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ON THE UPPER SEMI-LATTICE OF J_a^S-DEGREES

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S. C. Kleene developed the theory of recursive functionals of finite types in Kleene [5]. He proved that a set A of natural numbers is recursive in E if and only if A is hyperarithmetical, where E is the type 2 object defined by

$$E(\alpha) = \begin{cases} 0 & \text{if } \exists n[\alpha(n) = 0] , \\ 1 & \text{otherwise} . \end{cases}$$

By relativizing this result to a set B of natural numbers, A is hyperarithmetical in B if and only if A is recursive in E and B. Therefore, E-degrees coincide with hyperdegrees. A type 2 object F is said to be *normal* if E is recursive in F. The theory of recursive functions based on a normal type 2 object is an excellent generalization of the theory of hyperarithmetical functions. Hinman [4] is a good exposition of the theory of recursive functionals based on a normal type 2 object. It is natural to investigate F-degrees for a normal type 2 object F as a generalization of hyperdegrees. In this article, we shall discuss the upper semi-lattice of E_1-degrees and more generally of J_a^S-degrees, where E_1 is Tugué's object defined in Tugué [13] and $J_a^S(a \in O^S)$ are type 2 objects defined in Platek [6] which are obtained from E by consecutive applications of the superjump S.

The necessary preliminaries are given in § 1. Transfinite iterations of the F-jump are considered in § 2. In § 3, by using Cohen's forcing method, independent degrees are discussed. § 4 is devoted to the existence of minimal degrees. In § 5, we show the existence of incomparable degrees whose infimum does not exist.

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§ 1.

Let F be an arbitrary normal type 2 object, which we fix throughout § 1 and § 2. We let $\mathfrak{a} = (a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_k)$. A partial functional $\phi(\mathfrak{a})$ is said to be *partial F-recursive* if there exists an index e such that $\phi(\mathfrak{a}) \simeq \{e\}^F(\mathfrak{a})$. If ϕ is total, we omit the adjunct "partial". A predicate P is said to be *F-recursive* if its representing functional K_P is F-recursive.

The following three lemmas are very useful.

LEMMA 1.1[1] (*S-m-n* Theorem). *For each m, there exists a primitive recursive function* S^m *such that*

$$\{S^m(e, b_1, \cdots, b_m)\}^F(\mathfrak{a}) \simeq \{e\}^F(b_1, \cdots, b_m, \mathfrak{a}) .$$

LEMMA 1.2[1] (F-Recursion Theorem). *If* $\psi(e, \mathfrak{a})$ *is partial F-recursive, then there exists a number e such that*

$$\{e\}^F(\mathfrak{a}) \simeq \psi(e, \mathfrak{a}) .$$

LEMMA 1.3 (Substitution Theorem: cf. Hinman [4; VI. § 21]). *There exists a primitive recursive function* $\gamma(z, w)$ *such that for all* z, w *and* $\mathfrak{a}$

$$\{\gamma(z, w)\}^F(\mathfrak{a}) \simeq \{z\}^F(\mathfrak{a}, \lambda t\{w\}^F(t, \mathfrak{a})) .$$

If $\{e\}^F(\mathfrak{a})$ is defined, the computation of $\{e\}^F(\mathfrak{a})$ is represented in the form of a well-founded tree, whose length we denote by $|e:\mathfrak{a}|^F$. $|e:\mathfrak{a}|^F$ is a countable ordinal. If $\{e\}^F(\mathfrak{a})$ is undefined, then we let $|e:\mathfrak{a}|^F = \infty\ (=\aleph_1)$. The following lemma of Gandy's is fundamental in the recursion theory based on normal objects.

LEMMA 1.4 (Stage Comparison Theorem: cf. Hinman [4; VI. 3.3]). *There exists a partial F-recursive functional* $\chi(z, \mathfrak{a}, w, \mathfrak{b})$ *such that if* $\{z\}^F(\mathfrak{a})\downarrow$ *or* $\{w\}^F(\mathfrak{b})\downarrow$, *then* $\chi(z, \mathfrak{a}, w, \mathfrak{b})\downarrow$ *and*

$$\chi(z, \mathfrak{a}, w, \mathfrak{b}) = \begin{cases} 0 & \text{if } |z:\mathfrak{a}|^F \leqq |w:\mathfrak{b}|^F , \\ 1 & \text{if } |z:\mathfrak{a}|^F > |w:\mathfrak{b}|^F , \end{cases}$$

where $\mathfrak{a} = (a_1, \cdots, a_m, \alpha_1, \cdots, \alpha_j)$, $\mathfrak{b} = (b_1, \cdots, b_n, \beta_1, \cdots, \beta_k)$ *and* "$\downarrow$" *means "is defined".*

A predicate P is said to be *F-semirecursive* if it is the domain of a partial F-recursive functional. Kleene proved that P is E-semirecursive

1) For the proofs of these lemmas, see Hinman [4; VI].

if and only if it is a Π_1^1 predicate. Using Lemma 1.4, Gandy obtained the following result.

LEMMA 1.5 (see Hinman [4; VI. 4.3–4.6]).

(i) *A predicate P is F-recursive if and only if both P and* $\neg P$ *are F-semirecursive.*

(ii) *If* $R(n, \alpha)$ *is F-semirecursive, then so are* $\forall n R(n, \alpha)$ *and* $\exists n R(n, \alpha)$.

(iii) *A partial functional* ϕ *is F-recursive if and only if its graph is F-semirecursive.*

From (i) and (ii) of the above lemma, we see that if $R(n, \alpha)$ is F-recursive, then $\forall n R(n, \alpha)$ and $\exists n R(n, \alpha)$ are also F-recursive. But we can prove this more directly from the definition of normality: let $\phi(n, \alpha)$ be the representing function of $R(n, \alpha)$. Then ϕ is F-recursive and hence the function $E(\lambda n \phi(n, \alpha))$ is F-recursive because E is F-recursive. It is obvious that $E(\lambda n \phi(n, \alpha))$ is the representing function of $\exists n R(n, \alpha)$.

If $u = \langle e, \langle a_1, \cdots, a_n \rangle \rangle$, we use $|u|^F$ instead of $|e: a_1, \cdots, a_n|^F$. Let $U^F = \{\langle e, \langle a_1, \cdots, a_n \rangle \rangle : \{e\}^F(a_1, \cdots, a_n) \downarrow \}$. Then $\sup \{|u|^F : u \in U^F\} = \omega_1[F]$, where $\omega_1[F]$ is the first non-F-recursive ordinal (see Hinman [4; VI. 4.17]). Obviously U^F is F-semirecursive. If $P \subset \omega$ is F-semirecursive, then there exists a number e such that $P(a)$ iff $\{e\}^F(a) \downarrow$. Then, $P(a)$ iff $\langle e, \langle a \rangle \rangle \in U^F$. Thus U^F is a complete F-semirecursive set.

Let σ be an ordinal. Define $L_F(\sigma)$ by:

$$L_F(0) = \{0\} ;$$

$L_F(\sigma + 1) = \{x \subset L_F(\sigma) : x$ is first order definable over the structure $\langle L_F(\sigma), \in, F \upharpoonright L_F(\sigma) \rangle$ with parameters from $L_F(\sigma)\}$;

$$L_F(\lambda) = \bigcup \{L_F(\sigma) : \sigma < \lambda\} \qquad \text{if } \lambda \text{ is a limit ordinal .}$$

We use $\mathscr{M}_F(\sigma)$ to denote the structure $\langle L_F(\sigma), \in, F \upharpoonright L_F(\sigma) \rangle$. If $\mathscr{M}_F(\sigma)$ is a model of KP (Kripke-Platek set theory) formulated in the language $\{\in, F\}$, which we denote by $KP(F)$, then σ is said to be an *F-admissible ordinal.* We use $\tau_\nu[F]$ to denote the *ν-th F-admissible ordinal.* In particular, $\tau_0[F] = \omega$. For the basic knowledge of KP and admissible sets, see Barwise [2].

In [4; VIII], Hinman developed the theory of recursive functions of ordinal numbers. We can relativize it to F by adding the following (*) to the definition of $\Omega_{\kappa\lambda}$ in [4; VIII. 1.1]:

(*) for any b and β, if $(b, n, \boldsymbol{\mu}, \beta(n)) \in \Omega_{\kappa\lambda}(F)$ for all n, then $(\langle 5, k, b\rangle, \boldsymbol{\mu}, F(\beta)) \in \Omega_{\kappa\lambda}(F)$.

If we set $\{a\}^F_{\kappa\lambda}(\boldsymbol{\mu}) \simeq \nu$ iff $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\lambda}(F)$, then $\{a\}^F_{\kappa\lambda}$ defines a partial function of ordinal numbers. We define $\{a\}^F_\kappa$ and $\{a\}^F_{\infty\lambda}$ as in Hinman [4; VIII. 1.3]. A partial function of the form $\{a\}^F_\kappa$ ($\{a\}^F_{\infty\lambda}$) is said to be *κ-recursive in F* (*(∞, λ)-recursive in F*). Other notions such as "κ-semirecursive" are easily relativized to F by using $\{a\}^F_\kappa$ or $\{a\}^F_{\infty\lambda}$, so we omit to define them explicitly. We say that an ordinal κ is *F-recursively regular* if κ is closed under all partial functions (∞, κ)-recursive in F. This definition is equivalent to each of the following (a) and (b):

(a) *for all $a \in \omega$ and all $\boldsymbol{\mu} < \kappa$, $\{a\}^F_{\infty\kappa}(\boldsymbol{\mu}) \simeq \{a\}^F_\kappa(\boldsymbol{\mu})$;*

(b) *for all $a \in \omega$ and all $\rho, \boldsymbol{\mu} < \kappa$, if $\{a\}^F_\kappa(\pi, \boldsymbol{\mu})$ is defined for all $\pi < \rho$, then* $\sup_{\pi<\rho} \{a\}^F(\pi, \boldsymbol{\mu}) < \kappa$.

LEMMA 1.6. *If κ is an F-recursively regular ordinal $>\omega$, then κ is F-admissible and for every $P \subset \kappa$:*

(i) *P is κ-recursive in F in parameters if and only if it is Δ_1 on $\mathcal{M}_F(\kappa)$;*

(ii) *P is κ-semirecursive in F in parameters if and only if it is Σ_1 on $\mathcal{M}_F(\kappa)$.*

Conversely, if κ is an F-admissible ordinal $>\omega$, then κ is F-recursively regular.

Proof. Let κ be an F-recursively regular ordinal $>\omega$, then there exists a map C from κ onto $L_F(\kappa)$ which satisfies the following conditions (1) and (2):

(1) $\forall\mu < \kappa C(\mu) \subset C''\mu$ and $\forall\mu < \kappa\ \exists\nu < \kappa\ [\mu < \nu\ \&\ C(\nu) = C''\nu]$

(2) the predicates $C(\mu) \in C(\nu)$ and $C(\mu) = C(\nu)$ are κ-recursive in F.

For every Δ_0 formula $\Phi(v_1, \cdots, v_n)$ of the language $\{\in, F\}$, we see by induction on the length of Φ that the predicate $\Phi(C(\mu_1), \cdots, C(\mu_n))$ is κ-recursive in F. For example, if $\Phi(v_1, \cdots, v_n)$ is $\exists v_0 \in v_1 \Psi(v_0, v_1, \cdots, v_n)$, where Ψ is a Δ_0 formula, then

$$\Phi(C(\mu_1), \cdots, C(\mu_n)) \longleftrightarrow \exists\mu_0 < \mu_1[C(\mu_0) \in C(\mu_1)\ \&\ \Psi(C(\mu_0), C(\mu_1), \cdots, C(\mu_n))]\ .$$

Since the set of all predicates κ-recursive in F is closed under bounded quantifiers, $\Phi(C(\mu_1), \cdots, C(\mu_n))$ is κ-recursive in F by the induction hypothesis and (2). From this, the implications from the right to the left in (i) and (ii) are obvious. In order to prove that κ is F-admissible, it suffices to show that the Δ_0 Collection Axiom holds in $\mathcal{M}_F(\kappa)$. Let $\Phi(v_1, v_2, v_3)$

be a Δ_0 formula and $\sigma, \tau < \kappa$. Suppose that

(3) $\forall x \in C(\sigma) \exists y \in L_F(\kappa) \Phi(x, y, C(\tau))$.

We have to prove that for some $\rho < \kappa$,

(4) $\forall x \in C(\sigma) \exists y \in C(\rho) \Phi(x, y, C(\tau))$.

From (3), we have:

$$\forall \mu < \sigma \exists \nu < \kappa [C(\mu) \in C(\sigma) \longrightarrow \Phi(C(\mu), C(\nu), C(\tau))] .$$

We let:

$$f(\mu, \sigma, \tau) = \begin{cases} \min \{\nu < \kappa : C(\mu) \in C(\sigma) \longrightarrow \Phi(C(\mu), C(\nu), C(\tau))\} & \text{if } \mu < \sigma , \\ 0 \quad \text{otherwise.} \end{cases}$$

Then f is κ-recursive in F. Hence $\sup_{\mu<\sigma} f(\mu, \sigma, \tau) < \kappa$ by (b). From (1), there exists a $\rho < \kappa$ such that $C(f(\mu, \sigma, \tau)) \in C(\rho)$ for all $\mu < \sigma$. It is easy to see that (4) holds, and thus the Δ_0 Collection Axiom holds in $\mathcal{M}_F(\kappa)$.

