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ON THE STRUCTURE OF THE IDELE GROUP OF AN ALGEBRAIC NUMBER FIELD

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The purpose of this paper is to present the results of E. Artin and Furtwängler, with which they proved the principal ideal theorem, as a structure theorem of the idele group of an algebraic number field. Such treatment may be helpful to clarify the Arithmetic nature these results possess.

§1.

Let F be an algebraic number field (of finite degree over $\boldsymbol{Q}$), and let K/F and L/K be both finite abelian extensions. Suppose that L is a Galois extension of F, and that K is the maximal abelian extension of F contained in L. Then $G = \mathrm{Gal}(L/F)$ is metabelian, and $G' = \mathrm{Gal}(L/K)$ is the commutator subgroup of G.

Let us denote the Artin maps of K/F and L/K by $[\cdot, K/F]$ and $[\cdot, L/K]$ respectively. That is, for a prime ideal $\mathfrak{p}$ of F which is unramified in K/F, $[\mathfrak{p}, K/F]$ is the Frobenius automorphism of $\mathfrak{p}$ in $\mathrm{Gal}(K/F)$.

Let $\mathfrak{a}$ be an ideal of F. Then the extension of $\mathfrak{a}$ to an ideal of K is $\mathfrak{a}\cdot O_K$ where O_K is the maximal order of K.

THEOREM (Artin-Furtwängler). *Let L be a Galois extension of F, and suppose that $G = \mathrm{Gal}(L/F)$ is metabelian. Let K be the maximal abelian extension of F contained in L, and O_K the maximal order of K. Then, if an ideal $\mathfrak{a}$ of F is unramified in K/F, $[\mathfrak{a}\cdot O_K, L/K]$ is trivial.*

E. Artin showed that the map of $G/G' = \mathrm{Gal}(K/F)$ to $G' = \mathrm{Gal}(L/K)$ which gives

$$[\mathfrak{a}, K/F] \longmapsto [\mathfrak{a}\cdot O_K, L/K]$$

is the transfer (Verlagerung) $V_{G\to G'}$ of G/G' to G'. Then Furtwängler proved that $V_{G\to G'}$ is the trivial homomorphism of G/G' to G'. (See [1] and [3].)

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It may be worth to point out that this theorem is proved without using class field theory.

§ 2.

For an algebraic number field F, the ring of adeles of F is denoted by F_A, and the idele group of F by $F_A^\times$. Let F_{ab} be the maximal abelian extension in the algebraic closure $\bar{F}$ of F, and put $\mathfrak{A}_F = \mathrm{Gal}(F_{ab}/F)$ and $\mathfrak{G}_F = \mathrm{Gal}(\bar{F}/F)$. Let $F_A^\times = F_f^\times \cdot F_\infty^\times$ be the decomposition of $F_A^\times$ into the product of its non-Archimedean part $F_f^\times$ and its Archimedean part $F_\infty^\times$. Let $F_{\infty+}^\times$ be the connected component of the unity of $F_\infty^\times$, and $F^\#$ the topological closure of $F^\times \cdot F_{\infty+}^\times$ in $F_A^\times$. Here and after, F and $F^\times$ are considered to be diagonally embedded in F_A and $F_A^\times$ respectively.

By class field theory, Artin map or canonical morphism

$$[\cdot, F]\colon F_A^\times \longrightarrow \mathfrak{A}_F$$

is an open, continuous and surjective homomorphism whose kernel is $F^\#$. Our basic reference on class field theory is Weil's book [8] though the notation slightly differs.

Let K be a finite Galois extension of F. Then $\mathrm{Gal}(K/F) = \mathfrak{G}_F/\mathfrak{G}_K$ where $\mathfrak{G}_K = \mathrm{Gal}(\bar{F}/K)$. The ring of adeles of K is naturally identified with the tensor product $K \otimes_F F_A = K_A$. Then the natural action of $\mathfrak{G}_F$ on K_A is the one defined through the K-factor of the product.

Let $\mathfrak{G}'_K$ be the commutator subgroup of $\mathfrak{G}_K$. Then $\mathfrak{A}_K = \mathrm{Gal}(K_{ab}/K) = \mathfrak{G}_K/\mathfrak{G}'_K$. Since $\mathfrak{G}_K$ is a normal subgroup of $\mathfrak{G}_F$, this $\mathfrak{G}_F$ acts on $\mathfrak{G}_K$ through inner automorphisms of $\mathfrak{G}_F$, and also on $\mathfrak{A}_K = \mathfrak{G}_K/\mathfrak{G}'_K$. More precisely, let ξ be an element of $\mathfrak{G}_K$. Then for $\lambda \in \mathfrak{G}_F$, the action of λ on $\xi \bmod \mathfrak{G}'_K$ is defined by

$$(\xi \bmod \mathfrak{G}'_K)^\lambda = \lambda^{-1} \cdot \xi \cdot \lambda \mod \mathfrak{G}'_K .$$

THEOREM 1. *For* $x \in K_A^\times$ *and* $\lambda \in \mathfrak{G}_F$,

$$[x^\lambda, K] = [x, K]^\lambda$$

where $[\cdot, K]\colon K_A^\times \to \mathfrak{A}_K = \mathrm{Gal}(K_{ab}/K)$ is Artin map for K.

