sin \varphi dr - \int_{\gamma_b} \sin \varphi dr + \int_{\gamma_c} \sin \varphi dr + \int_{\gamma_d} \sin \varphi dr = \frac{1}{\nu_0} \nu(G) .$$

Q.E.D.

COROLLARY 5.3. *The flow $\{S_t^f\}$ in § 3 has expanding and contracting* *transversal fibres. Further if $\{S_t^f\}$ has no point spectrum, then $\{S_t^f\}$ is a K-system.*

§6. Bernoulli flow

A flow $\{S_t\}$ is called a Bernoulli flow if every S_t $(t \neq 0)$ is a Bernoulli shift.

THEOREM 5. *Under the assumptions* (H-1)~(H-3) *and* (f-1)~(f-3), *if* $\{S_t^f\}$ *is a K-system and* α *is a finite partition such that* $\log \tilde{d}(w; \partial\alpha)$ *is integrable, then* α *is very weakly Bernoullian for* S_t^f $(t \neq 0)$. *Furthermore* $\{S_t^f\}$ *is a Bernoulli flow.*

Proof. For $\varepsilon > 0$, choose sufficiently small $\delta > 0$. Let δ_1 be a positive number with $\mu(E_1) > 1 - \delta$ where $E_1 \equiv \{w \in W;\ \tilde{d}(w; \partial\alpha) > \delta_1\}$. Fix a positive t. Then S_t^f is an ergodic transformation, since $\{S_t^f\}$ is a K-system by the assumption. Hence by Birkhoff's ergodic theorem, there exist a set E_2 with $\mu(E_2) > 1 - \delta$ and a natural number N_1 such that for all w in E_2 and all $n \geq N_1$

$$\frac{1}{n} \sum_{k=0}^{n-1} I_{E_1^c}(S_{-kt}^f w) \leq 2\delta$$

where $I_{E_1^c}$ is the indicator function of E_1^c. Then there exists a δ_2 with $\mu(E_3) > 1 - \delta$, where $E_3 \equiv \{w \in W;\ \inf_{0 \leq k \leq N_1 - 1} \tilde{d}(S_{-kt}^f w; \partial\alpha) > \delta_2\}$. Denote by $C^{(+)}(w)$ and $C^{(-)}(w)$ the connected components of w in the elements of $\bar{\zeta}^{(+)} \vee \zeta_\alpha^{(+)}$ and $\bar{\zeta}^{(-)} \vee \zeta_\alpha^{(-)}$ respectively, where $\zeta_\alpha^{(+)} = \bigvee_{k=0}^{\infty} S_{-kt}^f \alpha$, $\zeta_\alpha^{(-)} = \bigvee_{k=0}^{\infty} S_{kt}^f \alpha$ and $\bar{\zeta}^{(+)}$ (resp. $\bar{\zeta}^{(-)}$) is the partition into curves $\{\bar{\gamma}^{(+)}\}$ (resp. $\{\bar{\gamma}^{(-)}\}$). By the same reason in the proof of Theorem 3, there exists a positive number δ_3 such that $\mu(E_4) > 1 - \delta$ where $E_4 \equiv \{w \in W;\ \bar{\theta}(\Pi(C^{(+)}(w)); \Pi(w)) > \delta_3$, $\underline{\theta}(\Pi(C^{(+)}(w)); \Pi(w)) > \delta_3$, $\bar{\theta}(\Pi(C^{(-)}(w)); \Pi(w)) > \delta_3$ and $\underline{\theta}(\Pi(C^{(-)}(w)); \Pi(w)) > \delta_3\}$. There exists a positive δ_4 $(<\delta)$ such that $1/\delta_4 > \sup_{x \in E_5} \{|f^{(+)}(x)| + |f^{(-)}(x)|\}$ with $E_5 \equiv \{x \in M;\ |\cos\varphi(x)| > \delta_1\}$. Note that $\Pi^{-1}(E_5)$ is a subset of E_1.

For any x not in $\bigcup_{i=-\infty}^{\infty} T_*^i S$, there exists $\varepsilon_3 = \varepsilon_3(x) > 0$ such that for any y in ε_3-neighbourhood $U_{\varepsilon_3}(x)$

$$|f^{(+)}(x) - f^{(+)}(y)| < \delta_4\delta,\ |f^{(-)}(x) - f^{(-)}(y)| < \delta_4\delta$$

$$|f(x) - f(y)| < \delta_2\delta \qquad \text{and} \quad \left|\frac{\cos\varphi(y)}{\cos\varphi(x)}\right| < 1 + \delta\,.$$

Let $\ell_0 = \ell_0(\delta, 1, 1/4)$, $m_0 \geq \ell_0$ and $\varepsilon_2 = \varepsilon_2(x, m_0)$ be as in the proof of Theorem