Now let κ be an F-admissible ordinal $> \omega$. By using the Second Recursion Theorem in $\mathcal{M}_F(\kappa)$ (see Barwise [2; V. 2.3]), we shall show that the relation $\{a\}_\kappa^F(\boldsymbol{\mu}) \simeq \nu$ is Σ_1 on $\mathcal{M}_F(\kappa)$. Find a Σ_1 formula $\Psi(v_1, v_2, v_3, v_4)$ such that for all $\sigma, \boldsymbol{\mu}, \nu < \kappa$ and all $a \in \omega$,

$$\mathcal{M}_F(\kappa) \models \Psi(\sigma, a, \langle \boldsymbol{\mu} \rangle, \nu) \qquad \text{iff } (a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}^\sigma(F) ,$$

where $\Omega_{\kappa\kappa}^\sigma(F)$ is the σ-th stage of the inductive definition of $\Omega_{\kappa\kappa}(F)$. Such a Σ_1 formula Ψ can be obtained by writing down the definition of $\Omega_{\kappa\kappa}^\sigma(F)$. For example,

$$\Psi(\sigma, \langle 5, k, b \rangle, \langle \boldsymbol{\mu} \rangle, \nu) \quad \text{iff } \exists \beta \forall n \in \omega \exists \tau < \sigma [\Psi(\tau, b, \langle n, \boldsymbol{\mu} \rangle, \beta(n)) \,\&\, F(\beta) = \nu] .$$

Let $X = \bigcup_{\sigma<\kappa} \Omega_{\kappa\kappa}^\sigma(F)$. We show that $X = \Omega_{\kappa\kappa}(F)$. Let $(a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}(F)$. By induction on $\min \{\sigma | (a, \boldsymbol{\mu}, \nu) \in \Omega_{\kappa\kappa}^\sigma(F)\}$, we prove that $(a, \boldsymbol{\mu}, \nu) \in X$. Except for the case where $(a)_0 = 3$, 4 or 5, the proof is straightforward. We consider the case where $(a)_0 = 3$. Other cases can be treated similarly. Let $b \in \omega, \rho, \boldsymbol{\mu} < \kappa$ and assume that $\forall \pi < \rho \exists \xi (b, \pi, \boldsymbol{\mu}, \xi) \in X$ Then,

$$\mathcal{M}_F(\kappa) \models \forall \pi < \rho \exists \xi \exists \tau \Psi(\tau, b, \langle \pi, \boldsymbol{\mu} \rangle, \xi) .$$

But since κ is F-admissible, there exist $\eta, \sigma < \kappa$ such that

$$\mathcal{M}_F(\kappa) \models \forall \pi < \rho \exists \xi < \eta \exists \tau < \sigma \Psi(\tau, b, \langle \pi, \boldsymbol{\mu} \rangle, \xi) .$$

This means that $(\langle 3, k, b \rangle, \rho, \boldsymbol{\mu}, \nu) \in X$, where $\nu = \sup_{\pi<\rho} \{b\}_\kappa^F(\pi, \boldsymbol{\mu})$.

To see that κ is F-recursively regular, we must show (b). But it has been proved in the above.

The implications from the left to the right in (i) and (ii) are clear since the relation $\{a\}_\kappa^F(\boldsymbol{\mu}) \simeq \nu$ is Σ_1 on $\mathscr{M}_F(\kappa)$. Q.E.D.

LEMMA 1.7. *$\omega_1[F]$ is the first F-recursively regular ordinal larger than ω, and for every $P \subset \omega^n$:*

(i) *P is F-recursive if and only if P is Δ_1 on $\mathscr{M}_F(\omega_1[F])$:*

(ii) *P is F-semirecursive if and only if it is Σ_1 on $\mathscr{M}_F(\omega_1[F])$.*

Proof. By a simple application of the ordinary recursion theorem, we have a primitive recursive function f such that

(c) $\{f(a)\}_\kappa^F(\boldsymbol{m}) \simeq \{a\}^F(\boldsymbol{m})$

for all F-recursively regular ordinal $\kappa > \omega$. Since $\omega_1[F] = \sup\{|u|^F : u \in U^F\}$, all the computations in $\omega_1[F]$ can be coded by elements of U^F. That is, there exists a primitive recursive function g such that

(d) $\{a\}_{\omega_1[F]}^F(|u_1|^F, \cdots, |u_k|^F) \simeq |\{g(a)\}^F(u_1, \cdots, u_k)|^F$

for all $a \in \omega$ and all $u_1, \cdots, u_k \in U^F$. The proof of this assertion is same as that of VIII. 4.2 in Hinman [4] except for the case where $\{a\}_{\omega_1[F]}^F(\boldsymbol{\mu}) \simeq F(\lambda n\{b\}_{\omega_1[F[}^F(n, \boldsymbol{\mu}))$. So we consider here only this new case. g is defined by induction on the length of the computation of $\{a\}_{\omega_1[F]}^F(\boldsymbol{\mu})$. We assume that $g(b)$ is already defined and satisfies:

$$\{b\}_{\omega_1[F]}^F(|v|^F, |\boldsymbol{u}|^F) \simeq |\{g(b)\}^F(v, \boldsymbol{u})|^F$$

for all $\boldsymbol{u}$, $v \in U^F$. Also assume that $\{a\}_{\omega_1[F]}^F(\boldsymbol{\mu}) \simeq F(\lambda n\{b\}_{\omega_1[F]}^F(n, \boldsymbol{\mu},))$. Let $\alpha: \omega \to U^F$ be a primitive recursive function such that $|\alpha(n)|^F = n$ for all n. By Lemma 1.4, the relation $|u|^F = n$ is F-recursive, and hence the function β defined by

$$\beta(u) = \begin{cases} n & \text{if } |u|^F = n \\ \uparrow & \text{if } |u|^F > \omega \end{cases}$$

is partial F-recursive. Let γ be a primitive recursive function such that

$$\{\gamma(w)\}^F(\boldsymbol{u}) \simeq F(\lambda n\beta(\{w\}^F(\alpha(n), \boldsymbol{u}))).$$

Such a γ exists by virtue of Lemma 1.3. We set $g(a) = \gamma(g(b))$, where $b = (a)_2$.

Using (d), it is seen that $\omega_1[F]$ is an F-recursively regular ordinal (cf. Hinman [4; VIII. 4.4]). From (c) and (d), we have that for all $P \subset \omega^n$,

(e) P is F-semirecursive iff it is $\omega_1[F]$-semirecursive in F.

and thus

(f) P is F-recursive iff it is $\omega_1[F]$-recursive in F.

Every function defined on $\omega_1[F]$ and with constant value $< \omega_1[F]$ is $\omega_1[F]$-recursive: let $\prec \subset \omega \times \omega$ be a well-ordering on ω which is F-recursive. Then the function h defined by

$$h(n, \mu) = \nu \text{ iff } n \text{ is the } \nu\text{-th number in the order } \prec$$

is $\omega_1[F]$-recursive. If ρ is the length of $\prec$, then $\rho = \sup \{h(n, \mu): n \in \omega\}$. Thus the function with constant value ρ is $\omega_1[F]$-recursive.

In view of (e), (f) and Lemma 1.6, we have (i) and (ii).

Let κ be an arbitrary F-recursively regular ordinal $> \omega$, and $\prec \subset \omega \times \omega$ be an F-recursive well-ordering. Then, by (c), $\prec$ is κ-recursive in F, and so $\prec \in \mathscr{M}_F(\kappa)$ by Lemma 1.6. Hence the order type of $\prec$ is less than κ. Thus $\omega_1[F] \leqq \kappa$. Q.E.D.

For any set $A \subset \omega$, we use $L_F(\sigma, A)$, $\mathscr{M}_F(\sigma, A)$ and $\omega_1[F,A]$ instead of $L_{\langle F,A\rangle}(\sigma)$, $\mathscr{M}_{\langle F,A\rangle}(\sigma)$ and $\omega_1[\langle F, A\rangle]$, respectively. Relativizing the above lemma to A, we have the following corollary.

Corollary 1.8. *For every set $B \subset \omega$, B is F-recursive in A if and only if $B \in L_F(\omega_1[F, A], A)$.*

The *superjump* $S(F)$ of F is a type 2 functional defined by

$$S(F)(e, \alpha) = \begin{cases} 0 & \text{if } \{e\}^F(\alpha)\downarrow , \\ 1 & \text{otherwise} . \end{cases}$$

Let e be an index such that

$$\{e\}^F(n, \alpha) = \begin{cases} 0 & \text{if } F(\alpha) = n , \\ \uparrow & \text{otherwise} . \end{cases}$$

Then $F(\alpha) = \mu n[S(F)(S^1(e, n), \alpha) = 0]$, and thus F is $S(F)$-recursive uniformly for F. An ordinal κ is said to be *F-recursively inaccessible* if κ is F-admissible and is the limit of F-admissible ordinals $< \kappa$. Recall that $\tau_k[F]$ is the κ-th F-admissible ordinal.

Lemma 1.9. *An ordinal κ is F-recursively inaccessible if and only if $\tau_\kappa[F] = \kappa$.*

Proof. Let κ be an F-recursively inaccessible ordinal. For each $\nu < \kappa$ such that $\tau_\nu[F] < \kappa$, let $f(\nu) = \tau_\nu[F]$. As easily seen, f is Σ_1 on $\mathscr{M}_F(\kappa)$. Suppose that $\kappa < \tau_\kappa[F]$, then domain $(f) < \kappa$. This implies that range $(f) \in L_F(\kappa)$

by Σ Replacement in $\mathscr{M}_F(\kappa)$. Hence there exists an F-admissible ordinal $\sigma < \kappa$ such that $\bigcup \text{range}(f) < \sigma$. This is a contradiction, and we have $\kappa = \tau_k[F]$.

The converse implication is trivial. Q.E.D.

LEMMA 1.10. *$\omega_1[S(F)]$ is the first F-recursively inaccessible ordinal.*

Proof. Put $\kappa = \omega_1[S(F)]$. Since F is recursive in $S(F)$, by a simple application of the Recursion Theorem, we have a primitive recursive function f such that

$$\{f(a)\}^{S(F)}_{\infty\kappa}(\boldsymbol{\mu}) \simeq \{a\}^F_{\infty\kappa}(\boldsymbol{\mu}) .$$

Hence κ is closed under all partial functions (∞, κ)-recursive in F. Thus κ is an F-admissible ordinal. By VIII. 4.12 in [4], κ is the limit of F-admissible ordinals $< \kappa$. Let ρ be an arbitrary F-recursively inaccessible ordinal. We want to show $\kappa \leqq \rho$. Using the Recursion Theorem, we can find a primitive recursive function g such that

$$\{g(a, e)\}^F_\rho(\boldsymbol{m}, \boldsymbol{\mu}) \simeq \{a\}^F(\boldsymbol{m}, \lambda n\{e\}^F_\rho(n, \boldsymbol{\mu})) .$$

The existence proof of g is quite similar to the proof of the Substitution Theorem (cf. Hinman [4; VI. § 2]), so we consider only the following case as an example:

$$\{a\}^F(\boldsymbol{m}, \alpha) \simeq F(\lambda j\{b\}^F(j, \boldsymbol{m}, \alpha)) .$$

Assume that $g(b, e)$ is already defined and satisfies

$$\{g(b, e)\}^F_\rho(j, \boldsymbol{m}, \boldsymbol{\mu}) \simeq \{b\}^F(j, \boldsymbol{m}, \lambda n\{e\}^F_\rho(n, \boldsymbol{\mu})) .$$

Since the predicate $\nu < \omega$ is ρ-recursive in F, there exists an index d such that

$$\{d\}^F_\rho(\boldsymbol{m}, \boldsymbol{\mu}) \simeq \begin{cases} F(\lambda j\{g(b, e)\}^F_\rho(j, \boldsymbol{m}, \boldsymbol{\mu})) & \text{if } \lambda n\{e\}^F_\rho(n, \boldsymbol{\mu}) \text{ is a total function from } \omega \text{ to } \omega \\ \uparrow \ \text{otherwise} . \end{cases}$$

Such a d may be computed from a, e and an index for g. Thus we let $g(a, e)$ be such an index d.

Now we claim that there is a primitive recursive function h such that

$$\{h(a, e)\}^F_\rho(\boldsymbol{\mu}) \simeq S(F)(a, \lambda n\{e\}^F_\rho(n, \boldsymbol{\mu})) .$$

From this, by using the Recursion Theorem, we have a primitive recursive function k which satisfies:

$$\{k(a)\}^F_\rho(\boldsymbol{m}) \simeq \{a\}^{S(F)}(\boldsymbol{m}) .$$

Therefore, as in the proof of Lemma 1.7, we see that $\omega_1[S(F)] \leqq \rho$.

We return to the proof of our claim. Recall that ρ is F-recursively inaccessible. For each $\boldsymbol{\mu} < \rho$, we let $\pi(\boldsymbol{\mu})$ be the least F-admissible ordinal larger than $\max(\omega, \boldsymbol{\mu})$. Then π is ρ-recursive in F, and

$$\begin{aligned} S(F)(a, \lambda n\{e\}^F_\rho(n, \boldsymbol{\mu})) = 0 & \\ \longleftrightarrow \{a\}^F(\lambda n\{e\}^F_g(n, \boldsymbol{\mu}))\downarrow & \\ \longleftrightarrow \{g(a, e)\}^F_\rho(\boldsymbol{\mu})\downarrow & \\ \longleftrightarrow \{g(a, e)\}^F_{\pi(\boldsymbol{\mu})}(\boldsymbol{\mu})\downarrow & . \end{aligned}$$

The last clause can be written by

$$\exists \xi < \pi(\boldsymbol{\mu}) R(g(a, e), \boldsymbol{\mu}, \xi)$$

where R is a relation $(\infty, 0)$-recursive in F. This is a generalization of the usual Enumeration Theorem (for the proof we may refer to Hinman [4; VIII. 2.6]). Let c be an index such that

$$\{c\}^F_\rho(z, \boldsymbol{\mu}) = \begin{cases} 0 & \text{if } \min\{\xi < \pi(\boldsymbol{\mu}) : R(z, \boldsymbol{\mu}, \xi)\} < \pi(\boldsymbol{\mu}) , \\ 1 & \text{otherwise.} \end{cases}$$

And let $h(a. e) = S^1(c, g(a, e))$. Then it is easy to see that this h has the desired property. Q.E.D.