This theorem is well known. But a proof will be given in § 6 for the completeness.

§ 3.

Now our intended result is ready to be shown. Generalization will be done in the next section. Note that K does not have to be an abelian extension of F in this theorem.

THEOREM 2. *Let F be an algebraic number field and K a finite Galois extension of F. If an open subgroup U of $K_A^\times$ satisfies*

(i) $U \supset K^\#$

(ii) $U^\sigma = U$ *for any* $\sigma \in \mathrm{Gal}(K/F)$

(iii) $U \cdot N_{K/F}^{-1}(F^\#) = K_A^\times$

then $U \supset F_A^\times$.

Here $N_{K/F}\colon K_A^\times \to F_A^\times$ is the norm map of K over F.

Proof. First we reduce the theorem to the case that K is an abelian extension of F. Let M be the maximal abelian extension of F contained in K. Then

$$F^\times \cdot N_{M/F}(M_A^\times) = F^\times \cdot N_{K/F}(K_A^\times) .$$

Put $V = M^\times \cdot N_{K/M}(U)$. Then V is an open subgroup of $M_A^\times$, and contains $M^\#$. It is obvious that $V^\tau = V$ for $\tau \in \mathrm{Gal}(M/F)$. Since

$$F^\times \cdot N_{M/F}(V) = F^\times \cdot N_{K/F}(U) = F^\times \cdot N_{K/F}(K_A^\times) = F^\times \cdot N_{M/F}(M_A^\times)$$

it is easy to see that

$$V \cdot N_{M/F}^{-1}(F^\#) = V \cdot N_{M/F}^{-1}(F^\times) = M_A^\times .$$

It follows, moreover, from (i) and (ii) that U contains V as a subgroup. Hence it is sufficient to show that V contains $F_A^\times$. Therefore we may assume that K itself is an abelian extension of F.

Now let L be the class field of K corresponding to U. Then

$$U = K^\times \cdot N_{L/K}(L_A^\times) .$$

By Theorem 1, condition (ii) implies that L is a Galois extension of F. From (iii), it follows that K is the maximal abelian extension of F contained in L.

For a prime ideal $\mathfrak{P}$ of K, let $O_{K,\mathfrak{P}}$ be the $\mathfrak{P}$-adic completion of O_K, and $O_{K,\mathfrak{P}}^\times$ the group of units of $O_{K,\mathfrak{P}}$. Then $O_{K,\mathfrak{P}}^\times$ is canonically regarded as a subgroup of $K_A^\times$. Since U is open, the number of such prime ideals $\mathfrak{P}$ that $O_{K,\mathfrak{P}}^\times \not\subset U$ is finite. Let S be the set of all such prime ideals of K. For each $\mathfrak{P} \in S$, fix an integer $e(\mathfrak{P})$ such that

$$1 + \mathfrak{P}^{e(\mathfrak{P})} \cdot O_{K,\mathfrak{P}} \subset U$$

and

$U_S = \prod_{\mathfrak{P} \notin S} O_{K,\mathfrak{P}}^{\times} \times \prod_{\mathfrak{P} \in S} (1 + \mathfrak{P}^{e(\mathfrak{P})} \cdot O_{K,\mathfrak{P}}) \times K_{\infty+}^{\times}$

$K_{A(S)}^{\times}$ = the subgroup of $K_A^{\times}$ generated by U_S and all $K_{\mathfrak{P}}^{\times}$ for $\mathfrak{P} \notin S$

$K_S^{\times} = K^{\times} \cap K_{A(S)}^{\times}$

$\mathfrak{M} = \prod_{\mathfrak{P} \in S} \mathfrak{P}^{e(\mathfrak{P})} \times$ product of all infinite places of K

$I_L(S)$ = the group of ideals of L prime to $\mathfrak{M}$

$I_K(S)$ = the group of ideals of K prime to $\mathfrak{M}$

$\mathfrak{S}_K(M)$ = the Strahl ideal class group modulo $\mathfrak{M}$.