2. Then for any $w = (\bar{t}, \bar{r}, \bar{\varphi}, \bar{v})$ in E_3 and for any $\delta_0 > 0$, there exists a subset $\tilde{G}$ of W which is constructed as follows: There exists a K-quadrilateral G in $U_{\varepsilon_3}(\bar{x}) \cap U_{\varepsilon_2}(\bar{x})$ with $\bar{x} \equiv (\bar{t}, \bar{r}, \bar{\varphi}) = \Pi(w)$ such that $T_*^{-m_0}$ and $T_*^{m_0}$ are continuous on G, $T_*^{-m_0}G$ and $T_*^{m_0}G$ are K-quadrilaterals, $\gamma_a(G)$ (resp. $\gamma_b(G)$) is a segment of the fibre $\gamma^{(c)}(\bar{x})$ (resp. $\gamma^{(e)}(\bar{x})$), and

$$\theta(\gamma_a(G)) = \theta(\gamma_b(G)) < \min\{\delta_0, \delta_1, \delta_2, \delta_3\}/(1 + c_2) .$$

Put $\tilde{\gamma}_a \equiv \tilde{\gamma}^{(+)}(w) \cap \Pi^{-1}(G)$. For $\bar{w}$ in W, put

$$D(\bar{w}) \equiv \left\{ (\bar{t}, r, \varphi, v) \; ; \; \begin{array}{l} 1 - \delta \leq \dfrac{\varphi - \bar{\varphi}}{r - \bar{r}} \dfrac{1}{\chi^{(c)}(x)} \leq 1 + \delta \\ f^{(+)}(\bar{x}) - \delta \leq \dfrac{v - \bar{v}}{\varphi - \bar{\varphi}} \leq f^{(+)}(\bar{x}) + \delta \end{array} \right\}$$

with $\bar{w} = (\bar{t}, \bar{r}, \bar{\varphi}, \bar{v})$. Define $\tilde{G}$ by

$$\tilde{G} = \bigcup_{-a \leq s \leq a} S_s^f \bigcup_{\bar{w} \in \tilde{\gamma}_b} D(\bar{w}) \cap \Pi^{-1}(G) ,$$

with $a = \theta(\gamma_a(G))$. As stated in § 3, there exist subsets $G^{(c,\delta)}$ and $G^{(e,\delta)}$ of G which satisfy (C-1), (C-2) and (C-3). Put

$$\tilde{G}^{(c,\delta)} \equiv \left\{ w \in \Pi^{-1}(G^{(c,\delta)}) \cap \tilde{G}; \; \tilde{\gamma}^{(+)}(w) \cap \bigcup_{|s| \leq a} S_s^f \tilde{\gamma}_b \neq \phi \right\} .$$

Then one can see the inequality

$$\mu(\tilde{G}^{(c,\delta)} \cap \Pi^{-1}(G^{(e,\delta)})) \geq (1 + 2\delta)^{-4}(1 - 2\delta)\mu(\tilde{G}) ,$$

since $1 - \delta \leq \chi^{(c)}(x)/\chi^{(c)}(\bar{x}) \leq 1 + \delta$ for x in G. By the covering theorem of Vitali, there exists a finite family $\mathscr{G}$ of $\tilde{G}$'s which satisfies

$$\mu(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \tilde{G}) \geq (1 - \delta)\mu(\tilde{G}) ,$$
$$\Pi^{-1}(E_5) \supset \tilde{G} ,$$
$$\mu\Big(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \Pi^{-1}(E_5) - \bigcup_{\tilde{G} \in \mathscr{G}} \tilde{G}\Big) < \delta ,$$
$$\tilde{G} \cap \tilde{G}' = \phi \qquad \text{if } \tilde{G} \neq \tilde{G}' .$$

Put

$$E(\tilde{G}) \equiv \{ w \in \tilde{G}^{(c,\delta)}; \; \tilde{\gamma}^{(+)}(w) \cap E_4 \cap \tilde{G} \neq \phi \} .$$

Then $B \cap E(\tilde{G})$ is $\tilde{\zeta}^{(+)}|_{\tilde{G}}$-measurable for every element B of $\bigvee_{k=N'}^{N''} S_{-kt}^f \alpha$ with $N'' > N' \geq 0$. By Lemma 3.1, the estimate

$$\begin{aligned}
&\mu(B \cap E(\tilde{G}) \cap \Pi^{-1}(G^{(e,\delta)}))/(1+\delta) \\
&\quad \le \int_{\gamma_a(G)\cap G^{(e,\delta)}} d\rho(\gamma) \int_{\bar{v}-a-\delta}^{\bar{v}+a+\delta} dt \int_{B\cap E(\tilde{G})\cap S^f_{-t}\gamma_b(G)} -\mu_0 \cos\varphi d\sigma(\varphi) \\
&\quad \le \mu(B \cap E(\tilde{G}) \cap \Pi^{-1}(G^{(e,\delta)}))/(1-\delta)
\end{aligned}$$