§ 2.

In this section, we define F-degrees and the F-jump, which are generalizations of hyperdegrees and the hyperjump. We shall extend Shoenfield's notation system for $\omega_1[F]$ to that for $\omega_1[S(F)]$, and generalize a result of Richter [7].

DEFINITION 2.1. For any $A, B, \subset \omega$, $A \leqslant_F B$ means that A is F-recursive in B. This is a reflexive and transitive relation. Thus we can consider F-degrees. That is, A and B have the same F-degrees if $A \leqslant_F B$ and $B \leqslant_F A$, which we denote by $A \equiv_F B$. We use $\deg_F(A)$ to denote the F-degree of A.

We use $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \cdots$ as variables for F-degrees. $\boldsymbol{a}|\boldsymbol{b}$, $\boldsymbol{a} \leqslant \boldsymbol{b}$ and $\boldsymbol{a} < \boldsymbol{b}$

are defined as for Turing degrees.

DEFINITION 2.2. $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. When $\boldsymbol{a} = \deg_F(A)$ and $\boldsymbol{b} = \deg_F(B)$, we denote the F-degree of $A \oplus B$ by $\boldsymbol{a} \cup \boldsymbol{b}$.
$\boldsymbol{a} \cup \boldsymbol{b}$ is the least upper bound of $\boldsymbol{a}$ and $\boldsymbol{b}$.

DEFINITION 2.3. The F-jump A' of A is defined by

$$A' = \{e \in \omega : S(F)(e, K_A) = 0\} .$$

If $\boldsymbol{a} = \deg_F(A)$, we denote the F-degree of A' by $\boldsymbol{a}'$.
From the following lemma, the above definition is well-defined.

LEMMA 2.4. *If $A \leqslant_F B$, then $A' \leqslant_F B'$. Moreover, A' is many-one reducible to B'.*

Proof. Let e be an index such that $K_A(n) = \{e\}^F(n, K_B)$, and f be a primitive recursive function such that

$$\{f(a)\}^F(K_B) \simeq \{a\}^F(\lambda n\{e\}^F(n, K_B)) .$$

Then,

$$a \in A' \longleftrightarrow f(a) \in B .$$

Thus A' is many-one reducible to B'. Q.E.D.

THEOREM 2.5. *For any F-degree $\boldsymbol{a}$, $\boldsymbol{a} < \boldsymbol{a}'$.*

Proof. Let A be such that $\boldsymbol{a} = \deg_F(A)$, and e be an index such that

$$\{e\}^F(a, \alpha) \simeq \begin{cases} 0 & \text{if } \alpha(a) = 0 , \\ \uparrow & \text{otherwise} , \end{cases}$$

Now we have

$$a \in A \longleftrightarrow \{e\}^F(a, K_A)\downarrow \longleftrightarrow \{S^1(e, a)\}^F(K_A)\downarrow \longleftrightarrow S^1(e, a) \in A' .$$

Hence A is many-one reducible to A'. Suppose that $A' \leqq_F A$. We let:

$$\phi(a) \simeq \begin{cases} 0 & \text{if } S^1(a, a) \notin A' , \\ \uparrow & \text{otherwise} . \end{cases}$$

Then ϕ is partial F-recursive in A. Take an index d such that $\{d\}^F(a, K_A) \simeq \phi(a)$, then

$$\phi(d)\downarrow \longleftrightarrow S^1(d, d) \notin A' \longleftrightarrow \{d\}^F(d, K_A)\uparrow \longleftrightarrow \phi(d)\uparrow .$$

This is a contradiciton. Thus $A' \not\leq_F A$. Q.E.D.

Clearly A' is F-semirecursive in A. If a predicate $P(a)$ is F-semirecursive in A, then there is an index e such that

$$P(a) \longleftrightarrow \{e\}^F(a, K_A) \downarrow \longleftrightarrow S^1(e, a) \in A' .$$

Therefore P is many-one reducible to A'. Consequently A' is a complete F-semirecursive-in-A set. We use $\mathbf{0}$ to denote the F-degree of F-recursive sets. Then $\mathbf{0}' = \deg_F(U^F)$.

THEOREM 2.6. *If A is F-semirecursive, then the F-degree of A is $\mathbf{0}$ or $\mathbf{0}'$.*

Proof. Let f be a recursive function such that

$$a \in A \longleftrightarrow f(a) \in U^F .$$

If A is not F-recursive, then $\sup\{|f(a)|^F\colon a \in A\} = \omega_1[F]$ by the Hierarchy Theorem (Hinman [4; VI. 4.11]). From this, we have

$$u \in U^F \longleftrightarrow \exists a(a \in A \text{ and } |u|^F \leqq |f(a)|^F) .$$

By lemma 1.4, U^F is F-recursive in A. Q.E.D.

THEOREM 2.7. *For every $A \subset \omega$, $\mathbf{0}' \leqslant \deg_F(A)$ if and only if $\omega_1[F] < \omega_1[F, A]$.*

Proof. Suppose $\mathbf{0}' \leqslant \deg_F(A)$. Let $\prec \subset \omega \times \omega$ be a well-ordering of order type $\omega_1[F]$ which is F-semirecursive. Then $\prec$ is F-recursive in A. Hence $\omega_1[F] < \omega_1[F, A]$. Conversely, suppose that $\omega_1[F] < \omega_1[F, A]$. Then there exists a $v \in U^{F,A}$ such that

$$|u|^F < |v|^{F,A}$$

for all $u \in U^F$. From this we have

$$u \in U^F \longleftrightarrow |u|^F < |v|^{F,A} .$$

By Lemma 1.4, U^F is F-recursive in A. Thus $\mathbf{0}' \leqslant \deg_F(A)$. Q.E.D.

COROLLARY 2.8. *If $\boldsymbol{a} < \mathbf{0}'$, then $\boldsymbol{a}' = \mathbf{0}'$.*

Proof. By lemma 2.4, we see that $\mathbf{0}' \leqslant \boldsymbol{a}'$. Take a set $A \subset \omega$ such that $\boldsymbol{a} = \deg_F(A)$. Then $\omega_1[F, A] = \omega_1[F]$ by Theorem 2.7, so $L_F(\omega_1[F, A], A) \in L_F(\omega_1[F, U^F], U^F)$. Since A' is Σ_1 on $L_F(\omega_1[F, A], A)$, we have that $A' \in L_F(\omega_1[F, U^F], U^F)$. Thus $A' \leqslant_F U^F$ by Corollary 1.8. Q.E.D.

For any normal type 2 object F, Shoenfield defined a notation system O^F for $\omega_1[F]$ and a hierarchy $\{H_a^F: a \in O^F\}$ for F-recursive functions (see Shoenfield [10]). We shall devote the remaining part of this section to extending this system to that for $\omega_1[S(F)]$.

Definition 2.9. For each ordinal, σ, N_σ^F and $H^F(a)$ are defined by induction on σ. (Till further notice, the superscript F will be deleted throughout this section.) If $a \in N_\sigma$, we let $|a| = \sigma$. Let $C_\sigma = \cup \{N_\tau: \tau < \sigma\}$.

Stage 0. $N_0 = \{1\}$. $H(1) = \omega$.

Stage $\sigma + 1$. $N_{\sigma+1} = \{2^a: a \in N_\sigma\}$. $H(2^a) = \{x \in \omega: \lambda n\{(x)_0\}^{H(a)}(n)$ is total and $F(\lambda n\{(x)_0\}^{H(a)}(n)) = (x)_1\}$.

Stage λ(limit). Assume that N_σ is defined for all $\sigma < \lambda$.

Case 1. There is an ordinal $\sigma < \lambda$ such that for some $a \in N_\sigma$ and $e \in \omega$,

(i) $\lambda n\{e\}^{H(a)}(n)$ is total;

(ii) for all n, $\{e\}^{H(a)}(n) \in C_\lambda$;

(iii) $\{e\}^{H(a)}(0) = a$ and $|\{e\}^{H(a)}(n)| < |\{e\}^{H(a)}(n+1)|$ for all n;

(iv) $\lambda = \sup\{|\{e\}^{H(a)}(n)|: n \in \omega\}$.

Then, $N_\lambda = \{3^a \cdot 5^e: \exists \sigma < \lambda\ [a \in N_\sigma\ \&\ a$ and e satisfy the above conditions (i)—(iv)]$\}$. $H(3^a \cdot 5^e) = \{x \in \omega: (x)_1 \in H(\{e\}^{H(a)}((x)_0))\}$.

Case 2. Case 1 does not hold, but there is an ordinal $\sigma < \lambda$ such that for some $a \in N_\sigma$, $7^a \notin C_\lambda$. Let $\sigma < \lambda$ be the least ordinal such that $7^a \notin C_\lambda$ for some $a \in N_\sigma$. Then, $N_\lambda = \{7^a: a \in N_\sigma\}$, and $H(7^a) = \{x \in \omega: (x)_0 \in C_\lambda\ \&\ (x)_1 \in H((x)_0)\}$.

Remark 2.10. (a) Except for Case 2 of Stage λ, the definition is analogous to that of Shoenfield [10]. We have avoided defining $<_o$. But this change is not essential as noted in Platek [6].

(b) $C_{|7|}$ is the set of notations for ordinals $\omega_1[F]$ defined in Platek [6; p. 260].

(c) $N_\sigma \cap N_\tau = 0$ if $\sigma \neq \tau$.

(d) The function $\sigma \mapsto \{x \in \omega: (x)_0 \in N_\sigma\ \&\ (x)_1 \in H((x)_0)\}$ is Σ_1 definable in $KP(F)$ + the Axiom of Infinity.

Definition 2.11. For any ordinal σ:

$$\mathcal{O}_0 = \{\bar{n}: n \in \omega\}, \quad \text{where } \bar{0} = 1 \text{ and } \overline{n+1} = 2^{\bar{n}}, \quad H_0 = \omega;$$

$$\mathcal{O}_\sigma = \bigcup \{C_{|7^a|}: |a| < \sigma\}, \quad H_\sigma = \{x \in \omega: (x)_0 \in \mathcal{O}_\sigma\ \&\ (x)_1 \in H((x)_0)\} \quad \text{for } \sigma > 0.$$

Let $\mathcal{O} = \bigcup\{\mathcal{O}_\sigma : \sigma$ is an ordinal$\}$. We let $|\mathcal{O}_\sigma| = \sup\{|a| : a \in \mathcal{O}_\sigma\}$, and $|\mathcal{O}| = \sup\{|a| : a \in \mathcal{O}\}$.

LEMMA 2.12 (Uniqueness Theorem). *There exists primitive recursive functions f and g such that*:

(i) *if $b \in \mathcal{O}$, then $\lambda x\{f(b)\}^{H(b^+)}(x)$ is the representing function of the set $\{a \in \mathcal{O} : |a| < |b|\}$, where $b^+ = 2^b$;*

(ii) *if $a, b \in \mathcal{O}$ and $|a| \leqq |b|$, then $H(a)$ is recursive in $H(b)$ with index $g(a, b)$.*

Proof. The functions f and g are defined by the Recursion Theorem over $\mathcal{O}$ as in Shoenfield [10]. All cases not involving a notation of the form 7^d can be treated as in [10] and we consider here only the new cases.

Case 1. $a = 7^d$ and b is not of the form 7^e: exactly as in Shoenfield [10].

Case 2. a is not of the form 7^d and $b = 7^e$:

(i) $|a| < |b| \leftrightarrow \exists x(\langle a, x\rangle \in H(b))$. By Shoenfield [10; (2), p. 104], the right hand side of the equivalence is recursive in $H(b^+)$.

(ii) In this case, $|a| < |b|$. Hence $x \in H(a) \leftrightarrow \langle a, x\rangle \in H(b)$, so $H(a)$ is recursive in $H(b)$.

Case 3. $a = 7^d$ and $b = 7^e$:

(i) As in Case 2.

(ii) Note that $|a| < |b|$ iff $|d| < |e|$ and that $|a| = |b|$ iff $|d| = |e|$. Since $|e^+| < |b|$, $H(e^+)$ is computable from $H(b)$ as in Case 2, so by the induction hypothesis it can be checked from $H(b)$ whether $|d| < |e|$ or $|d| = |e|$. If $|d| < |e|$, then

$$x \in H(a) \longleftrightarrow \langle a, x\rangle \in H(b)\ ,$$

and if $|d| = |e|$, then $H(a) = H(b)$. Thus $H(a)$ is computable from $H(b)$.

Q.E.D.

LEMMA 2.13. *There exists a primitive recursive function $a \oplus b$ such that for any $a, b \in \mathcal{O}$*:

(i) $a \oplus b \in \mathcal{O}$;

(ii) $|a \oplus b| \geqq \max\{|a|, |b|\}$;

(iii) $b \neq 1 \rightarrow |a| < |a \oplus b|$.