Here $K_{\mathfrak{P}}$ is the $\mathfrak{P}$-adic completion of K, and $K_{\mathfrak{P}}^{\times}$ is its multiplicative group. For prime $\boldsymbol{P}$ of L, let $L_{\boldsymbol{P}}$ be the $\boldsymbol{P}$-adic completion, and $L_{\boldsymbol{P}}^{\times}$ the multiplicative group of $L_{\boldsymbol{P}}$. Put

$L_{A(S)}^{\times}$ = the subgroup of $L_A^{\times}$ generated by $\prod_{\boldsymbol{P} \cap K \in S} O_{L,\boldsymbol{P}}^{\times}$ and all $L_{\boldsymbol{P}}^{\times}$ for $\boldsymbol{P} \cap K \notin S$.

For idele x of K (resp. of L, of F), denote the corresponding ideal of K (resp. of L, of F) by $\mathscr{I}_K(x)$ (resp. $\mathscr{I}_L(x)$, $\mathscr{I}_F(x)$). Then we have exact sequences

$$1 \longrightarrow U_S \longrightarrow K_{A(S)}^{\times} \xrightarrow{\mathscr{I}_K} I_K(S) \longrightarrow 1$$
$$1 \longrightarrow K_S^{\times} \cdot U_S \longrightarrow K_{A(S)}^{\times} \longrightarrow \mathfrak{S}_K(S) \longrightarrow 1$$
$$L_{A(S)}^{\times} \cap N_{L/K}^{-1}(K_{A(S)}^{\times}) \xrightarrow{\mathscr{I}_L} I_L(S) \longrightarrow 1\ .$$

Furthermore, for $x \in L_{A(S)}^{\times} \cap N_{L/K}^{-1}(K_{A(S)}^{\times})$,

$$\mathscr{I}_K(N_{L/K}(x)) = N_{L/K}(\mathscr{I}_L(x))$$

and, for $x \in F_A^{\times} \cap K_{A(S)}^{\times}$,

$$\mathscr{I}_K(x) = \mathscr{I}_F(x) \cdot O_K\ .$$

Now apply Artin-Furtwängler theorem to this case. Then, (by Hilbert theory), one can easily conclude that, for $x \in F_A^{\times} \cap K_{A(S)}^{\times}$, there exist $a \in K_S^{\times}$ and $y \in L_{A(S)}^{\times} \cap N_{L/K}^{-1}(K_{A(S)}^{\times})$ such that

$$\mathscr{I}_K(x) = \mathscr{I}_K(a) \cdot N_{L/K}(\mathscr{I}_L(y))\ .$$

Therefore

$$x = a \cdot N_{L/K}(y) \cdot u$$

with some $u \in U_S$. Since U contains all of $K_S^{\times}$, $N_{L/K}(L_A^{\times})$ and U_S, it has been shown that

$$F_A^\times \cap K_{A(S)}^\times \subset U .$$

Because S is a finite set of prime ideals of K, one can easily see by Chinese remainder theorem that $(F_A^\times \cap K_{A(S)}^\times)\cdot F^\times = F_A^\times$. Since U contains $F^\times$,

$$F_A^\times = (F_A^\times \cap K_{A(S)}^\times)\cdot F^\times \subset U\cdot F^\times = U .$$

The proof is done.

§ 4. Generalization

THEOREM 3. *Let F be an algebraic number field, and K a finite Galois extension of F. For an open subgroup U of $K_A^\times$ satisfying*

(i) $U \supset K^\#$

(ii) $U^\sigma = U$ *for any* $\sigma \in \mathrm{Gal}\,(K/F)$

put $m = [K_A^\times : U\cdot N_{K/F}^{-1}(F^\#)]$. *Then*

$$(F_A^\times)^m = \{a^m \mid a \in F_A^\times\} \subset U .$$

Proof. Let L be the abelian extension of K corresponding to $U\cdot N_{K/F}^{-1}(F^\#)$. Then $m = [L : K]$, and

$$K^\times \cdot N_{L/K}(L_A^\times) = U\cdot N_{K/F}^{-1}(F^\#) .$$

Put $V = N_{L/K}^{-1}(U)$. Then

$$L_A^\times = V\cdot N_{L/F}^{-1}(F^\#)$$

since

$$\begin{aligned} F^\times \cdot N_{L/F}(L_A^\times) &= F^\times \cdot N_{K/F}(K^\times \cdot N_{L/K}(L_A^\times)) \\ &= F^\times \cdot N_{K/F}(U\cdot N_{K/F}^{-1}(F^\#)) \\ &= F^\times \cdot N_{K/F}(U) \\ &= F^\times \cdot N_{L/F}(V) . \end{aligned}$$

Obviously L is a Galois extension of F. Theorem 2, therefore, is applicable to L/F and V, and implies that $V \supset F_A^\times$. Hence for any $a \in F_A^\times$

$$a^m = N_{L/K}(a) \in U .$$

The proof is completed.

