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CUSP FORMS OF WEIGHT 3/2

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Introduction

In this paper we deal with the problem (C) in § 4 of [4]. Let I_k be the Shimura mapping in [4] of $S_k(4N, \chi)$ into $\mathfrak{G}_{k-1}(N', \chi^2)$ (see p. 458). The problem (C) can be stated as follows: $I_3(f)$ is a cusp form if and only if $\langle f, h\rangle = 0$ for all $h \in U$, where U is the vector space spanned by every theta series of $S_3(4N, \chi)$ associated with some Dirichlet character.

Further, Niwa [2] proved that $2N$ can be taken as N' under the assumption that $k \geqq 7$; that is $I_k(S_k(4N, \chi)) \subseteqq \mathfrak{S}_{k-1}(2N, \chi^2)$.

§ 1 and § 2 are preparatory sections. In § 1 we show a characterization of integral modular cusp forms by means of the holomorphy of certain Dirichlet series. In § 2 we shall extend Niwa's result to the case, where the weight $k/2$ is not less than 3/2. In particular, we show that $I_3(S_3(4N, \chi)) \subseteqq \mathfrak{G}_2(2N, \chi^2)$ there.

In § 3, by using those results in § 1 and § 2, we prove the following theorem.

THEOREM. *If N is odd and square-free. Then the following two statements are equivalent.*

(A) *$I_3(f)$ is a cusp form.*

(B) *For every odd Dirichlet character ψ, $\langle f, h(z: \psi)\rangle = 0$.*
where $h(z; \psi)$ is a theta series associated with ψ defined in Lemma 3.1 *in* § 3.

Moreover, as an application of the above theorem we obtain the following:

THEOREM. *If N is odd and square-free and if χ_4, (defined in* § 3*), is trivial, then $I_3(S_3(4N, \chi)) \subseteqq \mathfrak{S}_2(2N, \chi^2)$.*

This theorem gives a partial answer to the problem (C) in [4].

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§1. A characterization of cusp forms

Let N be a positive integer and let χ be a Dirichlet character modulo N. Put

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c \equiv 0 \pmod{N}\right\}.$$

We consider an integral modular form $f(z)$ satisfying $f(\gamma(z))=\chi(d)(cz+d)^k f(z)$ for all $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. We denote by $\mathfrak{G}_k(N,\chi)$ the space of integral modular forms of Neben-type χ and of weight k with respect to $\Gamma_0(N)$ and by $\mathfrak{S}_k(N,\chi)$ the subspace of cusp forms in $\mathfrak{G}_k(N,\chi)$. In §2 and §3 we shall treat modular forms of half integral weight. As the definition of such modular forms and their basic properties, we may refer to Shimura [4].

Let $f(z)=\sum_{n=1}^{\infty} a_n e(nz)$ be the Fourier expansion of $f \in \mathfrak{G}_k(N,\chi)$ at ∞, where $e(z)=\exp(2\pi iz)$ and let ψ be a Dirichlet character. We now form the Dirichlet series

$$L(s;f,\psi)=\sum_{n=1}^{\infty} \psi(n) a_n n^{-s}.$$

Then we can prove the following theorem.

THEOREM 1. *Suppose that N is square-free. Then the following two statements are equivalent to each other:*

(A) *$f(z)$ is a cusp form.*

(B) *For every Dirichlet character ψ, $L(s;f,\psi)$ is holomorphic at $s=k$.*

To prove this theorem, we need some preparations. Let $L(s,\phi)$ be the Dirichlet L-function associated with a Dirichlet character ϕ. The following lemma is well-known.

LEMMA 1.1. *If ϕ is trivial, then $L(s,\phi)$ is a simple pole at $s=1$. If ϕ is non-trivial, then $L(s,\phi)$ is holomorphic at $s=1$ and $L(1,\phi)\neq 0$.*

Next we state some properties of Eisenstein series (cf. [1]). Let χ_1 (resp. χ_2) be a character modulo M_1 (resp. M_2) with $\chi=\chi_1\chi_2$. And let $\{\chi_1, \chi_2, \ell\}$ be a triplet satisfying $\ell M_1 M_2 | N$ and the following condition:

(*) If $k = 2$ and both χ_1 and χ_2 are trivial, $M_1 = 1$ and M_2 is square-free. If otherwise, χ_1 and χ_2 are primitive.