Moreover for every $\sigma < |\mathcal{O}|$:

(iv) $a, b \in \mathcal{O}_{\sigma+1} \rightarrow a \oplus b \in \mathcal{O}_{\sigma+1}$.

Proof. We define $a \oplus b$ by recursion on $b \in \mathcal{O}$ as follows:

(1) $a \oplus 1 = a$.

(2) $a \oplus 2^b = 2^{a \oplus b}$.

(3) $a \oplus 3^b \cdot 5^e = 3^{a \oplus b} \cdot 5^{\theta(p,a,b,e)}$, where p is an index of $\oplus$ and θ is a primitive recursive function such that

$$\{\theta(p, a, b, e)\}^{H(a \oplus b)}(n) \simeq a \oplus \{e\}^{H(b)}(n) .$$

Such a θ exists by Lemma 1.1 and Lemma 2.11.

(4) $a \oplus 7^b = 3^{(7^b)^+} \cdot 5^{\pi(a,b)}$, where π is a primitive recursive function such that

$$\{\pi(a, b)\}^{H((7^b)^+)}(n) \simeq \begin{cases} 7^b \oplus \overline{n+1} & \text{if } |a| < |7^b| , \\ 7^b \oplus 2 & \text{if } |a| \geqq |7^b| \text{ and } n = 0 , \\ a \oplus \overline{n+1} & \text{if } |a| \geqq |7^b| \text{ and } n > 0 . \end{cases}$$

Such a π exists by Lemma 2.11.

(5) In the case where b is not of the form 1, $2^{(b)_0}$, $3^{(b)_1} \cdot 5^{(b)_2}$ or $7^{(b)_3}$, we set $a \oplus b = 0$.

(i)—(iv) are easily proved by induction on $b \in \mathcal{O}$. Q.E.D.

COROLLARY 2.14. *There exists a primitive recursive function β such that for any $\sigma < |\mathcal{O}|$, $a \in \mathcal{O}_{\sigma+1}$ and any $e \in \omega$, if $\{e\}^{H(a)}$ is a total function from ω into $\mathcal{O}_{\sigma+1}$, then $\beta(a, e) \in \mathcal{O}_{\sigma+1}$ and $|\{e\}^{H(a)}(n)| < |\beta(a, e)|$ for all $n \in \omega$.*

Proof. Let d be an index of the partial function ϕ recursive in $H(a)$ defined by:

$$\phi(a, e, 0) \simeq a, \text{ and } \phi(a, e, n+1) \simeq \phi(a, e, n) \oplus (\{e\}^{H(a)}(n) \oplus 2) .$$

We let $\beta(a, e) = 3^a \cdot 5^{S^2(d,a,e)}$. It is easy to see that β has the desired properties. Q.E.D.

LEMMA 2.15. *There exists a primitive recursive function h such that for any $\sigma < |\mathcal{O}|$ and any $a \in \mathcal{O}_{\sigma+1}$, $\lambda x\{h(a)\}^F(x, H_\sigma)$ is the representing function of $H(a)$.*[2]

Proof. h is defined by recursion on $\mathcal{O}_{\sigma+1}$. Except for the case where $a = 7^b$ for some b, the definition is same as Shoenfield [10]. If $a = 7^b$, then $a \in \mathcal{O}_\sigma$ or $H(a) = H_\sigma$. If $a \in \mathcal{O}_\sigma$, then $H(a) = \{x: \langle a, x\rangle \in H_\sigma\}$. Therefore, $H(a)$ is recursive in $\mathcal{O}_\sigma$ and H_σ. $\mathcal{O}_\sigma$ is F-recursive in H_σ since $\mathcal{O}_\sigma^* = \{x: \exists y\langle x, y\rangle$

2) We identify a set with its representing function.

$\in H_\sigma\}$ and F is normal. Thus $H(a)$ is F-recursive in H_σ. Q.E.D.

Remark 2.16. We can take the above h such that

$$a \in \mathcal{O}_{\sigma+1} \longleftrightarrow \lambda x\{h(a)\}^F(x) \text{ is total}.$$

Thus $\mathcal{O}_{\sigma+1}$ is F-semirecursive in H_σ.

LEMMA 2.17. *For each $\sigma < |\mathcal{O}|$, there exists a primitive recursive function θ_σ and a partial recursive function χ_σ such that if $\{z\}^F(a_1, \cdots, a_n, H_\sigma)\downarrow$, then*

(i) $\theta_\sigma(z, \langle a_1, \cdots, a_n\rangle) \in \mathcal{O}_{\sigma+1}$

(ii) $\{z\}^F(a_1, \cdots, a_n, H_\sigma) = \chi_\sigma(z, \langle a_1, \cdots, a_n\rangle, H(\theta_\sigma(z, \langle a_1, \cdots, a_n\rangle)))$.

Proof. Let η be the representing function of H_σ. Except for the case where $\{z\}^F(a_1, \cdots, a_n, \eta) = \eta(a_1)$, θ_σ and χ_σ are defined as in Shoenfield [10] and we consider only this new case. If σ is a successor ordinal, then there exists a, $b \in \mathcal{O}$ such that $H_\sigma = H(7^b)$. We let $\theta_\sigma(z, \langle a_1, \cdots, a_n\rangle) = 7^b$ and $\chi_\sigma(z, \langle a_1, \cdots, a_n\rangle, \alpha) = \alpha(a_1)$. If $\sigma = |3^b \cdot 5^d|$ for some b and d, then we let e be an index such that

$$\{e\}^{H(b)}(0) = b \text{ and } \{e\}^{H(b)}(n+1) = 7^{\{d\}^{H(b)}(n)}.$$

Then it can be seen that $3^b \cdot 5^e \in \mathcal{O}_{\sigma+1}$ and $x \in \mathcal{O}_\sigma$ iff $|x| < |3^b \cdot 5^e|$. We let $\theta_\sigma(z, \langle a_1, \cdots, a_n\rangle) = (3^b \cdot 5^e)^+$. By Lemma 2.12, there exists a partial recursive function ϕ such that $\lambda x\phi(x, H((3^b \cdot 5^e)^+))$ is the representing function of H_σ. Let $\chi_\sigma(z, \langle a_1, \cdots, a_n\rangle, \alpha) \simeq \phi(a_1, \alpha)$. In the case where $\sigma = |7^b|$ for some b, by Lemma 2.12, we see that the set $\{x : |x| < \sigma\}$ is recursive in $H((7^b)^+)$. Therefore there exists an index e such that if we put $a = (7^b)^+$, then $3^a \cdot 5^e \in \mathcal{O}_{\sigma+1}$ and $\mathcal{O}_\sigma = \{x : |x| < |3^a \cdot 5^e|\}$. So we let $\theta_\sigma(z, \langle a_1, \cdots, a_n\rangle) = 3^a \cdot 5^e$ and $\chi_\sigma(z, \langle a_1, \cdots, a_n\rangle, \alpha) \simeq \phi(a_1, \alpha)$, where ϕ is a partial recursive function mentioned above. Q.E.D.

Remark 2.18. Examining the above proof, we see that if $\theta_\sigma(z, \langle a_1, \cdots, a_n\rangle) \in \mathcal{O}_{\sigma+1}$, then $\{z\}^F(a_1, \cdots, a_n, H_\sigma)\downarrow$. From this and Remark 2.16, $\mathcal{O}_{\sigma+1}$ is a complete set F-semirecursive in H_σ. Thus $\deg_F(\mathcal{O}_{\sigma+1}) = (\deg_F(H_\sigma))'$.

DEFINITION 2.19. For each ordinal $\sigma < |\mathcal{O}|$, we denote the F-degree of H_σ by $\mathbf{0}^{(\sigma)}$.

LEMMA 2.20. *For each $\sigma < |\mathcal{O}|$, $\mathbf{0}^{(\sigma+1)} = (\mathbf{0}^{(\sigma)})'$, and $|\mathcal{O}_{\sigma+1}| = \omega_1[F, H_\sigma]$.*

Proof. Since $\mathcal{O}_{\sigma+1} = \{a : \exists x\langle a, x\rangle \in H_{\sigma+1}\}$, it holds that $\mathcal{O}_{\sigma+1}$ is F-recursive in $H_{\sigma+1}$. By Definition 2.11,

$$x \in H_{\sigma+1} \longleftrightarrow (x)_0 \in \mathcal{O}_{\sigma+1} \ \& \ (x)_1 \in H((x)_0) .$$

If $(x)_0 \in \mathcal{O}_{\sigma+1}$, then

$$(x)_1 \in H((x)_0) \longleftrightarrow \{h((x)_0)\}^F((x)_1, H_\sigma) = 0 ,$$

where h is as in Lemma 2.15. Therefore $H_{\sigma+1}$ is F-semirecursive in H_σ, so $H_{\sigma+1} \leqq {}_F\mathcal{O}_{\sigma+1}$.

If $a \in \mathcal{O}_{\sigma+1}$, then the relation $\{\langle x, y\rangle : |x| < |y| < |a|\}$ is a prewellordering F-recursive in H_σ of order type $|a|$ by 2.12 and 2.15. Therefore we have that $|a| < \omega_1[F, H_\sigma]$, and $|\mathcal{O}_{\sigma+1}| \leqq \omega_1[F, H_\sigma]$. Conversely, let $\prec \subset \omega \times \omega$ be an arbitrary well-ordering F-recursive in H_σ. Then, by Lemma 2.17, $\prec$ is recursive in $H(a)$ for some $a \in \mathcal{O}_{\sigma+1}$, and hence the order type of $\prec$ is less than $\omega_1[H(a)]$, where $\omega_1[H(a)]$ is the first non-$H(a)$-recursive ordinal. Let $O^{H(a)}$ be the Church-Kleene notation system relativized to $H(a)$. It is easy to obtain a recursive function f such that if $x \in O^{H(a)}$, then $f(x) \in \mathcal{O}_{\sigma+1}$ and $|x|_O^{H(a)} \leqq |f(x)|$. Therefore, $\omega_1[H(a)] \leqq |\mathcal{O}_{\sigma+1}|$, so the order type of $\prec$ is less than $|\mathcal{O}_{\sigma+1}|$ and we have that $\omega_1[F, H_\sigma] \leqq |\mathcal{O}_{\sigma+1}|$. Thus $\omega_1[F, H_\sigma] = |\mathcal{O}_{\sigma+1}|$.

Q.E.D.

THEOREM 2.21. *$|\mathcal{O}| \leqq \omega_1[S(F)]$ and for any $\sigma < |\mathcal{O}|$;*

(i) $\mathbf{0}^{(0)} = \mathbf{0}$ and $\mathbf{0}^{(\sigma+1)} = (\mathbf{0}^{(\sigma)})'$;

(ii) $\mathbf{0}^{(\sigma)} = \sup\{\mathbf{0}^{(\nu)} : \nu < \sigma\}$ if σ is a limit ordinal.

Proof. By induction on $\sigma < |\mathcal{O}|$, we first prove that $|\mathcal{O}_{\sigma+1}| = \tau_{\sigma+1}[F]$ if $\sigma < \omega$ and $|\mathcal{O}_{\sigma+1}| = \tau_\sigma[F]$ if $\sigma \geqq \omega$, that $|\mathcal{O}_{\sigma+1}| < \omega_1[S(F)]$, and that $|\mathcal{O}_\sigma| = \sup\{\tau_\nu[F] : \nu < \sigma\}$ if σ is a limit ordinal.

Case 1. $\sigma < \omega$: it is clear if $\sigma = 0$. By the induction hypothesis, $H_\sigma = \{x \in \omega : |(x)_0| < \tau_\sigma[F]$ and $(x)_1 \in H((x)_0)\}$ if $\sigma > 0$, and $H_0 = \omega$. Hence $H_\sigma \in L_F(\tau_{\sigma+1}[F])$ from Remark 2.10. (d). By Lemma 2.20, $|\mathcal{O}_{\sigma+1}| = \omega_1[F, H_\sigma] = \tau_{\sigma+1}[F] < \omega_1[S(F)]$.

Case 2. σ is a successor ordinal $\geqq \omega$: exactly as Case 1.

Case 3. $\sigma = |3^a \cdot 5^e|$ for some a and e: let $\xi(n) = |\{e\}^{H(a)}(n)|$. Then $|\mathcal{O}_\sigma| = \sup\{\tau_{\xi(n)}[F] : n \in \omega_1\} \leqq \omega_1[S(F)]$ by the induction hypothesis. Since $|a| < \omega_1[S(F)]$, we have that $H(a) \in L_F(\omega_1[S(F)])$ by Remark 2.10(d). Hence the function $n \mapsto \tau_{\xi(n)}[F]$ is Δ_1 on $\mathcal{M}_F(\omega_1[S(F)]$, so $|\mathcal{O}_\sigma| < \omega_1[S(F)]$. By Lemma 1.10, $|\mathcal{O}_\sigma|$ is not F-recursively inaccessible, and therefore $|\mathcal{O}_\sigma| < \tau_\sigma[F]$. By Remark 2.10(d), $H_\sigma \in L(\tau_\sigma[F])$, so $|\mathcal{O}_{\sigma+1}| = \tau_\sigma[F] < \omega_1[S(F)]$ as in Case 1.

Case 4. $\sigma = |7^b|$ for some $b \in \mathcal{O}$: put $a = (7^b)^+$, then by Lemma 2.12, the set $\{x: |x| < \sigma\}$ is recursive in $H(a)$. Hence there exists a function f recursive in $H(a)$ such that $|f(n)| < \sigma$ for all $n \in \omega$ and $\sigma = \sup_{n<\omega} f(n)$. Then $|\mathcal{O}_\sigma| = \sup\{\tau_{|f(n)|}[F]: n \in \omega\}$ by the induction hypothesis. The rest is as in Case 3.