COROLLARY. *The notation and the assumptions being as in the theorem, let n be the largest common divisor of m and the degree $[K : F]$. Then*

$$(U\cdot N_{K/F}^{-1}(F^{\#}))\cap F_A^{\times}=(U\cdot X)\cap F_A^{\times}$$

where $X=\{x\in N_{K/F}^{-1}(F^{\#})\,|\,x^n\in U\}$.
Therefore especially

$$(U\cdot N_{K/F}^{-1}(F^{\#}))\cap F_A^{\times}=U\cap F_A^{\times}$$

if n *is prime to the index* $[U\cdot N_{K/F}^{-1}(F^{\#})\colon U]$.

Proof. Put $d=[K\colon F]$. For $a\in(U\cdot N_{K/F}^{-1}(F^{\#}))\cap F_A^{\times}$, choose $u\in U$ and $v\in N_{K/F}^{-1}(F^{\#})$ so that $a=u\cdot v$. Then $a^d=N_{K/F}(a)=N_{K/F}(u)\cdot N_{K/F}(v)$. Condition (ii) implies that $N_{K/F}(u)\in U$. Since $N_{K/F}(v)\in F^{\#}$, we conclude that $a^d\in U\cap F_A^{\times}$. It follows from the theorem that a^m belongs to $U\cap F_A^{\times}$. Therefore a^n belongs to $U\cap F_A^{\times}$ where $n=(m,d)$. Since $a^n=u^n\cdot v^n$, we see that $v\in X$. The proof is done.

§ 5. Remarks on $F^{\#}$

Let F be an algebraic number field of finite degree d over $\boldsymbol{Q}$, and $d=r_1+2\cdot r_2$ where r_1 is the number of real Archimedean primes of F. Put $r=r_1+r_2-1$. Let E_+ be the multiplicative group of all the totally positive units of F. (We exclude the roots of 1 in F from E_+ when $r_1=0$.) Then E_+ is a free $\boldsymbol{Z}$-module of rank r.

Let E_{+f} be the projection of E_+ to the non-Archimedean part $F_f^{\times}$ of $F_A^{\times}$, and $\overline{E_{+f}}$ the topological closure of E_{+f} in $F_f^{\times}$.

PROPOSITION 1. *The closure* $F^{\#}$ *of* $F^{\times}\cdot F_{\infty+}^{\times}$ *in* $F_A^{\times}$ *is equal to* $\overline{E_{+f}}\cdot F^{\times}\cdot F_{\infty+}^{\times}$. *Moreover, for every positive integer* n,

$$\overline{E_{+f}}=E_{+f}\cdot\{x^n\,|\,x\in\overline{E_{+f}}\}$$

$$F^{\#}=F^{\times}\cdot\{x^n\,|\,x\in F^{\#}\}\,.$$

(*See Shimura* [7], 2.2.)

PROPOSITION 2. (1) $F^{\times}\cap\{x^n\,|\,x\in F^{\#}\}=\{a^n\,|\,a\in F^{\times}\}$.
(2) *For* $x\in F^{\#}$, $x^n=1\Rightarrow x\in F^{\times}\cdot F_{\infty+}^{\times}$.
(See [6], 3.1.)

PROPOSITION 3. *As topological groups,* $\overline{E_{+f}}$ *is isomorphic to the direct product of r copies of* $\tilde{\boldsymbol{Z}}=\prod_{p,\,\mathrm{prime}}\boldsymbol{Z}_p$ *where* $\boldsymbol{Z}_p$ *is the ring of p-adic integers.*

Proof. By Chevalley [2], the topology induced on free $\boldsymbol{Z}$-module E_{+f} of rank r is the one defined by taking all the subgroups of finite index as the basis of the neighbourhood of 0. Therefore $\overline{E_{+f}}$ is isomorphic to the completion $\tilde{Z}^r$.

PROPOSITION 4. *Let K be a finite extension of F (not necessarily Galois). Then*

$$N_{K/F}^{-1}(F^{\#})/K^{\#}\cdot N_{K/F}^{-1}(1) \cong N_{K/F}(K_A^{\times}) \cap F^{\times}/N_{K/F}(K^{\times}) .$$

Proof. Put $N=N_{K/F}$, and $d=[K:F]$. First we see $N^{-1}(F^{\#})=N^{-1}(F^{\times})\cdot F^{\#}$. For $x\in N^{-1}(F^{\#})$, choose $a\in F^{\times}$ and $b\in F^{\#}$ by Prop. 1 so that $N(x)=a\cdot b^d$. Put $y=x\cdot b^{-1}$. Then $N(y)=a\in F^{\times}$, and $x=y\cdot b$.