We consider the sequence $\{a_n(\chi_1, \chi_2)\}_{n=1}^{\infty}$ determined by

$$L(s, \chi_1)L(s - k + 1, \chi_2) = \sum_{n=1}^{\infty} a_n(\chi_1, \chi_2)n^{-s}. \tag{1.1}$$

Let $E(z: \chi_1, \chi_2)$ be the modular form associated with the Dirichlet series (1.1). We summarize well-known facts as the following lemma (cf. [1]).

LEMMA 1.2 (Hecke). *Consider triplets $\{\chi_1, \chi_2, \ell\}$ satisfying the condition (*). Then modular forms $E(\ell z: \chi_1, \chi_2)$ are linearly independent and*

$$\mathfrak{G}_k(N, \chi) = \mathfrak{E}_k(N, \chi) \oplus \mathfrak{S}_k(N, \chi),$$

where $\mathfrak{E}_k(N, \chi)$ denotes the vector space spanned by the above modular forms over $\mathbf{C}$. Moreover, $E(\ell z: \chi_1, \chi_2)$ is an eigenfunction of Hecke operators $T(n)((n, N) = 1)$ and $E(\ell z: \chi_1, \chi_2)T(n) = a_n(\chi_1, \chi_2)E(\ell z: \chi_1, \chi_2)$.

Here we note that $\{a_n(\chi_1, \chi_2)\}_{n=1}^{\infty}$ has the following property:

$$\text{If } a_n(\chi_1, \chi_2) = a_n(\chi_1', \chi_2')((n, N) = 1), \qquad \text{then } \chi_i = \chi_i'(i = 1, 2).$$

Now we can give a proof of Theorem 1. It is easy to derive (B) from (A) (cf. [3]). Next we assume (B). For the simplicity, we suppose that $k > 2$ or if $k = 2$, χ is non-trivial. We can put

$$f(z) = \sum_{\chi_1, \chi_2, \ell} c(\ell: \chi_1, \chi_2)E(\ell z: \chi_1, \chi_2) + g(z), \tag{1.2}$$

where $g(z)$ is a cusp form.

If $\{\chi_1, \chi_2\}$ is fixed, it is sufficient to verify

$$c(\ell: \chi_1, \chi_2) = 0 \qquad \text{for every } \ell(\ell M_1 M_2 | N). \tag{**}$$

We shall prove this by means of induction with respect to the number t of prime factors of ℓ. First we consider the case $t = 0$. By virtue of (1.2), we have

$$\begin{aligned} L(s: f, 1_N\bar{\chi}_2) = \sum_{\chi_1', \chi_2'} & c(1: \chi_1', \chi_2')L(s, 1_N\bar{\chi}_2\chi_1')L(s - k + 1, 1_N\bar{\chi}_2\chi_2') \\ & + L(s: g, 1_N\bar{\chi}_2), \end{aligned}$$

where 1_N is the trivial character modulo N. If $(\chi_1', \chi_2') \neq (\chi_1, \chi_2)$, then $L(s, 1_N\bar{\chi}_2\chi_1')L(s - k + 1, 1_N\bar{\chi}_2\chi_2')$ is holomorphic at $s = k$ and, if otherwise, $L(s, 1_N\bar{\chi}_2\chi_1')L(s - k + 1, 1_N\bar{\chi}_2\chi_2')$ has a simple pole at $s = k$. Since both

$L(s: f, 1_N\bar{\chi}_2)$ and $L(s: g, 1_N\bar{\chi}_2)$ are holomorphic at $s = k$, we have $c(1: \chi_1\chi_2) = 0$. Therefore (**) holds for $t = 0$.