(i) is clear from Lemma 2.20.

(ii) Let σ be a limit ordinal $< |\mathcal{O}|$. If $H_\nu \leqslant_F A$ for all $\nu < \sigma$, then $\tau_\sigma[F] \leqq \omega_1[F, A]$, so $H_\sigma \in L_F(\omega_1[F, A], A)$. Thus $H_\sigma \leqslant_F A$. Q.E.D.

Theorem 2.22. *$|\mathcal{O}| = \omega_1[S(F)]$ and for any $A \subset \omega^n$, A is $S(F)$-recursive if and only if A is recursive in $H(a)$ for some $a \in \mathcal{O}$.*

Proof. If, $\sigma < |\mathcal{O}|$, then by Remark 2.10 and Lemma 2.21, $H_\sigma \in L_F(\omega_1[S(F)])$. Hence each $H(a)$ with $a \in \mathcal{O}$ is $S(F)$-recursive. Thus if A is recursive in $H(a)$ for some $a \in \mathcal{O}$, then A is $S(F)$-recursive.

In order to show the converse implication, we define a primitive recursive function θ and a partial recursive function χ such that if $\{z\}^{S(F)}(a_1, \cdots, a_n)\downarrow$, then

(a) $\theta(z, \langle a_1, \cdots, a_n\rangle) \in \mathcal{O}$;

(b) $\chi(z, \langle a_1, \cdots, a_n\rangle, H(\theta(z, \langle a_1, \cdots, a_n\rangle)) = \{z\}^{S(F)}(a_1, \cdots, a_n)$.

We define these functions by the Recursion Theorem. We consider only the case where $\{z\}^{S(F)}(a_1, \cdots, a_n) = S(F)(a_1, \lambda m\{w\}^{S(F)}(m, a_1, \cdots, a_n))$. Other cases can be treated as in Shoenfield [10]. Note that the function β defined in the proof of Corollary 2.14 has the following property: if $a \in \mathcal{O}$ and $\{e\}^{H(a)}$ is a total function from ω into $\mathcal{O}$, then $\beta(a. e) \in \mathcal{O}$ and $|\{e\}^{H(a)}(n)| < |\beta(a. e)|$ for all $n < \omega$. By the induction hypothesis and the above note, we can find a $b \in \mathcal{O}$ calculated from w, $\langle a_1, \cdots, a_n\rangle$ and index of θ such that $|\theta(w, \langle m, a_1, \cdots, a_n\rangle)| < |b|$ for all m. By Lemma 2.12 and the induction hypothesis, we can compute $\lambda m\{w\}^{S(F)}(m, a_1, \cdots. a_n)$ from an index of χ and $H(b)$. Hence $\{z\}^{S(F)}(a_1, \cdots, a_n)$ is calculated from an index of χ and $H(7^b)$. So we may take $\theta(z, \langle a_1, \cdots, a_n\rangle) = 7^b$.

As in Shoenfield [10], we can show that if A is $S(F)$-recursive, then A is recursive in some $H(a)$ with $a \in \mathcal{O}$.

Suppose that $\sigma = |\mathcal{O}| < \omega_1[S(F)]$, then for any $S(F)$-recursive set A, we have that $A \in L_F(\tau_\sigma[F])$. This is absurd. Q.E.D.

§ 3.

Iterating the superjump operation S to E, we can define normal type

2 objects $E_1, E_2, \cdots, E_n, \cdots, E_\omega$. In this section, we consider the E_n-degrees and the E_ω-degrees. The results in this section and the next two sections can be easily extended to the J_a^S-degrees, where $J_a^S (a \in O^S)$ are the type 2 objects defined by Platek [6].

DEFINITION 3.1. A type 2 object E_n is defined by recursion on n:

$$E_0 = E \text{ and } E_{n+1} = S(E_n) = \{\alpha \in \omega^\omega : \{\alpha(0)\}^{E_n}(\lambda m \alpha(m+1)) \downarrow\} .$$

We let $E_\omega = \{\langle n, \alpha \rangle : \alpha \in E_n\}$.

It is well-known that E_1 in this definition is essentially same as Tugue's object E_1.

DEFINITION 3.2 (Aczel and Hinman). For any ordinal κ:

(a) κ is 0-recursively inaccessible iff $\kappa > \omega$ and κ is admissible;

(b) κ is $n+1$-recursively inaccessible iff κ is n-recursively inaccessible and the limit of n-recursively inaccessible ordinals $< \kappa$;

(c) κ is ω-recursively inaccessible iff κ is n-recursively inaccessible for all n.

LEMMA 3.4 (Aczel and Hinman [1]). *For each $\sigma \leqq \omega$, $\omega_1[E_\sigma]$ is the first σ-recursively inaccessible ordinal and for all $P \subset \omega^k$:*

(i) *P is E_σ-recursive if and only if $P \in L(\omega_1[E_\sigma])$;*

(ii) *P is E_σ-semirecursive if and only if P is Σ_1 on $L(\omega_1[E_\sigma])$, where $L(\nu)$ is the set of all constructible sets of order $< \nu$.*

DEFINITION 3.3. For any $A \subset \omega$:

$$L(0, A) = \omega;$$

$L(\sigma+1, A) = \{A\} \cup \{x \subset L(\sigma, A): x$ is first order definable over the structure $\langle L(\sigma, A), \in \rangle$ with parameters from $L(\sigma, A)\}$;

$$L(\lambda, A) = \bigcup \{L(\sigma, A): \sigma < \lambda\} \text{ if } \lambda \text{ is a limit ordinal.}$$

In the case where $A = 0$, $L(\sigma, A)$ is simply denoted by $L(\sigma)$. Following Sacks [9], we introduce a language $\mathscr{L}(\kappa, G)$ which is the syntactical counterpart of $L(\kappa, A)$.

DEFINITION 3.5. Let κ be an admissible ordinal $> \omega$. $\mathscr{L}_0(\kappa)$ is the following language:

unranked variables: $v_0, v_1, \cdots, v_n, \cdots$;

ranked variables: $v_0^\sigma, v_1^\sigma, \cdots, v_n^\sigma, \cdots$ $(\sigma < \kappa)$;

predicate symbol: $\in$;
logical symbols: $\neg, \vee, \exists$;
parentheses: (,).

$\mathscr{L}(\kappa, G)$ is the ramified language obtained by adding constant symbols G, $\bar{n}(n \in \omega)$ and the abstraction operator $\hat{}$ to $\mathscr{L}_0(\kappa)$. A set $C(\sigma)$ of constant terms of $\mathscr{L}(\kappa, G)$ is defined by recursion on σ:

$$C(0) = \{\bar{n}: n \in \omega\};$$

$C(\sigma + 1) = \{\hat{x}^\sigma \phi(x^\sigma, c_1, \cdots, c_n): c_1, \cdots, c_n \in \bigcup_{\tau \leqq \sigma} C(\tau)$ and $\phi(x^\sigma, y_1, \cdots, y_n)$ is a formula of $\mathscr{L}_0(\kappa)$ such that all quantified variables are of rank $\leqq \sigma\}$;

$$C(\lambda) = \bigcup \{C(\sigma): \sigma < \lambda\} \qquad \text{if } \lambda \text{ is a limit ordinal}.$$

Let $C = \bigcup \{C(\sigma): \sigma < \kappa\}$. The atomic formulas are of the form $s \in t$ where s and t are variables or elements of C. A formula of $\mathscr{L}(\kappa, G)$ is said to be ranked if it has no unranked quantifiers. If ϕ is a ranked formula or a formula of the form $(\exists v_i)\phi$, where ϕ is a ranked formula, then ϕ is said to be a Σ_1 formula of $\mathscr{L}(\kappa, G)$.

For each $c \in C$, $\rho(c)$ is the least σ such that $c \in C(\sigma)$. If ϕ is a ranked sentence of $\mathscr{L}(\kappa, G)$, then we let $\rho(\phi)$ be the greatest element of $\{\sigma: (\exists x^\sigma)$ occurs in $\phi\} \cup \{\rho(c): c$ occurs in $\phi\}$.

All the above syntactical notions are Δ_1 on $L(\kappa)$. For any $A \subset \omega$, $\mathscr{L}(\kappa, G)$ is interpreted by $L(\kappa, A)$ as usual. For each element of $L(\kappa, A)$ is denoted by an element of C. In particular, A is denoted by G. We identify 2^ω with $P(\omega)$, the power set of ω, and often use $L(\kappa, f)$ in place of $L(\kappa, \{n: f(n) = 0\})$.

Definition 3.6. We use $p, q, r, \cdots$ to represent finite sequences of 0's and 1's. The Cohen forcing relation $p \Vdash_\kappa \phi$ is defined as usual. For example,

$$p \Vdash_\kappa \bar{n} \in G \longleftrightarrow n < lh(p) \text{ and } p(n) = 0 .$$

If a real $f \in 2^\omega$ is generic with respect to this relation $\Vdash_\kappa$, we say that f is a Cohen real over $L(\kappa)$. That is, for every sentence ϕ of $\mathscr{L}(\kappa, G)$, there exists a $p \subset f$ such that $p \Vdash_\kappa \phi$ or $p \Vdash_\kappa \neg \phi$, where $p \subset f$ means that f is an extension of p.

Let κ be a countable admissible ordinal $> \omega$. The following lemma is proved in the standard way, so we omit prove it.

LEMMA 3.7. (i) *For any p, there exists a Cohen real f over $L(\kappa)$ such that $p \subset f$.*

If f is a Cohen real over $L(\kappa)$, then:

(ii) $L(\kappa, f) \models \phi$ *iff* $\exists p(p \subset f$ *and* $p \Vdash_\kappa \phi)$;

(iii) $L(\kappa, f)$ *is an admissible set and* $f \notin L(\kappa)$.

LEMMA 3.8. *Let κ and λ be admissible ordinals such that $\omega < \kappa < \lambda$. If f is a Cohen real over $L(\lambda)$, then it is also a Cohen real over $L(\kappa)$.*

Proof. For each sentence ϕ of $\mathscr{L}(\kappa, G)$, let ϕ^* be the ranked sentence of $\mathscr{L}(\lambda, G)$ obtained from ϕ by replacing each occurrence of an unranked quantifier $(\exists x)$ with a ranked quantifier $(\exists x^\kappa)$. Then for any p, $p \Vdash_\kappa \phi$ iff $p \Vdash_\lambda \phi^*$. If f is a Cohen real over $L(\lambda)$, then for each sentence ϕ of $\mathscr{L}(\kappa, G)$, there exists some $p \subset f$ such that $p \Vdash_\kappa \phi^*$ or $p \Vdash_\lambda \neg \phi^*$. Thus f is a Cohen real over $L(\kappa)$. Q.E.D.

Definition 3.2 and Lemma 3.3 can be relativized to any $f \in 2^\omega$.

THEOREM 3.9. *Let $\sigma \leqq \omega$. If f is a Cohen real over $L(\omega_1[E_\sigma])$, then $\deg_{E_\sigma}(f) > \mathbf{0}$ and $\omega_1[E_\sigma, f] = \omega_1[E_\sigma]$.*

Proof. From Lemma 3.3. (i) and Lemma 3.7. (iii), it is clear that $\deg_{E_\sigma}(f) > \mathbf{0}$. By the relativized form of Lemma 3.3, $\omega_1[E, f]$ is the first σ-recursively-in-f inaccessible ordinal. In order to show that $\omega_1[E_\sigma, f] = \omega_1[E_\sigma]$, it suffices to prove that for any n-recursively inaccessible ordinal. $\kappa < \aleph_1$, if f is a Cohen real over $L(\kappa)$, then κ is an n-recursively-in-f inaccessible ordinal. We show this assertion by induction on n. In the case where $n = 0$, it is obvious from Lemma 3.7 (iii). Suppose that κ is $n+1$-recursively inaccessible. Then κ is n-recursively inaccessible and there exists a sequence $\kappa_0 < \kappa_1 < \cdots < \kappa_i < \cdots$ of n-recursively inaccessible ordinals such that $\kappa = \sup\{\kappa_i : i \in \omega\}$. By the induction hypothesis and Lemma 3.8, κ and all κ_i are n-recursively-in-f inaccessible ordinals. Thus κ is an $n+1$-recursively-in-f inaccessible ordinal. Q.E.D.

DEFINITION 3.10. We say that a finite set $\{f_1, \cdots, f_n\} \subset 2^\omega$ is F-independent if $f_i \nleq_F \langle f_1, \cdots, f_{i+1}, f_{i+1}, \cdots, f_n \rangle$ for all i. *A* set $X \subset 2^\omega$ is F-independent if all finite subsets of X are F-independent.

THEOREM 3.11. *For each $\sigma \leqq \omega$, there exists an E_σ-independent set with cardinality of the continuum.*

Proof. We set $\kappa = \omega_1[E_\sigma]$. We consider the ramified language $\mathscr{L}(\kappa, G_1, \cdots,$

G_n) defined in the same way as $\mathscr{L}(\kappa, G)$. We can extend the forcing relation $\Vdash_\kappa$ to the language $\mathscr{L}(\kappa, G_1, \cdots, G_n)$. We denote this extended forcing relation by $\langle p_1, \cdots, p_n \rangle \Vdash_\kappa^n \phi$. It is well-known (cf. [11]) that if $\langle f_1, \cdots, f_n \rangle$ is generic with respect to $\Vdash_\kappa^n$, then

(1) each f_i is a Cohen real over $L(\kappa)$;

(2) $f_i \notin L(\kappa, f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n)$ for all i;

(3) $L(\kappa, f_1, \cdots, f_n)$ is an admissible set.