Next we show $N^{-1}(F^{\times})\cap K^{\#}=K^{\times}\cdot(N^{-1}(1)\cap K^{\#})$. Obviously the right is contained by the left. For $z\in K^{\#}$, suppose that $N(z)\in F^{\times}$. By Prop. 1 for K, choose $u\in K^{\times}$ and $v\in K^{\#}$ so that $z=u\cdot v^d$. Then $N(v)^d=N(z)\cdot N(u)^{-1}\in F^{\times}$. Therefore by Prop. 2, (1), we can find $a\in F^{\times}$ such that $N(v)^d=a^d$. Then $z=(u\cdot a)\cdot(a^{-1}\cdot v^d)$ with $u\cdot a\in K^{\times}$ and $N(a^{-1}v^d)=1$. Now

$$\begin{aligned} N^{-1}(F^{\#})/K^{\#}\cdot N^{-1}(1) &= N^{-1}(F^{\times})\cdot F^{\#}/K^{\#}\cdot N^{-1}(1) \\ &\cong N^{-1}(F^{\times})/N^{-1}(F^{\times})\cap(K^{\#}\cdot N^{-1}(1)) \\ &= N^{-1}(F^{\times})/(N^{-1}(F^{\times})\cap K^{\#})\cdot N^{-1}(1) \\ &= N^{-1}(F^{\times})/K^{\times}\cdot N^{-1}(1) \\ &\cong N(K_A^{\times})\cap F^{\times}/N(K^{\times}) . \end{aligned}$$

The proof is done.

§ 6. Proof of Theorem 1

Let K be a finite Galois extension of an algebraic number field F. Let the notation and the situation be as in § 2. We have to prove that canonical homomorphism $[\cdot, K]: K_A^{\times}\to\mathfrak{A}_K=\mathrm{Gal}(K_{ab}/K)$ of class field theory is compatible with the action of $\mathfrak{G}_F=\mathrm{Gal}(\bar{F}/F)$ (modulo $\mathfrak{G}_K$).

Let $\mathfrak{p}$ be a prime divisor of F, $F_{\mathfrak{p}}$ the completion of F at $\mathfrak{p}$, and $\bar{F}_{\mathfrak{p}}$ the algebraic closure of $F_{\mathfrak{p}}$. Fix an isomorphism ι of $\bar{F}$ into a subfield $\iota(\bar{F})$ of $\bar{F}_{\mathfrak{p}}$, which is identical on F. Put $\tilde{K}=\iota(K)\cdot F_{\mathfrak{p}}$. This is a Galois extension of $F_{\mathfrak{p}}$. Put $\mathfrak{G}_{\mathfrak{p}}=\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$ and $\tilde{\mathfrak{G}}=\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/\tilde{K})$. The latter is a normal subgroup of the former. Note that $\bar{F}_{\mathfrak{p}}=\iota(\bar{F})\cdot F_{\mathfrak{p}}$, $F_{\mathfrak{p},ab}=\iota(F_{ab})\cdot F_{\mathfrak{p}}$, and $\tilde{K}_{ab}=\iota(K_{ab})\cdot\tilde{K}$ where $F_{\mathfrak{p},ab}$ and $\tilde{K}_{ab}$ are the maximal abelian extension of $F_{\mathfrak{p}}$ and $\tilde{K}$ in $\bar{F}_{\mathfrak{p}}$ respectively. Hence the restriction of the action of $\mathfrak{G}_{\mathfrak{p}}$ on $\iota(\bar{F})$ gives an isomorphic embedding of $\mathfrak{G}_{\mathfrak{p}}$ into $\iota\circ\mathfrak{G}_F\circ\iota^{-1}$. Let $\mathfrak{Z}_{\mathfrak{p}}$ be the subgroup of $\mathfrak{G}_F$ corresponding to $\mathfrak{G}_{\mathfrak{p}}$. That is, $\iota\circ\mathfrak{Z}_{\mathfrak{p}}\circ\iota^{-1}=\mathfrak{G}_{\mathfrak{p}}$. We also have

$$\mathfrak{G}'_{\mathfrak{p}} = \mathfrak{G}_{\mathfrak{p}} \cap (\iota \circ \mathfrak{G}'_F \circ \iota^{-1})$$
$$\tilde{\mathfrak{G}}' = \tilde{\mathfrak{G}} \cap (\iota \circ \mathfrak{G}'_K \circ \iota^{-1})$$

where $\mathfrak{G}'_{\mathfrak{p}}$ and $\tilde{\mathfrak{G}}'$ are the commutator subgroups of $\mathfrak{G}_{\mathfrak{p}}$ and $\tilde{\mathfrak{G}}$ respectively.

