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TOPOLOGICALLY STABLE HOMEOMORPHISMS OF THE CIRCLE

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Introduction

In this paper, we study topologically stable homeomorphisms of the circle. Our results are the following:

THEOREM 1. *A homeomorphism of the circle is topologically stable if and only if it is topologically conjugate to some Morse-Smale diffeomorphism.*

THEOREM 2. *There exists a homeomorphism of the circle which has the pseudo-orbit-tracing-property but is not topologically stable.*

In [1], Bowen introduced the concept of the pseudo-oribt-tracing-property and essentially showed that expansive homeomorphisms with this property are topologically stable. Recently in [2], Morimoto has proved that the topological stability implies the pseudo-orbit-tracing-property. Theorem 2 above shows that expansiveness condition is necessary in Bowen's result.

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Notations and Definitions

Let X be a compact metric space with metric d. For continuous maps h_1, h_2 from X to itself, we set

$$\bar{d}(h_1, h_2) = \sup_{x \in X} d(h_1(x), h_2(x)) .$$

With this metric, the set of all continuous maps from X to itself is a metric space. Let f and g be homeomorphisms of X.

DEFINITION 1. We say that g is *topologically conjugate* (resp. *semi-conjugate*) to f by h, if h is a homeomorphism (resp. a continuous map)

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from X onto itself satisfying $h \circ g = f \circ h$.

If g is topologically conjugate (resp. semi-conjugate) to f by h, g^n is also topologically conjugate (resp. semi-conjugate) to f^n by h for any integer n.

DEFINITION 2. We call f *topologically stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every homeomorphism f' with $\bar{d}(f, f') < \delta$ is topologically semi-conjugate to f by some h satisfying $\bar{d}(h, \mathrm{id}_X) < \varepsilon$.

DEFINITION 3. A sequence $\{x_n\}_{n \in Z}$ of points in X is called a *δ-pseudo orbit* of f if $d(f(x_n), x_{n+1}) < \delta$ for any integer n, and is said to be *ε-traced* by a point x of X if $d(f^n(x), x_n) < \varepsilon$ for any integer n.

DEFINITION 4. We say that f has the *pseudo-orbit-tracing-property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of f is ε-traced by some point.

Consider S^1 as $\boldsymbol{R}/\boldsymbol{Z}$ and let d denote the standard metric on S^1. The set of all fixed points of f and the set of all periodic points of f are denoted by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$, respectively.

DEFINITION 5. A diffeomorphism f of S^1 is called *Morse-Smale* if $\mathrm{Per}(f)$ is non-empty and every element of $\mathrm{Per}(f)$ is hyperbolic, i.e., the differential of f^n at x is different from ± 1 for every periodic point x of f with period n.

DEFINITION 6. Let f be an orientation preserving homeomorphism of S^1. A fixed point x of f is said to be *topologically hyperbolic* if x is isolated in $\mathrm{Fix}(f)$ and $f(t) - t$ changes its sign at $t = x$. A periodic point x of any homeomorphism g of S^1 is called *topologically hyperbolic* if x is a topologically hyperbolic fixed point of g^{2n}, where n is a period of x.

Proof of Theorem 1

By a theorem of Nitecki [3], every Morse-Smale diffeomorphism is topologically stable. Since the topological stability is invariant under the topological conjugacy, every homeomorphism which is topologically conjugate to some Morse-Smale diffeomorphism is topologically stable.

It is easy to see that a homeomorphism f of S^1 is topologically conjugate to some Morse-Smale diffeomorphism if and only if it satisfies the following two conditions:

(a) $\mathrm{Per}(f)$ is non-empty and finite.

(b) Every element of $\mathrm{Per}(f)$ is topologically hyperbolic.

Hence, to prove Theorem 1, it suffices to show that every topologically stable homeomorphism of S^1 satisfies the above conditions (a) and (b). First we prove the following

LEMMA 1. *Suppose f and g are orientation preserving homeomorphisms of S^1 and g is topologically semi-conjugate to f by some h. Then if* $\mathrm{Per}(g)$ *is non-empty, so is* $\mathrm{Per}(f)$, *and there exists a constant C depending only on f such that if h satisfies $\bar{d}(h, \mathrm{id}_{S^1}) \leq C$, the cardinality of* $\mathrm{Per}(g)$ *is not less than that of* $\mathrm{Per}(f)$.

Proof. Since for every homeomorphism f' of S^1 there exists an integer n such that $\mathrm{Per}(f') = \mathrm{Fix}(f'^n)$, it is enough to prove Lemma 1 replacing $\mathrm{Per}(\ )$ by $\mathrm{Fix}(\ )$.

The proof of the first part is immediate. To prove the second part, we take a positive constant C satisfying the following two conditions:

(1) $C \leq 1/8$.

(2) If I is a closed interval in S^1 of length not greater than $4C$, then the length of $f(I)$ is not greater than $1/4$.