Let θ_n be the sentence of $\mathscr{L}(\kappa, G_1, \cdots, G_n)$ defined by:

$$\theta_n = \bigwedge_{i=1}^{n} [G_i \notin L(G_1, \cdots, G_{i-1}, G_{i+1}, \cdots, G_n)].$$

Let P be the set of all finite sequences of 0's and 1's. For any $p_1, \cdots, p_n \in P$, there are reals $f_1, \cdots, f_n$ such that $p_1 \subset f_1, \cdots, p_n \subset f_n$ and $\langle f_1, \cdots, f_n \rangle$ is generic with respect to $\Vdash_\kappa^n$. Then $L(\kappa, f_1, \cdots, f_n) \models \theta_n$ by (2). Hence there are $q_1, \cdots, q_n \in P$ such that $p_1 \subset q_1, \cdots, p_n \subset q_n$ and $\langle q_1, \cdots, q_n \rangle \Vdash_\kappa^n \theta_n$. Thus we have proved the following (4):

(4) $(\forall p_1, \cdots, \forall p_n \in P)(\exists q_1, \cdots, q_n \in P)[p_1 \subset q_1 \& \cdots \& p_n \subset q_n \& \langle q_1, \cdots, q_n \rangle \Vdash_\kappa^n \theta_n]$.

Let $\langle \phi_n^{(i)} : i \in \omega \rangle$ be an enumeration of all sentences of $\mathscr{L}(\kappa, G_1, \cdots, G_n)$.

We define a $p_s \in P$ for each $s \in P$.

Let $p_{\langle\rangle} = \langle\rangle$.

Assume that all p_s with $lh(s) = n$ are already defined. Put $m = 2^{n+1}$. By (4), we can find incompatible extensions p_s^0 and p_s^1 of p_s such that $\langle p^0_{\langle 0,\cdots,0\rangle}, p^1_{\langle 0,\cdots,0\rangle}, \cdots, p^0_{\langle 1,\cdots,1\rangle}, p^1_{\langle 1,\cdots,1\rangle} \rangle \Vdash_\kappa^m \theta_m$, where $\langle 0, \cdots, 0\rangle, \cdots, \langle 1, \cdots, 1\rangle$ is the enumeration of $\{s \in P : lh(s) = n\}$ in the lexicographical ordering. There exist extensions q_s^i of p_s^i ($lh(s) = n, i \leqq 1$) such that for each $k \leqq n$ and for any combination $q_1, \cdots, q_{2^k}$ of q_s^i's such that $q_1 \supset p_{\langle 0,\cdots,0\rangle}, \cdots, p_{2^k} \supset q_{\langle 1,\cdots,1\rangle}$ ($\langle 0, \cdots, 0\rangle, \cdots, \langle 1, \cdots, 1\rangle$ is the enumeration of $\{t \in P : lh(t) = k\}$ in the lexicographical ordering), $\langle q_1, \cdots, q_{2^k} \rangle$ decides $\phi_{2^k}^{(0)}, \cdots . \phi_{2^k}^{(n)}$.[3] We set $p_{s*\langle i\rangle} = q_s^i$, where $s*t$ is the concatenation of s and t.

Clearly $\{p_s : s \in P\}$ defines a perfect set $A \subset 2^\omega$. It is easy to verify that A is E_σ-independent. Q.E.D.

LEMMA 3.12. *If f is a Cohen real over $L(\omega_1[E_\sigma])$, then $f \oplus 0' \equiv_{E_\sigma} f'$, where $0'$ and f' are the E_σ-jumps of 0 and f, respectively.*

Proof. Put $\kappa = \omega_1[E_\sigma]$. It is trivial that $f \oplus 0' \leqslant_{E_\sigma} f'$. Let $\theta(x)$ be a

3) In any forcing relation $\Vdash$ we say that p decides ϕ if $p \Vdash \phi$ or $p \Vdash \neg\phi$.

Σ_1 formula of $\mathscr{L}(\kappa, G)$ such that

$$g' = \{n \in \omega : L(\kappa, g) \models \theta(n)\}$$

for all g with $\omega_1[E_\sigma, g] = \kappa$. Then,

$$n \in f' \longleftrightarrow (\exists p \subset f)[p \Vdash_\kappa \theta(\bar{n})] .$$

Since $\{\langle p, n\rangle : p \Vdash_\kappa \theta(\bar{n})\}$ is Σ_1 on $L(\kappa)$, we have that $f' \leqslant_{E_\sigma} f \oplus 0'$. Q.E.D.

THEOREM 3.13.[4] *There exist E_σ-degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $(\boldsymbol{a} \cup \boldsymbol{b})' = \boldsymbol{a}' \cup \boldsymbol{b}'$ and $\boldsymbol{a}|\boldsymbol{b}$.*

Proof. Put $\kappa = \omega_1[E_\sigma]$. Let $\langle f, g\rangle$ be generic over $L(\kappa)$ with respect to $\Vdash^2_\kappa$. Then both f and g are Cohen reals over $L(\kappa)$. Put $\boldsymbol{a} = \deg_{E_\sigma}(f)$ and $\boldsymbol{b} = \deg_{E_\sigma}(g)$, then $\boldsymbol{a}|\boldsymbol{b}$ (see the proof of Theorem 3.11). It is clear that $\boldsymbol{a}' \cup \boldsymbol{b}' \leqslant (\boldsymbol{a} \cup \boldsymbol{b})'$. Since $P \times P \simeq P$ in $L(\kappa)$, $f \oplus g$ is a Cohen real over $L(\kappa)$ (*cf.* [11]). By Lemma 3.12, $(\boldsymbol{a} \cup \boldsymbol{b})' = (\boldsymbol{a} \cup \boldsymbol{b}) \cup \boldsymbol{0}' = \boldsymbol{a} \cup (\boldsymbol{b} \cup \boldsymbol{0}') = \boldsymbol{a} \cup \boldsymbol{b}' \leqslant \boldsymbol{a}' \cup \boldsymbol{b}'$. Thus, $\boldsymbol{a}' \cup \boldsymbol{b}' = (\boldsymbol{a} \cup \boldsymbol{b})'$. Q.E.D.

Remark 3.14. We can take the above $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a}, \boldsymbol{b} < \boldsymbol{0}'$ (see the proof of Theorem 3.16).

An admissible ordinal κ is said to be *projectible* into λ if there exists an injection from κ into λ which is Δ_1 on $L(\kappa)$. The least $\lambda \leqq \kappa$ such that κ is projectible into λ is called the *projectum* of κ and denoted by κ^*. An admissible ordinal κ is called a *recursively Mahlo ordinal* if every closed unbounded subset of κ which is Δ_1 on $L(\kappa)$ contains an admissible ordinal. Aczel and Hinman [1] showed that every $\omega_1[J^S_a]$ is less than the first recursively Mahlo ordinal $> \omega$.

LEMMA 3.15. *If κ is less than the first recursively Mahlo ordinal $> \omega$, then κ is projectible into ω.*

Proof. If not, then κ^* is a recursively Mahlo ordinal $> \omega$, which is a contradiction (see Barwise [2; V. 7.25]). Q.E.D.

THEOREM 3.16.[4] *For each $\sigma \leqq \omega$, there are E_σ-degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a}' \cup \boldsymbol{b}' \leqslant \boldsymbol{a} \cup \boldsymbol{b} \leqslant 0'$.*

Proof. We put $\kappa = \omega_1[E_\sigma]$. Let $\theta(x)$ be a Σ_1 formula as in the proof of Lemma 3.12. We will construct two Cohen reals f and g over $L(\kappa)$ such that $f' \oplus g' \leqslant_{E_\sigma} f \oplus g \leqslant_{E_\sigma} 0'$. Let $\langle \phi_n : n \in \omega\rangle \in L(\kappa^+)$ be an enumeration of

4) These are analogues of Spector's results (see Spector [12]).

all sentences of $\mathscr{L}(\kappa, G)$, where κ^+ is the first admissible ordinal larger than κ. Such an enumeration exists because κ is projectible into ω by Lemma 3.15. By recursion on n, we define two sequences $\langle p_n : n \in \omega \rangle$ and $\langle q_n : n \in \omega \rangle$ of forcing conditions.

Stage 0. $p_0 = q_0 = \langle\ \rangle$.

Stage $2n+1$. Assume that p_{2n} and q_{2n} are already defined and have the same length. Let p' be an extension of p_{2n} such that p' decides ϕ_{2n}, say $p' = p_{2n} * s$. Put $q' = q_{2n} * s$. Let q'' be an extension of q' which decides ϕ_{2n}, say $q'' = q' * t$. We put $p'' = p' * t$. Then it is clear that both p'' and q'' decide ϕ_{2n}. See whether there exists an extension $\tilde{p}$ of p'' such that

$$\tilde{p} \Vdash_\kappa \theta(\bar{n}) .$$

If so, let $\tilde{p}$ be the least such extension. Define:

$$p_{2n+1} = \tilde{p} * \langle 0 \rangle \text{ and } q_{2n+1} = q'' * u * \langle 1 \rangle ,$$

where $\tilde{p} = p'' * u$. If not, define:

$$p_{2n+1} = p'' * \langle 1 \rangle \text{ and } q_{2n+1} = q'' * \langle 0 \rangle .$$

Stage $2n+2$. The definition for stage $2n+2$ is as for Stage $2n+1$ with the symbols p and q interchanged throughout.

It is easy to see that the above constructions of p_n and q_n are accomplished in $L(\kappa^+)$ since $\{\langle p, \phi \rangle : p \Vdash_\kappa \phi\}$ is a set in $L(\kappa^+)$. We let $f = \bigcup \{p_n : n \in \omega\}$ and $g = \bigcup \{q_n : n \in \omega\}$. Then f and g are Cohen reals over $L(\kappa)$. Let $i_0, i_1, \cdots, i_n, \cdots$ be the members of $\{i \in \omega : f(i) \neq g(i)\}$ in increasing order. Then,

$$f' = \{n \in \omega : L(\kappa, f) \models \theta(n)\} = \{n \in \omega : f(i_{2n}) = 0\} .$$

Hence, $f' \leqslant_{E_\sigma} f \oplus g \leqslant_{E_\sigma} 0'$. Similarly, $g' \leqslant_{E_\sigma} f \oplus g \leqslant_{E_\sigma} 0'$. Q.E.D.

COROLLARY 3.17. *There exist E_σ-degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that*

(i) $\boldsymbol{a}' \cup \boldsymbol{b}' \neq (\boldsymbol{a} \cup \boldsymbol{b})$;

(ii) $\boldsymbol{a} < \boldsymbol{0}'$, $\boldsymbol{b} < \boldsymbol{0}'$ *and* $\mathbf{a} \cup \mathbf{b} = \boldsymbol{0}'$.

Proof. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be as in Theorem 3.16.

(i) $\boldsymbol{a}' \cup \boldsymbol{b}' = \boldsymbol{a} \cup \boldsymbol{b} < (\boldsymbol{a} \cup \boldsymbol{b})'$.

(ii) Since $\boldsymbol{a}' \leqslant \boldsymbol{a} \cup \boldsymbol{b}$, we see that $\boldsymbol{0}' \leqslant \boldsymbol{a} \cup \boldsymbol{b}$ and hence $\boldsymbol{a} \cup \boldsymbol{b} = \boldsymbol{0}'$. If $\boldsymbol{a} = \boldsymbol{0}'$, then $\boldsymbol{0}'' = \boldsymbol{a}' \leqslant \boldsymbol{0}'$. This is a contradiction. Q.E.D.

§ 4.

In [3], Gandy and Sacks constructed a minimal hyperdegree by the forcing method with perfect sets. We shall extend this result to minimal J_a^S-degrees. As in the preceding section, we shall consider only the cases where $|a|^S \leqslant \omega$.

DEFINITION 4.1. A set P of finite sequences of 0's and 1's is called a tree if $p \in P \,\&\, q \subset p \rightarrow q \in P$, where $q \subset p$ means that p is an extension of q. A tree P is said to be perfect if $p \in P \rightarrow \exists q, r \in P[p \subset q \,\&\, p \subset r \,\&\, q$ and r are incompatible]. For each perfect tree P, we denote the set $\{f: (\forall n)\bar{f}(n) \in P\}$ by $[P]$.

Let κ be a countable admissible ordinal $> \omega$ which is projectible into ω. We use $P, Q, R, \cdots$ to represent perfect trees in $L(\kappa)$.

DEFINITION 4.2 (Gandy and Sacks). For any ranked sentence ϕ of $\mathscr{L}(\kappa, G)$,

(i) $P \Vdash \phi$ iff $(\forall f \in [P])L(\kappa, f) \models \phi$.

For unranked sentences ϕ and ψ:

(ii) $P \Vdash \neg \phi$ iff for all subtrees Q of P, $Q \nVdash \phi$;

(iii) $P \Vdash \phi \vee \psi$ iff $P \Vdash \phi$ or $P \Vdash \psi$.

If $\phi(x^\sigma)$ and $\psi(x)$ are unranked formulas, then:

(iv) $P \Vdash (\exists x^\sigma)\phi(x^\sigma)$ iff $P \Vdash \phi(c)$ for some $c \in C(\sigma)$;

(v) $P \Vdash (\exists x)\phi(x)$ iff $P \Vdash \phi(c)$ for some $c \in C$.