is obtained. Moreover, for y in $\gamma_a(G) \cap G^{(e,\delta)}$

$$1-\delta < \frac{\displaystyle\int_{\bar{v}-a-2\delta}^{\bar{v}+a+2\delta} dt \int_{B\cap E(\tilde{G})\cap S^f_{-t}\gamma^{(e)}(y)} \cos\varphi d\varphi}{\displaystyle\int_{\bar{v}-a-\delta}^{\bar{v}+a+\delta} dt \int_{B\cap E(\tilde{G})\cap S^f_{-t}\gamma^{(e)}(y)} \cos\varphi d\varphi} < 1+\delta$$

is obtained. Therefore, there exists a $((1+\delta)^3-1)$-measure preserving mapping ϕ from $\bigcup_{\tilde{G}\in\mathscr{G}}(E(\tilde{G}) \cap \Pi^{-1}(G^{(e,\delta)})) \cap B$ to $\bigcup_{\tilde{G}\in\mathscr{G}}(E(\tilde{G}) \cap \Pi^{-1}(G^{(e,\delta)}))$ such that ϕ maps $E(\tilde{G}) \cap \Pi^{-1}(\gamma^{(e)}(x)) \cap B$ to $E(\tilde{G}) \cap \Pi^{-1}(\gamma^{(e)}(x))$ for x in $\gamma_a(G) \cap G^{(e,\delta)}$. Let $\ell_i(w)$ be the name function of $S^f_{it}\,\alpha$. For z in $E_1 \cap E_2 \cap E_3 \cap E(\tilde{G}) \cap B$

$$\ell_i(z) = \ell_i(\phi z) \qquad \text{for } 1 \le i \le N_1 - 1$$

and for $n \ge N_1$

$$\frac{1}{n}\sum_1^n e(\ell_i(z) - \ell_i(\phi z)) \le 2\delta$$

hold, since

$$\begin{aligned}
&\tilde{d}(S^f_{-kt}z;\partial\alpha) > \delta_2 \qquad \text{for } 1 \le k \le N_1 - 1\,, \\
&\frac{1}{n}\sum_{k=0}^{n-1} I_{E_1^c}(S^f_{-kt}z) \le 2\delta\,, \\
&(\text{distance of } S^f_{-kt}z \text{ and } S^f_{-kt}\phi z) \le \min(\delta_1, \delta_2, \delta_3)
\end{aligned}$$

for $k \ge 0$. On the other hand, there exist an N_2 and a set E_6 such that $\mu(E_6) > 1-\delta$ and for all N', $N'' \ge N_2$ and all B in $\bigvee_{k=N'}^{N''} S^f_{-kt}\alpha$, $B \subset E_6$,

$$|\mu(E_1 \cap E_2 \cap E_3 \cap E(\tilde{G})) - \mu(E_1 \cap E_2 \cap E_3 \cap E(\tilde{G})|B)| \le \delta\mu(\tilde{G})$$

holds, since S^f_t is a K-system. Hence

$$\begin{aligned}
&\mu\Big(\bigcup_{\tilde{G}\in\mathscr{G}} (E_1 \cap E_2 \cap E_3 \cap E(\tilde{G}) \cap B)\Big) \\
&\quad \ge \sum_{G\in\mathscr{G}} \mu(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \tilde{G})\mu(B) - \delta\mu(B) \\
&\quad \ge (1-\delta)\mu\Big(\bigcup_{\tilde{G}\in\mathscr{G}} \tilde{G}\Big)\mu(B) - \delta\mu(B)
\end{aligned}$$

$$\geq (1-\delta)(\mu(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \Pi^{-1}(E_5)) - \delta)\mu(B) - \delta\mu(B)$$
$$\geq [(1-\delta)(1-6\delta) - \delta]\mu(B) .$$

Therefore by Lemma 2.4, the partition α is very weakly Bernoullian.

Since there exists an increasing sequence of finite partitions $\{\alpha_n\}$ such that $\log \tilde{d}(w; \partial\alpha_n)$ is integrable and α_n increases to ϵ as $n \to \infty$, S_t^f is a Bernoulli shift for fixed $t \neq 0$. Q.E.D.