Next suppose that (**) holds for $t = 0, 1, \cdots, n-1$ and n. We set $\ell = p\tilde{\ell}$, where $\tilde{\ell} = 1$ or $p_1p_2\cdots p_n$ and $p_1, p_2, \cdots, p_n$ are primes. Put $L = N/\ell M_1M_2$ and $\psi = 1_N\bar{\chi}_2$. By (1.2) and the assumption of the induction, we see

$$\begin{aligned} L(s: f, \psi) &= c(\ell: \chi_1, \chi_2)L(s: E(\ell z: \chi_1, \chi_2), \psi) \\ &\quad + \sum_{(\chi_1', \chi_2') \neq (\chi_1, \chi_2), \ell'} c(\ell': \chi_1', \chi_2')L(s: E(\ell' z: \chi_1', \chi_2'), \psi) \\ &\quad + L(s: g, \psi)\,. \end{aligned}$$

Now we have

$$L(s: E(\ell z: \chi_1, \chi_2), \psi) = \psi(\ell)\ell^{-s}L(s, 1_L\chi_1\bar{\chi}_2)L(s-k+1, 1_{LM_2})\,. \tag{1.3}$$

Since $L(s, E(\ell' z: \chi_1', \chi_2'), \psi) = \psi(\ell')(\ell')^{-s}L(s, \psi\chi_1')L(s-k+1, \psi\chi_2')$, $L(s, E(\ell' z: \chi_1', \chi_2'), \psi)$ is holomorphic at $s = k$. So we obtain $c(\ell: \chi_1, \chi_2) = 0$. Therefore we see that (**) holds for $t = n+1$. This completes the proof of Theorem 1.

§ 2. A complement to a result of Niwa [2]

First we recall the results of Niwa [2]. Let N be a positive integer and let χ be a Dirichlet character modulo $4N$. For an odd integer $k(\geqq 3)$, define by $k = 2\lambda + 1$ and put $\chi_1(*) = \chi(*)\left(\frac{-1}{*}\right)^{\lambda}$. We define f_λ on $\boldsymbol{R}^3$ by

$$f_\lambda(x_1, x_2, x_3) = (x_1 - ix_2 - x_3)^\lambda \exp\left((-2/N)(2x_1^2 + x_2^2 + 2x_3^2)\right).$$

We also define $\theta(z, g)$ on $\mathfrak{H} \times SL_2(\boldsymbol{R})$ by

$$\theta(z, g) = \sum_{(x_1, x_2, x_3) \in L} \bar{\chi}_1(x_1)v^{(3-k)/4} \exp\left(2\pi i(u/N)(x_2^2 - 4x_1x_3)\right)f_\lambda(\sqrt{v}\,\rho(g^{-1})x)\,,$$

where $z = u + iv$, $L = \boldsymbol{Z} \oplus N\boldsymbol{Z} \oplus (N/4)\boldsymbol{Z}$ and

$$\rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)x = x\begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad+bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix}.$$

Then we have

$$\theta(\sigma(z), g) = \bar{\chi}(d)\left(\frac{N}{d}\right)j(\sigma, z)^k\theta(z, g)$$

for every $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$. Here the Petersson inner product

$$F(g) = \int_{D_0(4N)} v^{k/2}\bar{\theta}(z, g)F(z)\frac{dudv}{v^2}$$

is well-defined, where $F(z) \in S_k\left(4N, \bar{\chi}\left(\frac{4N}{*}\right)\right)$ and $D_0(4N)$ is a fundamental region for $\Gamma_0(4N)$. The following lemma is due to [2] and [6].