Fix a set of representatives $S = \{\sigma_1, \cdots, \sigma_g\}$ of the left cosets of $\mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K$ in $\mathfrak{G}_F$. (Remember that $\mathfrak{G}_F$ acts on both of K_A and $\mathfrak{A}_K$ from the right.) For $\sigma \in \mathfrak{G}_F$, the representative in S of $\mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K \cdot \sigma$ is denoted by $[\sigma]$. Put

$$\iota(\sigma) = \iota \circ [\sigma]^{-1} \qquad (\sigma \in \mathfrak{G}_F) .$$

Then $\iota(\sigma)$ depends only on the coset $\mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K \cdot \sigma$. The family of pairs $\{(\iota(\sigma), K) | \sigma \in S\}$ is a set of all non-equivalent proper embeddings of K above $F_{\mathfrak{p}}$. That is, for any proper embedding (λ, L) of K above $F_{\mathfrak{p}}$, there are $\sigma \in S$ and isomorphism ρ of L over $F_{\mathfrak{p}}$ into $\tilde{K}$ such that $\iota(\sigma) = \rho \circ \lambda$. (See Weil [8], p. 51, Cor. 2.) Fix a set of representatives $R = \{\rho_1, \cdots, \rho_f\}$ of $\mathfrak{G}_{\mathfrak{p}}/\tilde{\mathfrak{G}} = \mathrm{Gal}(\tilde{K}/F_{\mathfrak{p}})$ where $\rho_i \in \mathfrak{G}_{\mathfrak{p}}$. Then for any two elements σ, τ of $\mathfrak{G}_F$, there is a unique element $\rho(\sigma, \tau)$ of R such that, restricted to K,

$$\iota(\sigma) \circ \tau|_K = \rho(\sigma, \tau) \circ \iota(\sigma\tau^{-1})|_K .$$

For σ and $\tau \in \mathfrak{G}_F$, define $\zeta(\sigma, \tau) \in \mathfrak{Z}_{\mathfrak{p}} \cdot \mathfrak{G}_K$ by

$$[\sigma] \cdot \tau^{-1} = \zeta(\sigma, \tau) \cdot [\sigma\tau^{-1}] .$$

Then

$$\rho(\sigma, \tau) \equiv \iota \circ \zeta(\sigma, \tau)^{-1} \circ \iota^{-1} \qquad \text{modulo } \tilde{\mathfrak{G}} .$$

For each $\sigma \in S$, put

$$\tilde{\mathfrak{G}}_\sigma = \sigma \circ \iota^{-1} \circ \tilde{\mathfrak{G}} \circ \iota \circ \sigma^{-1} = \iota^{-1} \circ [(\iota \circ \sigma^{-1} \circ \iota^{-1}) \cdot \tilde{\mathfrak{G}} \cdot (\iota \circ \sigma \circ \iota^{-1})] \circ \iota .$$

Then $\tilde{\mathfrak{G}}_\sigma$ is a subgroup of $\mathfrak{G}_K$ and is a conjugate of $\mathfrak{Z}_{\mathfrak{p}} \cap \mathfrak{G}_K$ in $\mathfrak{G}_F$. It is easy to see that the commutator subgroup $\tilde{\mathfrak{G}}'_\sigma$ of $\tilde{\mathfrak{G}}_\sigma$ coincides with $\tilde{\mathfrak{G}}_\sigma \cap \mathfrak{G}'_K$. Put

$$\mathfrak{A}_{\tilde{K}, \sigma} = \tilde{\mathfrak{G}}_\sigma / \tilde{\mathfrak{G}}'_\sigma .$$

This is considered as a subgroup of $\mathfrak{A}_K = \mathfrak{G}_K / \mathfrak{G}'_K$. The action of $\mathfrak{G}_F$ on $\mathfrak{A}_K$ maps the family $\{\mathfrak{A}_{\tilde{K}, \sigma} | \sigma \in S\}$ onto itself. Each $\mathfrak{A}_{\tilde{K}, \sigma}$ is isomorphic to $\mathfrak{A}_{\tilde{K}} = \tilde{\mathfrak{G}} / \tilde{\mathfrak{G}}'$.