Suppose g is topologically semi-conjugate to f by a map h with $\bar{d}(h, \mathrm{id}_{S^1}) \leq C$ and take a fixed point x of f. Since $h^{-1}(x)$ is a non-empty closed g-invariant subset of S^1 contained in $[x - C, x + C]$, we can define $\sup h^{-1}(x)$ and $\inf h^{-1}(x)$ without confusion. Put $B = [\inf h^{-1}(x), \sup h^{-1}(x)]$. Then we have

$$(\text{length of } B) \leq 2C$$

and

$$\begin{aligned}(\text{length of } g(B)) &\leq (\text{length of } h \circ g(B)) + 2\bar{d}(h, \mathrm{id}_{S^1}) \\ &\leq (\text{length of } f \circ h(B)) + 2C \\ &\leq \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.\end{aligned}$$

For the last inequality, we used the condition (2) and the estimation:

$$(\text{length of } h(B)) \leq (\text{length of } B) + 2\bar{d}(h, \mathrm{id}_{S^1}) \leq 4C.$$

Hence

$$(\text{length of } B) + (\text{length of } g(B)) \leq 2C + \tfrac{1}{2} \leq \tfrac{3}{4}.$$

By the invariance of $h^{-1}(x)$, both end points of $g(B)$ are in B and those of B in $g(B)$. Because the total length of S^1 is one, the above estimation shows that B is g-invariant. Since g is orientation preserving, $\sup h^{-1}(x)$ is a fixed point of g. Hence we have an injection from $\mathrm{Fix}(f)$

to Fix(g), which maps x to $\sup h^{-1}(x)$. This completes the proof of Lemma 1.

Now suppose that f is topologically stable. Since every homeomorphism of S^1 can be approximated by a diffeomorphism, by a theorem of Peixoto ([4]; p. 51), f is approximated by a Morse-Smale diffeomorphism. Thus there exists a Morse-Smale diffeomorphism which is topologically semi-conjugate to f. Therefore, by Lemma 1, Per(f) is non-empty and finite. (If f is orientation reversing, apply Lemma 1 to f^2.)

Next, take a periodic point x of f. If x is not topologically hyperbolic, we can eliminate this periodic point by a small perturbation, this contradicts Lemma 1. Therefore f satisfies the conditions (a) and (b), this completes the proof of Theorem 1.

Proof of Theorem 2

First consider a homeomorphism f_0 of $[0, 1]$ defined by

$$f_0(t) = \begin{cases} \frac{1}{2}t\,, & 0 \leq t \leq \frac{1}{4}\,, \\ \frac{3}{2}t - \frac{1}{4}\,, & \frac{1}{4} \leq t \leq \frac{3}{4}\,, \\ \frac{1}{2}t + \frac{1}{2}\,, & \frac{3}{4} \leq t \leq 1\,. \end{cases}$$

Let $p_n = 1/2^n$, $p_{-n} = 1 - 1/2^n$, $q_n = (1/2^{n+1})(1 + 1/2)$ and $q_{-n} = 1 - (1/2^{n+1})(1 + 1/2)$ for a positive integer n. Then the desired homeomorphism f of $S^1 = [0, 1]/\sim$ is given as follows:

$$f(0) = 0\,,$$

$$f(x) = \begin{cases} p_{n+1} + (1/2^{n+1})f_0(2^{n+1}(x - p_{n+1}))\,, & p_{n+1} \leq x \leq p_n\,, \\ p_{-n} + (1/2^{n+1})f_0(2^{n+1}(x - p_{-n}))\,, & p_{-n} \leq x \leq p_{-n-1}\,, \end{cases}$$

$$n = 1, 2, \cdots .$$

Since Fix(f) is an infinite set, Theorem 1 implies that f is not topologically stable. So we have only to show that f has the pseudo-orbit-tracing-property.

LEMMA 2. *For a real number k, let L_k be the linear map from $\boldsymbol{R}$ to itself defined by $L_k(x) = kx$. Suppose $\{x_n\}$ is a δ-pseudo-orbit of L_k with $x_0 \in [-M, M]$.*

(i) *If $0 < k < 1$ and $\delta \leq (1 - k)M$, then $x_n \in [-M, M]$ for every $n \geq 0$.*

(ii) *If $k > 1$ and $\delta \leq (k - 1)M$, then $x_n \in [-M, M]$ for every $n \leq 0$.*

The proof is immediate and is omitted.

Fix an arbitrary positive number ε and choose a positive integer n satisfying $1/2^n < \varepsilon$. Let $I = [p_{-n}, p_n]$, $J = [p_{n+1}, p_{-n-1}]$ and $J' = [p_{n+2}, p_{-n-2}]$. Then $J \subset J'$ and $I \cup J = S^1$. By the definition of f, there exists a homeomorphism $\tilde{f}$ of S^1, which is topologically conjugate to some Morse-Smale diffeomorphism and satisfies $\tilde{f}|_{J'} = f|_{J'}$.

Now we take a constant δ satisfying the following two conditions:

(1) Every δ-pseudo-orbit of $\tilde{f}$ can be $1/2^{n+2}$-traced by some point.

(2) $\delta \leq 1/2^{n+4}$.

Suppose $\{x_n\}$ is a δ-pseudo-orbit of f. Then we can show that $\{x_n\}$ is ε-traced by some point as follows:

Case 1. For every integer n, x_n is in I.

It is evident that $0 \in S^1$ ε-traces this sequence with respect to f.

Case 2. There exists m such that $x_m \notin I$.

By Lemma 2 together with the condition (2), this sequence does not jump over the intervals $[p_n - 1/2^{n+3}, p_n + 1/2^{n+3}]$ and $[p_{-n} - 1/2^{n+3}, p_{-n} + 1/2^{n+3}]$ in the positive direction and the intervals $[q_n - 1/2^{n+3}, q_n + 1/2^{n+3}]$ and $[q_{-n} - 1/2^{n+3}, q_{-n} + 1/2^{n+3}]$ in the negative direction. Therefore this sequence always stays in J, and is a δ-pseudo-orbit of $\tilde{f}$. By the condition (1), there exists a point x of S^1 which $1/2^{n+2}$-traces this sequence with respect to $\tilde{f}$. In particular x is in the $1/2^{n+2}$-neighborhood of x_0, hence in J'. It follows that x ε-traces this sequence with respect to f.

Thus we have shown that every δ-pseudo-orbit of f can be ε-traced by some point. This completes the proof of Theorem 2.

References

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