LEMMA 2.1. *The function $F(g)$ has the following properties*:

(1) *$F(g)(\in C^\infty(SL_2(\boldsymbol{R})))$ is an eigenfunction of the Casimir operator D_g, that is, $D_g F = \lambda(\lambda - 1)F$, where*

$$D_g = \frac{1}{4}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 + 2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + 2\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right),$$

(2) $F\left(g\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = \exp(2\lambda\theta\sqrt{-1})F(g),$

and

(3) $F(\gamma g) = \chi^2(d)F(g)$

$$\textit{for every } \gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \begin{pmatrix}2 & 0\\ 0 & 1/2\end{pmatrix}\Gamma_0(2N)\begin{pmatrix}1/2 & 0\\ 0 & 2\end{pmatrix}.$$

We define two functions $\Psi(w)$ and $\Phi(w)$ $(w = \xi + i\eta \in \mathfrak{H})$ by

$$\Psi(w) = F\left(\begin{pmatrix}2 & 0\\ 0 & 1/2\end{pmatrix}\begin{pmatrix}\eta^{1/2} & \xi\eta^{-1/2}\\ 0 & \eta^{-1/2}\end{pmatrix}\right)(4\eta)^{-\lambda}$$

and

$$\Phi(w) = \psi(-1/2Nw)(2N)^\lambda(-2Nw)^{-2\lambda}.$$

Before stating our result, we recall the definition of the Shimura mapping.

Let W be the isomorphism of $S_k\left(4N, \bar{\chi}\left(\frac{N}{*}\right)\right)$ onto $S_k(4N, \chi)$ defined by

$$G(z) = W(F(z)) = F(-1/4Nz)(4N)^{-k/4}(-iz)^{-k/2}$$

for all $F(z) \in S_k\left(4N, \bar{\chi}\left(\frac{N}{*}\right)\right)$. Then $G(z)$ has the Fourier expansion

$$G(z) = \sum_{n=1}^\infty a(n)e(nz)$$

at ∞. Determine the sequence $\{A(n)\}_{n=1}^\infty$ by the relation

$$\sum_{n=1}^\infty A(n)n^{-s} = L(s - \lambda + 1, \chi_1)\sum_{n=1}^\infty a(n^2)n^{-s},$$

where $G(z) = \sum_{n=1}^\infty a(n)e(nz)$. We can define the Shimura mapping $I_k(k \geqq 3)$ by

$$I_k(G(z)) = \sum_{n=1}^{\infty} A(n)e(nz) \qquad \text{for } G(z) \in S_k(4N, \chi)\,.$$

Shimura [4] showed $I_k(S_k(4N, \chi)) \subseteqq \mathfrak{G}_{k-1}(N', \chi^2)$ for some N' and he also conjectured that $2N$ is taken as N'. Now we define another mapping $\tilde{I}_k$ of $S_k(4N, \chi)$ into $C^\infty(\mathfrak{H})$ by $\tilde{I}_k(G(z)) = \Phi(w)$, where $G(z) = W(F(z))$. Then, under the condition $k \geqq 7$, the above conjecture was proved by Niwa [2] as follows.

THEOREM. *If $k \geqq 7$, then $\Phi(w)$ belongs to $\mathfrak{S}_{k-1}(2N, \chi^2)$ and*

$$\Phi(w) = \tilde{I}_k(G(z)) = cI_k(G(z))\,,$$

where

$$c = i^{k-1}N^{k/4}2^{(-9k+15)/4}\,\mathrm{Re}\,((2-i)^{(k-1/2)})\,.$$

Now we shall prove the following:

THEOREM 2. *If $k \geqq 3$, then $\Phi(w)$ belongs to $\mathfrak{G}_{k-1}(2N, \chi^2)$ and $\Phi(w) = \tilde{I}_k(G(z)) = cI_k(G(z))$. Moreover, if $k \geqq 5$, then $\Phi(w)$ belongs to $\mathfrak{S}_{k-1}(2N, \chi^2)$.*

Proof. First we prove that Φ is holomorphic on $\mathfrak{H}$. Though our method is adaptable to all the cases, we assume $k = 3$ for the simplicity. By virtue of Lemma 2.1 and by the invariance of the Casimir operator D_g, we have

$$\left\{\eta^2\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) - 2i\eta\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)\right\}\Phi(w) = 0\,. \tag{2.1}$$