Let us now consider the $\mathfrak{p}$-part of K_A. It is naturally identified with $K \otimes_F F_{\mathfrak{p}}$. Take copies of $\tilde{K}$ indexed by S. That is, put $\tilde{K}_\sigma = \tilde{K}$ for each $\sigma \in S$. Then the map $\iota(\sigma)\colon K \to \tilde{K}_\sigma$ for $\sigma \in S$ gives an $F_{\mathfrak{p}}$-linear isomorphism

$\eta_{\mathfrak{p}}$ of $K \otimes_F F_{\mathfrak{p}}$ onto the direct product $\prod_{\sigma \in S} \tilde{K}_\sigma$.

For $\sigma, \tau \in \mathfrak{G}_F$, and for $a \in K$,

$$\begin{aligned} \iota(\sigma)(a^\tau) &= (\iota(\sigma) \circ \tau)(a) = (\rho(\sigma, \tau) \circ \iota(\sigma\tau^{-1}))(a) \\ &= (\iota(\sigma\tau^{-1})(a))^{\rho(\sigma,\tau)} . \end{aligned}$$

Therefore it is easy to see the following:

For $x \in K \otimes_F F_{\mathfrak{p}}$, *let* $\eta_{\mathfrak{p}}(x) = (x_\sigma)_{\sigma \in S} \in \prod_\sigma \tilde{K}_\sigma$.

Then for $\tau \in \mathfrak{G}_F$,

$$\eta_{\mathfrak{p}}(x^\tau) = (y_\sigma)_{\sigma \in S} \in \prod_\sigma \tilde{K}_\sigma$$
$$y_\sigma = (x_{[\sigma\tau^{-1}]})^{\rho(\sigma,\tau)} .$$

Let χ be a (linear) character of $\mathfrak{G}_K$. It is automatically considered as a character of $\mathfrak{A}_K = \mathfrak{G}_K/\mathfrak{G}'_K = \mathrm{Gal}\,(K_{ab}/K)$. For $\lambda \in \mathfrak{G}_F$, define a character χ^λ of $\mathfrak{G}_K$ by

$$\chi^\lambda(\tau) = \chi(\lambda\tau\lambda^{-1}) \qquad (\tau \in \mathfrak{G}_K) .$$

Since $\mathfrak{G}_K$ is normal in $\mathfrak{G}_F$, this is well defined. Note that χ^λ depends only on λ modulo $\mathfrak{G}_K$.

For χ, we can associate characters $\chi_\sigma(\sigma \in S)$ of $\mathfrak{A}_{\tilde{K}} = \tilde{\mathfrak{G}}/\tilde{\mathfrak{G}}' = \mathrm{Gal}\,(\tilde{K}_{ab}/\tilde{K})$ through the isomorphisms of $\mathfrak{A}_{\tilde{K}}$ onto $\mathfrak{A}_{\tilde{K},\sigma}$ established above. Namely for $\mu \in \tilde{\mathfrak{G}}$,

$$\begin{aligned} \chi_\sigma(\mu) &= \chi(\sigma \circ \iota^{-1} \circ \mu \circ \iota \circ \sigma^{-1}) \\ &= \chi(\sigma^{-1} \cdot (\iota^{-1} \circ \mu \circ \iota) \cdot \sigma) \\ &= \chi^{\sigma^{-1}}(\iota^{-1} \circ \mu \circ \iota) . \end{aligned}$$

For a character χ of $\mathfrak{G}_K$, and for $x \in K \otimes_F F_{\mathfrak{p}}$ with $\eta_{\mathfrak{p}}(x) = (x_\sigma)_{\sigma \in S} \in \prod_\sigma \tilde{K}_\sigma$, the canonical pairing $(\chi, x)_{K,\mathfrak{p}}$ is defined by

$$(\chi, x)_{K,\mathfrak{p}} = \prod_{\sigma \in S} (\chi_\sigma, x_\sigma)_{\tilde{K}} ,$$

where each $(\chi_\sigma, x_\sigma)_{\tilde{K}}$ is the canonical pairing of local class field theory for $\tilde{K}_\sigma = \tilde{K}$.