We say that a real f is Sacks over $L(\kappa)$ if for any sentence ϕ of $\mathscr{L}(\kappa, G)$, there exists a perfect tree P in $L(\kappa)$ such that $f \in [P]$ and P decides ϕ.

LEMMA 4.3. *The following relation* Force_Σ *is* Σ_1 *on* $L(\kappa)$:

$\mathrm{Force}_\Sigma (P, \phi) \longleftrightarrow P$ *is a perfect tree in* $L(\kappa)$ & ϕ *is a* Σ_1 *sentence of* $\mathscr{L}(\kappa, G)$ & $P \Vdash \phi$.

Proof. From clause (v) of 4.2, ranked ϕ's need be considered. Since κ is projectible into ω,

$$L(\kappa) \models \text{every set is countable}\,.$$

Hence for any set $x \in L(\kappa)$, there exists a set $A \in L(\kappa) \cap P(\omega)$ such that $x \in L(\omega_1[A], A)$. Let $\Phi(x, y, z)$ be a Σ_1 formula such that for any $A \in L(\kappa) \cap P(\omega)$, if $P \in L(\omega_1[A], A)$ and ϕ is a ranked sentence of $\mathscr{L}(\kappa, G)$ with $\rho(\phi) < \omega_1[A]$, then

$$L(\omega_1[A], A) \models \Phi(P, \phi, A) \text{ iff } \forall f \in [P]\; L(\omega_1[A], f) \models \phi\,.$$

For the existence of such a formula Φ, see Gandy and Sacks [3; Lemma 1], where it is proved that if $P \in L(\omega_1)$ and ϕ is a ranked sentence of $\mathscr{L}(\omega_1, G)$, then $\forall f \in [P]\ L(\omega_1, f) \models \phi$ is a Π_1^1 relation of P and ϕ. It is well-known that every Π_1^1 relation is Σ_1 on $L(\omega_1)$. Relativizing this result to A, we can find a required Σ_1 formula $\Phi(x, y, z)$.
Then for any P and any ranked sentence ϕ, we have:

$$P \Vdash \phi \text{ iff } (\exists A \in L(\kappa) \cap P(\omega))[P, \phi \in L(\omega_1[A], A) \,\&\, L(\omega_1[A], A) \models \Phi(A, P, \phi)] .$$

Thus the relation $P \Vdash \phi$ restricted to ranked sentences ϕ is Σ_1 on $L(\kappa)$.
Q.E.D.

Lemma 4.4.[5] $(\forall\phi)(\forall P)(\exists Q \subset P)[Q$ decides $\phi]$.

Lemma 4.5.[5] *If f is a Sacks real over $L(\kappa)$, then*:
(i) $L(\kappa, f) \models \phi$ *iff* $(\exists P)[f \in [P]$ *and* $P \Vdash \phi]$;
(ii) $L(\kappa, f)$ *is admissible and* $f \notin L(\kappa)$;
(iii) $g \in L(\kappa, f) \longrightarrow g \in L(\kappa)$ *or* $f \in L(\kappa, g)$.

For every perfect tree P in $L(\kappa)$, Sacks defined the local forcing relation $p \Vdash_\kappa^P \phi$ where $p \in P$ and ϕ is a sentence of $\mathscr{L}(\kappa, G)$ (see Sacks [9: 2.8]). We say that a real f is P-Cohen over $L(\kappa)$ if f is generic with respect to $\Vdash_\kappa^P$. Obviously, every P-Cohen real belongs to $[P]$.

Lemma 4.6. *If κ is σ-recursively inaccessible and P is a perfect tree in $L(\kappa)$, then there exists a perfect tree $P^* \subset P$ in $L(\kappa^+)$ such that for any $f \in [P^*]$, f is P-Cohen over $L(\kappa)$ and κ is σ-recursively-in-f inaccessible, where κ^+ is the first admissible ordinal larger than κ.*

Proof. Note that $\{\langle p, \phi\rangle : p \in P \text{ and } p \Vdash_\kappa^P \phi\} \in L(\kappa^+)$. Let $\langle \phi_n : n \in \omega\rangle \in L(\kappa^+)$ be an enumeration of all sentences of $\mathscr{L}(\kappa, G)$. Such an enumeration exists since κ is projectible into ω. For each $s \in \mathrm{Seq}\,(2)$, we define $p_s \in P$ by recursion on $lh(s)$, where Seq (2) is the set of all finite sequences of 0's and 1's.

Let $p_{\langle\,\rangle} = \langle\,\rangle$.

Assume that $lh(s) = n$ and p_s is already defined. Let $p_{s*\langle 0\rangle} \in P$ and $p_{s*\langle 1\rangle} \in P$ be incompatible extensions of p_s such that both $p_{s*\langle 0\rangle}$ and $p_{s*\langle 1\rangle}$ decide ϕ_n.

We set $P^* = \{p \in P : (\exists s \in \mathrm{Seq}\,(2))[p \subset p_s \text{ or } p_s \subset p]\}$. Then P^* is a perfect subtree of P. Obviously the above construction of p_s $(s \in \mathrm{Seq}\,(2))$ can

5) See Gandy and Sacks [3].

be performed in $L(\kappa^+)$. Thus. we see the that $P^* \in L(\kappa^+)$. By a similar proof to that of Theorem 3.9, κ is σ-recursively-in-f inaccessible for all $f \in [P^*]$. Q.E.D.

THEOREM 4.7. *For each $\sigma \leqq \omega$, there exists a Sacks real f over $L(\omega_1[E_\sigma])$ such that $\omega_1[E_\sigma, f] = \omega_1[E_\sigma]$.*

Proof. Put $\kappa = \omega_1[E_\sigma]$. In the case where $\sigma = 0$, the existence of such an f is due to Gandy and Sacks [3].

Now we consider the case where $\sigma = m + 1$ for some $m \in \omega$. Let $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ be a sequence of m-recursively inaccessible ordinals such that $\kappa = \sup\{\kappa_n : n \in \omega\}$, and $\langle \phi_n : n \in \omega \rangle$ be an enumeration of all sentences of $\mathscr{L}(\kappa, G)$. We define a sequence $\langle n_i : i \in \omega \rangle$ of natural numbers and a sequence $\langle P_i : i \in \omega \rangle$ of perfect trees in $L(\kappa)$. We let $n_0 = 0$ and $P_0 = \mathrm{Seq}\,(2)$. Suppose that n_i and P_i are already defined. Let n_{i+1} is the least n such that $n > n_i$ and $P_i \in L(\kappa_n)$. By Lemma 4.6 and Lemma 3.15, there exists a perfect tree $Q \subset P_i$ such that $Q \in L(\kappa)$ and κ_{n_i} is m-recursively-in-f inaccessible for all $f \in [Q]$. Then, by Lemma 4.4, there is a perfect tree $R \subset Q$ such that $R \in L(\kappa)$ and R decides ϕ_i. We let $P_{i+1} = R$. Since 2^ω is compact, there exists an $f \in \bigcap\{[P_i] : i \in \omega\}$. It is easy to verify that such an f has the desired property.

In the case where $\sigma = \omega$, let $\kappa_0 < \kappa_1 < \cdots < \kappa_n < \cdots$ be a sequence of ordinals such that each κ_n is n-recursively inaccessible and $\kappa = \sup\{\kappa_n : n \in \omega\}$. By the same argument as above, there exists a Sacks real f over $L(\kappa)$ and a subsequence $\langle \kappa_{n_i} : i \in \omega \rangle$ of $\langle \kappa_n : n \in \omega \rangle$ such that κ_{n_i} is n_i-recursively-in-f inaccessible for each $i \in \omega$, so κ is ω-recursively-in-f inaccessible.

Q.E.D.

In the above proof, every $f \in \bigcap\{[P_i] : i \in \omega\}$ is P_i-Cohen over $L(\kappa_{n_i})$ and hence does not satisfy the minimality condition:

$$g \in L(\kappa_{n_i}, f) \longrightarrow g \in L(\kappa_{n_i}) \text{ or } f \in L(\kappa_{n_i}, g)\,.$$

But we can construct a sequence $\langle P_i : i \in \omega \rangle$ such that the minimality condition holds as follows: by induction on $\sigma \leqq \omega$, we can show that for every σ-recursively inaccessible ordinal κ less than the first recursively Mahlo ordinal and for every perfect tree $P \in L(\kappa)$, there exists a perfect tree $Q \subset P$ such that $Q \in L(\kappa^+)$ and every $f \in [Q]$ is a Sacks real over $L(\kappa)$. Then the construction of $\langle P_i : i \in \omega \rangle$ is similar to that in the proof.

Moreover we can construct $\langle P_i : i \in \omega \rangle$ such that $\bigcap\{[P_i] : i \in \omega\}$ is a perfect set, and so there are $2^{\aleph_0}$ f's which satisfy the condition of Theorem 4.7.

COROLLARY 4.8. *For each $\sigma \leqq \omega$, there exists a minimal E_σ-degree $\boldsymbol{a}$ such that $\boldsymbol{a} < \boldsymbol{0}'$.*

Proof. Let f be a Sacks real over $L(\omega_1[E_\sigma])$ such that $\omega_1[E_\sigma, f] = \omega_1[E_\sigma]$. As is known from the proof of Theorem 4.7. we can take such an f so that $f \in L(\omega_1[E_\sigma]^+)$. We let $\boldsymbol{a} = \deg_{E_\sigma}(f)$. Then $\boldsymbol{a}$ is a minimal E_σ-degree by Lemma 4.5, and $\boldsymbol{a} \leqslant \boldsymbol{0}'$. By Corollary 3.17, there is an E_σ-degree between $\boldsymbol{0}$ and $\boldsymbol{0}'$. Thus we see that $\boldsymbol{a} < \boldsymbol{0}'$. Q.E.D.

§ 5.

In this section. we shall prove that for each $\sigma \leqq \omega$, the set $\{\boldsymbol{0}^{(\nu)} : \nu < \omega_1[E_{\sigma+1}]$ of E_σ-degrees does not have the least upper bound. In the case where $\sigma = 0$, this result was proved by Sacks in [9]. We use the forcing method with absolutely pointed perfect trees.

DEFINITION 5.1. A perfect tree P is said to be E_σ-pointed if:

$$(\forall f \in [P])[P \leqslant_{E_\sigma} f] .$$

When we require that:

$$(\forall f \in [P])[P \in L(\omega_1[E_\sigma, f])] ,$$

we say that P is absolutely E_σ-pointed. Obviously, absolute E_σ-pointedness implies E_σ-pointedness.

Let κ be a σ-recursively inaccessible ordinal projectible into ω.

LEMMA 5.2. *Let $P \in L(\kappa)$ be an E_σ-pointed perfect tree and $X \in L(\kappa)$ be a subset of ω such that $P \leqslant_{E_\sigma} X$. Then there exists an absolutely E_σ-pointed perfect tree $Q \in L(\kappa)$ such that $Q \subset P$ and $X \leqslant_{E_\sigma} Q$.*

Proof. By Proposition 2.12 of Sacks [9], there is a $Y \in L(\kappa) \cap P(\omega)$ such that X is recursive in Y and $Y \in L(\omega_1[Y]) \subset L(\omega_1[E_\sigma, Y])$. Hence there exists an E_σ-pointed perfect tree $Q \in L(\kappa)$ such that $Q \subset P$ and $Q \equiv_{E_\sigma} Y$ (Sacks [9; 2.3]). Take an $f \in [Q]$ to see that Q is absolutely E_σ-pointed. Since Q is E_σ-pointed, we see that $\omega_1[E_\sigma, Q] \leqslant \omega_1[E_\sigma, f]$. On the other hand, $Q \in L(\omega_1[E_\sigma, Y])$. Since $Q \equiv_{E_\sigma} Y$ and $Y \in L(\omega_1[E_\sigma, Y])$. Consequently, $Q \in L(\omega_1[E_\sigma, f])$. Q.E.D.

LEMMA 5.3. *The set of all absolutely E_σ-pointed perfect trees in $L(\kappa)$ is Σ_1 on $L(\kappa)$.*

Proof. Firstly we shall show that for each $\tau < \kappa$, there exists a ranked formula $\theta_\tau(x)$ of $\mathscr{L}(\kappa, G)$ such that

$$L(\kappa, f) \models \theta_\tau(\nu) \longleftrightarrow \nu \text{ is } \sigma\text{-recursively-in-}f \text{ inaccessible}$$

for all $f \in 2^\omega$ and $\nu < \tau$. As an example, we consider the case $\sigma = \omega$.

Let Φ be the predicate defined by

$$\Phi(\nu, f) \longleftrightarrow \nu > \omega \ \& \ \nu \text{ is a limit ordinal } \& \ \forall\phi(\phi \text{ is an axiom of } KP \longrightarrow L(\nu, f) \models \phi) .$$

Then $\Phi(\nu, f)$ says that ν is 0-recursively-in-f inaccessible. Note that all quantifiers in $\Phi(\nu, f)$ can be restricted to $L(\tau, f)$ whenever $\nu < \tau$. Similarly. for each n, the predicate which says that $\nu < \tau$ and ν is n-recursively-in-f inaccersible is represented by a bounded formula all of whose quantifiers are restricted to $L(\tau, f)$. Let Ψ be the following predicate:

$$\begin{aligned}\Psi(s, n, \nu, f, i) \longleftrightarrow\ & s \text{ is a function } \& \operatorname{dom}(s) = (n+1) \times \tau \ \& \operatorname{rng}(s) \subset 2 \\ & \& \ i \in 2 \ \& \ s(n, \nu) = i \ \& \ \forall\alpha < \tau[s(0, \alpha) = 0 \longleftrightarrow \Phi(\alpha, f)] \\ & \& \ \forall j < n \ \forall\alpha < \tau[s(j+1, \alpha) = 0 \longleftrightarrow s(j, \alpha) = 0 \\ & \& \ \forall\beta < \alpha \ \exists\gamma < \alpha(\beta \leqq \gamma \ \& \ s(j, \gamma) = 0)] .\end{aligned}$$

Then it is easily seen that for all $\nu < \tau$ and $f \in 2^\omega$,

$$\begin{aligned}& \nu \text{ is } \omega\text{-recursively-in-}f \text{ inaccessible} \\ & \quad \longleftrightarrow \forall n < \omega \exists s \in L(\tau, f) \Psi(s, n, \nu, f. 0) .\end{aligned}$$

From this, we can obtain a required ranked formula $\theta_\tau(x)$. The sequence $\langle \theta_\tau(x) \colon \tau < \kappa \rangle$ is Σ_1 on $L(\kappa)$.