COROLLARY 6.1. *A Sinai billiard system is a Bernoulli flow.*

COROLLARY 6.2. *If $\{S_t^f\}$ has no point spectrum, then $\{S_t^f\}$ is a Bernoulli flow.*

THEOREM 6. *The flow $\{S_t\}$ given in* § 3 *is a Bernoulli flow, if the assumptions* (H-1)~(H-3) *are fulfilled and if $\{S_t\}$ has no point spectrum.*

Appendix

The properties of the partitions $\alpha^{(c)}$ and $\alpha^{(e)}$ have been shown in [5]. Now some of them will be stated. Under the suitable numbering the followings are true, here denote as $a(j) = O(j^b)$ if

$$0 < \varliminf_{j\to\infty} |a(j)| j^{-b} \leq \varlimsup_{j\to\infty} |a(j)| j^{-b} < \infty :$$

(i) $T_* X_j^{(e)} = X_j^{(c)}$,

(ii) $\tau(x) = O(j)$ for $x \in X_j^{(e)}$ (resp. $X_j^{(c)}$).

Further the following figure is also true.

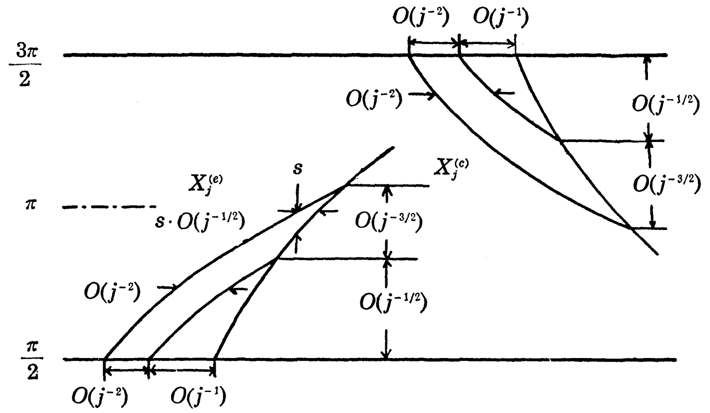

Fig. 2

These show that $\log d(x; \partial\alpha^{(e)})$ is integrable. The condition (f–1) and (i) imply that the distance between $w = (x, v)$ in $\Pi^{-1}(X_j^{(e)})$ and the boundary W^* is greater than $O(j^{-1/2})d(x;\partial\alpha^{(e)})^{1/2}(f(x)-v)$ if $f(x)-v\leq O(j^{1/2})d(x;\partial\alpha^{(e)})^{1/2}$. Moreover,

$$\sup_{x,y\in X_j^{(e)}}(f(x)-f(y))\leq O(j)$$

is shown. Hence one can easily obtain that $\log\tilde{d}(w;\partial\beta)$ is integrable.

REFERENCES

[1] L. M. Abramov, On the entropy of a flow, Dokl. Akad. Nauk SSSR **128** (1959), 873–876.=Amer. Math. Soc. Transl. (2) **49** (1966), 167–170.

[2] N. A. Friedman and D. S. Ornstein, On isomorphism of weak Bernoulli transformations, Advances in Math. **5** (1970), 365–394.

[3] G. Gallavotti and D. S. Ornstein, Billiards and Bernoulli schemes, Commun. math. Phys. **38** (1974), 83–101.

[4] Sh. Ito, H. Murata and H. Totoki, Remarks on the isomorphism theorems for weak Bernoulli transformations in general case, Publ. RIMS, Kyoto Univ. **7** (1972), 541–580.

[5] I. Kubo, Perturbed billiard systems I. The ergodicity of the motion of a particle in a compound central field, Nagoya Math. J. **61** (1976), 1–57.

[6] ——, The ergodicity of the motion of a particle in a potential field, International symposium on mathematical problems in theoretical physics, Jan. 1975 Kyoto Japan, 141–148.

[7] D. S. Ornstein, Two Bernoulli shifts with infinite entropy are isomorphic, Advances in Math. **5** (1970), 339–348.

[8] ——, Imbedding Bernoulli shifts in flows, Lecture Notes in Math. 160 [L. Sucheston (editor): Contribution to ergodic theory and probability] Springer-Verlag (1970), 178–218.

[9] D. S. Ornstein and B. Weiss, Geodesic flows are Bernoullian, Israel J. Math. **14** (1973), 184–198.

[10] V. A. Rohlin and Ya G. Sinai, Construction and properties of invariant measurable partitions, Dokl. Akad. Nauk SSSR **141** (1961), 1038–1041.=Soviet Math. Dokl. **2** (1961), 1611–1614.

[11] Ya. G. Sinai, On the foundations of the ergodic hypothesis for a dynamical system of statistical mechanics, Dokl. Akad. Nauk SSSR **153** (1963), 1261–1264.=Soviet Math. Dokl. **4** (1963), 1818–1822.

[12] ——, Dynamical systems with elastic reflections, Uspehi Math. Nauk **25** (1970), 141–192.=Russian Math. Surveys **25** (1970), 137–189.

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