Now $\Phi(w)$ has the Fourier expansion

$$\Phi(w) = \sum_{m=-\infty}^{\infty} a_m(\eta)\exp(2\pi i m\xi)$$

at ∞. So $a_m(\eta)$ is a solution of the differential equation

$$\left\{\frac{d^2}{d\eta^2} + \frac{2}{\eta}\frac{d}{d\eta} + (-4\pi^2m^2 + 4\pi m/\eta)\right\}a_m(\eta) = 0\,. \tag{2.2}$$

Therefore, we obtain

$$a_m(\eta) = \begin{cases} b_m\exp(-2\pi m\eta) + c_m u_m(\eta), & \text{if } m \neq 0,\\ b_0 + c_0\eta^{-1}\,, & \text{if } m = 0\,,\end{cases}$$

where

$$u_m(\eta) = \begin{cases} \exp(-2\pi m\eta)\int_1^\eta \eta^{-2}\exp(4\pi m\eta)d\eta\,, & \text{if } m > 0\,, \\ \exp(-2\pi m\eta)\int_\eta^\infty \eta^{-2}\exp(4\pi m\eta)d\eta\,, & \text{if } m < 0\,. \end{cases}$$

By integration by parts, we have the following asymptotic behaviors of $u_m(\eta)$:

$$(2.3) \qquad |u_m(\eta)| \geqq (4\pi m - \pi)^{-1}\exp(-2\pi m\eta)|\exp((4\pi m - \pi)\eta) - \exp(4\pi m - \pi)|$$

for $m > 0$,

$$(2.3)' \qquad u_m(\eta) = -\exp(2\pi m\eta)/4\pi m\eta^2 + \alpha_m(\eta) \qquad \text{for } m < 0\,,$$

where $|\alpha_m(\eta)| \leqq \exp(2\pi m\eta)(1/8\pi^2|m^2|\eta^3 + 15/32\pi^3|m^3|\eta^4)$.
Moreover we have

$$(2.3)'' \qquad \eta\Phi(w) = O(\eta + \eta^{-1})(\eta \longrightarrow 0 \text{ and } \eta \longrightarrow \infty)$$

uniformly in ξ. Since

$$\int_0^1 \eta^2|\Phi(w)|^2\, d\xi = \sum_{m=-\infty}^{\infty} |a_m(\eta)|^2\eta^2\,,$$

we obtain from (2.3)''

$$(2.4) \qquad |a_m(\eta)| \leqq M((\eta + \eta^{-1})\eta^{-1})\,,$$

where M is independent of m and η. Hence, by (2.3) and (2.3)', we have $c_m = 0 (m > 0)$ and $b_m = 0 (m < 0)$. Consequently, we see

$$(2.5) \qquad \begin{aligned} \Phi(w) = &\sum_{m=1}^{\infty} b_m \exp(-2\pi m\eta)\exp(2\pi i m\xi) \\ &+ \sum_{m=1}^{\infty} c_{-m}u_{-m}(\eta)\exp(-2\pi i m\xi) + a_0(\eta)\,. \end{aligned}$$

By (2.4), we have $|a_m(1/|m|)| \leqq M(1 + m^2)$. Hence we obtain $b_m = O(m^\nu)$ $(m \to \infty)$ and $c_{-m} = O(m^\nu)$ $(m \to \infty)$ for some $\nu > 0$. We see that $\Phi(i\eta)$ has the following asymptotic behavior:

$$(2.6) \qquad \Phi(i\eta) = \begin{cases} O(\eta^{-\mu}) \ (\eta \to +\infty)\,, & \text{for all } \mu > 0\,, \\ O(\eta^{\mu}) \ \ (\eta \to 0)\,, & \text{for some } \mu > 0\,, \end{cases}$$

(see pp. 158–159 in [2] and [4]). In particular, we see $a_0(\eta) = 0$. Hence we see

$$\Phi(w) = \sum_{m=1}^{\infty} b_m \exp(-2\pi m\eta) \exp(2\pi i m\xi) \tag{2.5$'$}$$
$$+ \sum_{m=1}^{\infty} c_{-m} u_{-m}(\eta) \exp(-2\pi i m\xi) .$$