Now, for each perfect tree in $L(\kappa)$, we let $h(P) = \min\{\nu < \kappa \colon P \in L(\nu)\}$. Obviously, h is Σ_1 on $L(\kappa)$. Let ϕ_P be a ranked sentence of $\mathscr{L}(\kappa, G)$ such that for any $f \in 2^\omega$,

$$L(\kappa, f) \models \phi_P \longleftrightarrow \forall\nu < h(P) \ (\nu \text{ is not } \sigma\text{-recursively-in-}f \text{ inaccessible}) .$$

Such a ϕ_P can be constructed using $\theta_{h(P)}(x)$. Therefore, the function $P \mapsto \phi_P$ is Σ_1 on $L(\kappa)$. By the definition of $h(P)$, we have:

$$\begin{aligned}L(\kappa, f) \models \phi_P & \longleftrightarrow h(P) \leqq \omega_1[E_\sigma, f] \\ & \longleftrightarrow P \in L(\omega_1[E_\sigma, f]) .\end{aligned}$$

Consequently, for every perfect tree P in $L(\kappa)$,

$$P \text{ is absolutely } E_\sigma\text{-pointed} \longleftrightarrow \forall f \in [P] L(\kappa, f) \models \phi_P .$$

But the right hand side means $P \Vdash \phi_P$ see Definition 4.2). Thus the lemma follows from Lemma 4.3. Q.E.D.

We define a forcing relation $P \Vdash^\sigma \phi$ as in 4.2 except that P varies through absolutely E_σ-pointed perfect trees in $L(\kappa)$. We use $P, Q, R, \cdots$ to represent absolutely E_σ-pointed perfect trees in $L(\kappa)$.

DEFINITION 5.4. We say that a real f is σ-Sacks over $L(\kappa)$ if for each sentence ϕ of $\mathscr{L}(\kappa, G)$, there exists a P such that $f \in [P]$ and P decides ϕ(i.e., $P \Vdash^\sigma \phi$ or $P \Vdash^\sigma \neg \phi$).

It is well-known ([8]) that the above definition is equivalent to: f is σ-Sacks real over $L(\kappa)$ if and only if $f \in \cup \{[P]: P \in D\}$ for any dense set D of absolutely E_σ-pointed perfect trees which is definable in $L(\kappa)$, where we say that D is dense if $(\forall P)(\exists Q)[Q \in D \ \& \ Q \subset P]$.

From 4.3 and 5.3. we obtain the following lemma.

LEMMA 5.5. *The relation* $P \Vdash^\sigma \phi$ *restricted to* Σ_1 *sentences* ϕ *of* $\mathscr{L}(\kappa, G)$ *is* Σ_1 *on* $L(\kappa)$.

LEMMA 5.6. $(\forall \phi)(\forall P)(\exists Q)[Q \subset P \ \& \ Q$ *decides* $\phi]$.

For the proofs of this lemma and the following lemma, see Sacks [9]. Although his proofs are for hyperdegrees, we can easily modify them for E_σ-degrees by using Lemma 5.2.

LEMMA 5.7. *If* f *is a* σ*-Sacks real over* $L(\kappa)$, *then*:

(i) $L(\kappa, f) \models \phi$ *if and only if* $(\exists P)[f \in [P]$ *and* $P \Vdash^\sigma \phi]$;

(ii) $L(\kappa, f)$ *is an admissible set and* $f \notin L(\kappa)$;

(iii) $g \in L(\kappa, f) \longrightarrow g \in L(\kappa)$ *or* $(\exists X)[X \in L(\kappa) \cap P(\omega)$ *and* $f \leqslant_{E_\sigma} g, X]$.

THEOREM 5.8. *If* κ *is a* σ*-recursively inaccessible ordinal projectible into* ω, *then there exists a* σ*-Sacks real* f *over* $L(\kappa)$ *such that* $\omega_1[E_\sigma, f] = \kappa$.

Proof. The proof is similar to that of Theorem 4.7, but we must arrange for all perfect trees to be absolutely E_σ-pointed. Let δ be an arbitrary ordinal less than κ. Then, by Lemma 5.2, the set $\{P \in L(\kappa): P$ is absolutely E_σ-pointed and $\delta < \omega_1[E_\sigma, P]\}$ is dense and definable in $L(\kappa)$. Consequently, for every σ-Sacks real f over $L(\kappa)$, it holds that $\kappa \leqq \omega_1[E_\sigma, f]$. Hence we need to show the existence of a σ-Sacks real f over $L(\kappa)$ such that κ is σ-recursively-in-f inaccessible. In the case where $\sigma = 0$, it is clear from Lemma 5.7. We consider the case where $\sigma = m + 1$ for some $m < \omega$. Let $\delta_0 < \delta_1 < \cdots < \delta_n < \cdots$ be a sequence of ordinals such that $\kappa = \sup\{\delta_n : n \in \omega\}$. For each $n < \omega$, we define an m-recursively inaccessible ordinal κ_n and an absolutely E_{m+1}-pointed perfect tree P_n by recursion on n. Let $P_0 = \mathrm{Seq}(2)$ and $\kappa_0 = \min\{\nu < \kappa : \nu$ is m-recursively inaccessible$\}$. Assume that κ_n and P_n are already defined and that $\kappa_n = \min\{\nu < \kappa : \nu$ is m-recursively inaccessible and $P_n \in L(\nu)\}$. By Lemma 4.6, there exists a perfect tree $Q \subset P_n$ such that $Q \in L(\kappa_n^+)$ and κ_n is m-recursively-in-f inaccessible for all $f \in [Q]$. Note that $\kappa_n^+ < \omega_1[E_{m+1}, f]$ for all $f \in [P_n]$ because P_n is absolutely E_{m+1}-pointed. Consequently, Q is also absolutely E_{m+1}-pointed. In view of Lemma 5.2, there exists an absolutely E_{m+1}-pointed perfect tree $R \subset Q$ such that $R \in L(\kappa)$ and $\delta_n < \omega_1[E_{m+1}, R]$. From Lemma 5.6, we can find an absolutely E_{m+1}-pointed perfect tree $P_{n+1} \subset R$ such that $P_{n+1} \in L(\kappa)$ and P_{n+1} decides ϕ_n, where ϕ_n is the n-th sentence of $\mathscr{L}(\kappa, G)$ in an enumeration of all sentences of $\mathscr{L}(\kappa, G)$ which we fix throughout the proof. We set $\kappa_{n+1} = \min\{\nu < \kappa : \nu$ is m-recursively inaccessible and $P_{n+1} \in L(\nu)\}$.

It is easy to verify that for any $f \in \bigcap\{[P_n] : n < \omega\}$, f is an $m+1$-Sacks real over $L(\kappa)$ and each κ_n is m-recursively-in-f inaccessible, and hence κ is $m+1$-recursively-in-f inaccessible.

By the same way as in the case where $\sigma < \omega$, we can construct a sequence $\langle P_n : n \in \omega\rangle$ of absolutely E_ω-pointed perfect trees and a sequence $\langle \kappa_n : n \in \omega\rangle$ of ordinals such that $\kappa = \sup\{\kappa_n : \mathrm{n} \in \omega\}$, and that for every $f \in \bigcap\{[P_n] : n \in \omega\}$, f is an ω-Sacks real over $L(\kappa)$ and each κ_n is n-recursively-in-f inaccessible. Hence, $\omega_1[E_\omega, f] = \kappa$ for all $f \in \bigcap\{[P_n] : n \in \omega\}$. Q.E.D.

THEOREM 5.9. *For each $\sigma \leqq \omega$, there are E_σ-degrees $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ such that:*

(i) $\mathbf{0}^{(\nu)} < \boldsymbol{a}_i$ *for all* $\nu < \omega_1[E_{\sigma+1}]$ *and all* $i \leqq 1$;

and that for any E_σ-degree $\boldsymbol{b}$:

(ii) $\boldsymbol{b} \leqslant \boldsymbol{a}_0$ *and* $\boldsymbol{b} \leqslant \boldsymbol{a}_1 \longrightarrow (\exists\nu < \omega_1[E_{\sigma+1}])[\boldsymbol{b} \leqslant \mathbf{0}^{(\nu)}]$;

(iii) $(\forall i \leqq 1)(\exists\nu < \omega_1[E_{\sigma+1}])[\boldsymbol{b} < \boldsymbol{a}_i \longrightarrow \boldsymbol{b} \leqslant \mathbf{0}^{(\nu)}]$.

Proof. We set $\kappa = \omega_1[E_{\sigma+1}]$ and consider the ramified language $\mathscr{L}(\kappa, G_0, G_1)$ defined in the same way as $\mathscr{L}(\kappa, G)$. For each pair $\langle P_0, P_1\rangle$ of absolutely E_σ-pointed perfect trees in $L(\kappa)$ and for each sentence ϕ of $\mathscr{L}(\kappa, G_0, G_1)$, we define a forcing relation $\langle P_0, P_1\rangle \Vdash^{\sigma\prime} \phi$. For ranked sentence ϕ, $\langle P_0, P_1\rangle \Vdash^{\sigma\prime} \phi$ iff $(\forall f_0 \in [P_0])(\forall f_1 \in [P_1])[L(\kappa, f_0, f_1) \models \phi]$. For unranked sentences, the definition of $\Vdash^{\sigma\prime}$ is similar to 4.2. It is well-known (cf. [11]) that if $\langle f_0, f_1\rangle$ is generic with respect to $\Vdash^{\sigma\prime}$, then

(1) f_i is σ-Sacks over $L(\kappa)$ $(i = 0, 1)$;

(2) $L(\kappa, f_0) \cap L(\kappa, f_1) = L(\kappa)$.

By the same way as Theorem 5.9, we see that there exists a pair $\langle f_0, f_1\rangle$ of reals such that $\langle f_0, f_1\rangle$ is generic with respect to $\Vdash^{\sigma\prime}$ and $\omega_1[E_\sigma, f_0] = \omega_1[E_\sigma, f_1] = \kappa$. We set $\boldsymbol{a}_0 = \deg_{E_\sigma}(f_0)$ and $\boldsymbol{a}_1 = \deg_{E_\sigma}(f_1)$. Recall that $\boldsymbol{0}^{(\nu)} = \deg_{E_\sigma}(H_\nu)$, where $\{H_\nu : \nu < \kappa\}$ is the hierarchy for E_σ-recursive sets obtained in § 2. By Lemma 5.7, $\boldsymbol{0}^{(\nu)} < \boldsymbol{a}_i$ $(\nu < \kappa, i \leqq 1)$. If $\boldsymbol{b} \leqslant \boldsymbol{a}_0$ and $\boldsymbol{b} \leqslant \boldsymbol{a}_1$, then $\boldsymbol{b} \leqslant \boldsymbol{0}^{(\nu)}$ for some $\nu < \kappa$ by (2). Thus we have proved (i) and (ii). (iii) is clear from Lemma 5.7. Q.E.D.

COROLLARY 5.10. *For each $\sigma \leqq \omega$, there are E_σ-degree $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ such that $\{\boldsymbol{a}_0, \boldsymbol{a}_1\}$ does not have the greatest lower bound.*

Proof. LET $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ be as in Theorem 5.9. If $\boldsymbol{b} \leqslant \boldsymbol{a}_0$ and $\boldsymbol{b} \leqslant \boldsymbol{a}_1$, then $\boldsymbol{b} \leqslant \boldsymbol{0}^{(\nu)}$ for some $\nu < \omega_1[E_{\sigma+1}]$. Then $\boldsymbol{b} < \boldsymbol{0}^{(\nu+1)}$ and $\boldsymbol{0}^{(\nu+1)} \leqslant \boldsymbol{a}_i$ $(i = 0, 1)$. Thus $\boldsymbol{b}$ is not the greatest lower bound of $\{\boldsymbol{a}_0, \boldsymbol{a}_1\}$. Q.E.D.

COROLLARY 5.11. *For each $\sigma \leqq \omega$, the set $\{\boldsymbol{0}^{(\nu)} : \nu < \omega_1[E_{\sigma+1}]\}$ does not have the least upper bound.*

Proof. Let $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ be as in Theorem 5.9. Then each $\boldsymbol{a}_i$ is an upper bound of the set $\{\boldsymbol{0}^{(\nu)} : \nu < \omega_1[E_{\sigma+1}]\}$. If $\boldsymbol{b} \leqslant \boldsymbol{a}_i$ $(i = 0, 1)$, then $\boldsymbol{b}$ can not be an upper bound of $\{\boldsymbol{0}^{(\nu)} : \nu < \omega_1[E_{\sigma+1}]\}$ as is known from the proof of 5.10. Q.E.D.

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