Let λ be an element of $\mathfrak{G}_F$. For $x \in K \otimes_F F_{\mathfrak{p}}$ with $\eta_{\mathfrak{p}}(x) = (x_\sigma)_{\sigma \in S}$, we had $\eta_{\mathfrak{p}}(x^\lambda) = (y_\sigma)$ with $y_\sigma = (x_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)}$. On the other hand, $(\chi^\lambda)_\sigma(\mu) = \chi^{\lambda\sigma^{-1}}(\iota^{-1} \circ \mu \circ \iota)$ for $\mu \in \tilde{\mathfrak{G}}$. Since $\sigma\lambda^{-1} = \zeta(\sigma, \lambda)[\sigma\lambda^{-1}]$,

$$(\chi^\lambda)_\sigma(\mu) = \chi^{[\sigma\lambda^{-1}]^{-1}\zeta(\sigma,\lambda)^{-1}}(\iota^{-1} \circ \mu \circ \iota)$$

$$
\begin{aligned}
&= \chi^{[\sigma\lambda^{-1}]^{-1}}(\zeta(\sigma,\lambda)^{-1}\cdot(\iota^{-1}\circ\mu\circ\iota)\cdot\zeta(\sigma,\lambda)) \\
&= \chi^{[\sigma\lambda^{-1}]^{-1}}(\zeta(\sigma,\lambda)\circ\iota^{-1}\circ\mu\circ\iota\circ\zeta(\sigma,\lambda)^{-1}) \\
&= \chi^{[\sigma\lambda^{-1}]^{-1}}(\iota^{-1}\circ\rho(\sigma,\lambda)^{-1}\circ\mu\circ\rho(\sigma,\lambda)\circ\iota) \\
&= \chi_{[\sigma\lambda^{-1}]}(\rho(\sigma,\lambda)^{-1}\circ\mu\circ\rho(\sigma,\lambda)) \\
&= \chi_{[\sigma\lambda^{-1}]}(\rho(\sigma,\lambda)\cdot\mu\cdot\rho(\sigma,\lambda)^{-1}) \\
&= (\chi_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)}(\mu)\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\chi^\lambda, x^\lambda)_{K,\mathfrak{p}} &= \prod_{\sigma\in S}((\chi^\lambda)_\sigma, y_\sigma)_{\tilde{K}} \\
&= \prod_{\sigma\in S}((\chi_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)}, (x_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)})_{\tilde{K}}\,.
\end{aligned}
$$

Since $\rho(\sigma,\lambda)\in\mathfrak{G}_\mathfrak{p} = \mathrm{Gal}(\bar{F}_\mathfrak{p}/F_\mathfrak{p})$, and since $\tilde{K}$ is a Galois extension of $F_\mathfrak{p}$, we have $\tilde{K}^{\rho(\sigma,\lambda)} = \tilde{K}$.
Therefore

$$((\chi_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)}, (x_{[\sigma\lambda^{-1}]})^{\rho(\sigma,\lambda)})_{\tilde{K}} = (\chi_{[\sigma\lambda^{-1}]}, x_{[\sigma\lambda^{-1}]})_{\tilde{K}}\,.$$

(See Weil [8], p 223, Cor. 5.) This shows that

$$(\chi^\lambda, x^\lambda)_{K,\mathfrak{p}} = (\chi, x)_{K,\mathfrak{p}}\,.$$

Since this is true for any prime divisor of F,

$$(\chi^\lambda, x^\lambda)_K = (\chi, x)_K$$

for $x\in K_A^\times$, $\lambda\in\mathfrak{G}_F$ and a character χ of $\mathfrak{G}_K$. Here $(\chi, x)_K$ is the canonical pairing of K.

The canonical morphism

$$[\cdot, K]\colon K_A^\times \longrightarrow \mathfrak{A}_K = \mathfrak{G}_K/\mathfrak{G}_K' = \mathrm{Gal}(K_{ab}/K)$$

is defined so that

$$(\chi, x)_K = \chi([x, K])$$

for any $x\in K_A^\times$ and any χ. For each $[x,K]\in\mathfrak{A}_K$, choose $[x,K]^*\in\mathfrak{G}_K$ so that $[x,K]^*$ modulo $\mathfrak{G}_K'$ is $[x,K]$. Then for $\lambda\in\mathfrak{G}_F$,

$$(\chi^\lambda, x^\lambda)_K = \chi^\lambda([x^\lambda, K]^*) = \chi(\lambda\cdot[x^\lambda, K]^*\cdot\lambda^{-1})\,.$$

Therefore

$$\chi([x, K]) = \chi(\lambda\cdot[x^\lambda, K]^*\cdot\lambda^{-1})$$

for any χ. This implies that

$$[x, K] = \lambda \cdot [x^{\lambda}, K]^{*} \cdot \lambda^{-1} \text{ modulo } \mathfrak{G}'_K .$$

Equivalent to say,

$$\lambda^{-1} \cdot [x, K]^{*} \cdot \lambda \equiv [x^{\lambda}, K]^{*} \qquad \text{modulo } \mathfrak{G}'_K .$$

This is what Theorem 1 claims. The proof is done.

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