By virtue of (2.6), $\Phi(i\eta)\,\eta^{\ell-1}$ belongs to $L_1(\boldsymbol{R}^+)$ for a sufficiently large $\ell > 0$. Let $\Omega(s)$ be the Mellin transformation of $\Phi(i\eta)$, that is

$$\Omega(s) = \int_0^{\infty} \Phi(i\eta)\eta^{s-1} d\eta .$$

Here we note that $\Phi(i\eta)$ is a function with bounded variation on all compact sets of $\boldsymbol{R}^+$ and $\Phi(i\eta) = 1/2(\Phi(i(\eta+0)) + \Phi(i(\eta-0)))$ for all $\eta > 0$. Hence the Mellin inversion formula gives

$$\Phi(i\eta) = \frac{1}{2\pi i} \int_{\ell - i\infty}^{\ell + i\infty} \Omega(s)\eta^{-s} ds . \tag{2.7}$$

On the other hand, by the same computations as those of [2], we have

$$\Omega(s) = c(2\pi)^{-s}\Gamma(s)L(s, \chi_1) \sum_{n=1}^{\infty} a(n^2)n^{-s} ,$$
$$= (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} a'_n n^{-s} ,$$

where $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ and $c \neq 0$. Consequently, we obtain

$$\Phi(i\eta) = \sum_{n=1}^{\infty} a'_n \exp(-2\pi n\eta) . \tag{2.5$''$}$$

Therefore, by (2.5)$'$, to prove the holomorphy of $\Phi(w)$ it is sufficient to show that $c_{-m} = 0 (m \geqq 1)$. We assume $c_{-m_0} \neq 0$ and $c_{-m} = 0$ for all $m (< m_0)$. Then, by (2.5)$'$ and (2.5)$''$, we see

$$\begin{aligned} &\sum_{m > m_0} c_{-m} u_{-m}(\eta)/H_{m_0}(\eta) + c_{-m_0} u_{-m_0}(\eta)/H_{m_0}(\eta) \\ &\quad = \sum_{n=1}^{\infty} (a'_n - b_n) \exp(-2\pi n\eta)/H_{m_0}(\eta), \end{aligned} \tag{2.8}$$

where $H_{m_0}(\eta) = \exp(-2\pi m_0 \eta)/4\pi m_0 \eta^2$.

We note that the series of both sides of (2.8) are uniformly convergent on $[1, \infty)$. Set $t = \exp(-2\pi\eta)$ $(\eta > 0)$. The right hand side of (2.8) equals

$$\frac{m_0}{\pi} (\log t)^2 \sum_{n=1}^{\infty} (a'_n - b_n) t^{(n - m_0)} .$$

By virtue of (2.3)$'$, we see that the left hand side of (2.8) converges to c_{m_0} as $\eta \to +\infty$. Hence we have

$$\lim_{\substack{t\to 0\\ t>0}} \left\{ \frac{m_0}{\pi} (\log t)^2 \sum_{n=1}^{\infty} (a_n' - b_n) t^{(n-m_0)} \right\} = c_{-m_0} (\neq 0)\,.$$

This is a contradiction and we obtain the holomorphy of $\Phi(w)$. Since the remainders of our assertions can be proved in the same manner as that of [2], we omit the proof.

§3. Shimura mapping in the case of weight 3/2

First we shall prove the following:

THEOREM 3. *Let N be odd and square-free and suppose $k = 3$. Then the following two statements are equivalent*:

(A) *$\Phi(w)$ is a cusp form.*

(B) $< G(z),\ h(z: \bar{\psi}) > = 0$ *for every Dirichlet character ψ with trivial $\chi\left(\frac{-1}{*}\right)\psi$, where $\langle\, ,\rangle$ denotes the Petersson inner product.*

To show this, we prepare two lemmas.

LEMMA 3.1. *Let χ be a Dirichlet character modulo N. Define $\nu \in \{0, 1\}$ by $\chi(-1) = (-1)^\nu$. Then $h(z: \chi) = 1/2 \sum_{m=-\infty}^{\infty} \chi(m) m^\nu e(m^2 z)$ belongs to $G_{2\nu+1}(4N^2, \chi')$, where $\chi' = \chi\left(\frac{-1}{*}\right)^\nu$.*

Proof. If χ is primitive, this lemma was proved by Shimura [4]. If χ is not primitive, we set $\chi = 1_L \phi$, where L is square-free and ϕ is the primitive character associated with χ. Clearly L and the conductor of ϕ are coprime. Then we can prove the above lemma by means of induction with respect to the number of prime factors of L. We may omit the details of the proof. (Recalling that $G_3(4N, \chi) = 0$ if $\chi(-1) = -1$, we assume $\chi(-1) = 1$.)

LEMMA 3.2. *Let ψ be a character modulo M. Define $\hat{L}(s, \psi)$ by*

$$\hat{L}(s, \psi) = L(s: \Phi, \psi) = \sum_{n=1}^{\infty} \psi(n) A(n) n^{-s}\,.$$

If $\chi \neq \bar{\psi}_1$, then $\hat{L}(s, \psi)$ is holomorphic at $s = 2$, and if otherwise, $\hat{L}(s, \psi)$ has a simple pole at $s = 2$. Furthermore, in the latter case ($\chi = \bar{\psi}_1$), $\mathrm{Res}_{s=2}\, \hat{L}(s, \psi)$ equals $c' < G,\ h(z: \bar{\psi}) >$ for some $c' (\neq 0)$.

Proof. The method of the proof is the same as that of [4]. For a constant $\sigma > 0$, we have

$$\int_0^\infty \int_0^1 G(z)\bar{h}(z:\bar{\psi})y^{s-1}dxdy \tag{3.1}$$
$$= (4\pi)^{-s}\Gamma(s) \sum_{m=1}^{\infty} \psi(m)a(m^2)m^{\nu-2s},$$

where $s \in \boldsymbol{C}(\mathrm{Re}\, s > \sigma)$ and ν is defined by $\psi(-1) = (-1)^\nu$. Set $\tilde{M} = \ell.c.m(4M^2, 4N)$. We define $B(z, s)$ by $B(z, s) = G(z)\bar{h}(z:\bar{\psi})y^{s+1}$. By virtue of Lemma 3.1, we see

$$B(\gamma(z), s) = \left(\frac{-1}{d}\right)\psi\chi(d)(cz + d)^{1-\nu}|cz + d|^{2\nu-1-2s}B(z, s)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\tilde{M})$. Hence the left hand side of (3.1) equals

$$\int_D B(z, s)\left\{ \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \left(\frac{-1}{d}\right)\psi\chi(d)(cz + d)^{1-\nu}|cz + d|^{2\nu-1-2s}\right\}\frac{dxdy}{y^2},$$

where $\Gamma = \Gamma_0(\tilde{M})$ and D is a fundamental region for $\Gamma_0(\tilde{M})$. Hence we obtain

$$\begin{aligned} &L(2s - \nu, \Phi(w), \psi) \\ &\quad = L(2s - \nu - \lambda + 1, \psi\chi_1) \sum_{n=1}^{\infty} \psi(n)a(n^2)n^{-2s+\nu} \\ &\quad = \frac{1}{2}(4\pi)^s\Gamma(s)^{-1} \int_D B(z, s)L(2s - \nu - \lambda + 1, \psi\chi_1) \\ &\quad\quad \times \left\{ \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \psi\chi_1(d)(cz + d)^{1-\nu}|cz + d|^{2\nu-1-2s}\right\}\frac{dxdy}{y^2}. \end{aligned}$$

Now it is easy to see

$$\begin{aligned} &L(2s - \nu - \lambda + 1, \psi\chi_1) \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} \psi\chi_1(d)(cz + d)^{1-\nu}|cz + d|^{2\nu-1-2s} \\ &\quad = \frac{1}{2} \sum_{m,n}{}' \psi\chi_1(n)(\tilde{M}mz + n)^{1-\nu}|\tilde{M}mz + n|^{2\nu-1-2s}. \end{aligned}$$

We set $c(z, s)$ by

$$c(z, s) = \sum_{m,n}{}' \psi\chi_1(n)(\tilde{M}mz + n)^{1-\nu}|\tilde{M}mz + n|^{\nu-1-s}.$$

The following lemma is well-known (see Shimura [5]).

LEMMA 3.3. *$c(z, s)$ is holomorphic at $s = 2$, if $\psi\chi_1$ is non-trivial, $c(z, s)$*

has a simple pole at $s=2$ *and* $\operatorname{Res}_{s=2} c(z, s) = c''y^{-1}$ *for some* $c''(\neq 0)$, *if otherwise.*

Using the Lemma 3.3, we obtain Lemma 3.2. By Theorem 1, Theorem 2 and Lemma 3.2, we can easily prove Theorem 3 and we may omit the details of the proof.

Let N be odd and square-free and let χ be a character modulo $4N$. We define the isomorphism ϕ of $(Z/4NZ)^\times$ onto $(Z/4Z)^\times \times (Z/NZ)^\times$ by $\phi(a) = (a, a)$ for all $a \in (Z/4NZ)^\times$. Define χ_4 by $\chi_4(a) = \chi(\phi^{-1}(a, 1))$ for all $a \in (Z/4Z)^\times$. Under the above notations, we can prove the following theorem as an application of Theorem 3.

THEOREM 4. *Suppose that* χ_4 *is trivial. Then* $I_3(S_3(4N, \chi)) \subsetneqq \mathfrak{S}_2(2N, \chi^2)$.

Proof. Let $\{f_1, f_2, \cdots, f_n\}$ be a base of $S_3(4N, \chi)$ over C with $T_{3,\chi}^{4N}(p^2)f_i = w_p^{(i)}f_i (1 \leqq i \leqq n)((p, 4N) = 1)$. By Theorem 3, it is sufficient to show $\langle f_i, h(z: \bar{\psi})\rangle = 0$ for all characters ψ with $\bar{\psi} = \chi_1$ and for all i. Now assume $\langle f_{i_0}, h(z: \bar{\psi}_0)\rangle \neq 0$ for some $\psi_0 (\operatorname{mod} M)$ and some i_0. We set $\tilde{M} = \ell.c.m(4N, 4M^2)$. Then we have

$$\begin{aligned} w_p^{(io)}&\langle f_{i_0}, h(z: \bar{\psi}_0)\rangle \\ &= \langle T_{3,\chi}^{\tilde{M}}(p^2)f_{i_0}, h(z: \bar{\psi}_0)\rangle \\ &= \langle f_{i_0}, (T_{3,\chi}^{\tilde{M}}(p^2))^* h(z: \bar{\psi}_0)\rangle \\ &= \langle f_{i_0}, \bar{\chi}(p^2)T_{3,\chi}^{\tilde{M}}(p^2)h(z: \bar{\psi}_0)\rangle \\ &= \langle f_{i_0}, \bar{\chi}_1(p)(p+1)h(z: \bar{\psi}_0)\rangle \\ &= \chi_1(p)(p+1)\langle f_{i_0}, h(z: \bar{\psi}_0)\rangle \end{aligned}$$

for all primes p with $(p, \tilde{M}) = 1$.

By the above assumption, we obtain $w_p^{(io)} = \chi_1(p)(p+1)$ for all primes $p((p, \tilde{M}) = 1)$. Therefore, by the definition of the Shimura mapping, we see $T(p)I_3(f_{i_0}) = \chi_1(p)(p+1)I_3(f_{i_0})$ for all primes $p((p, \tilde{M}) = 1)$. Here we note that $I_3(f_{i_0})$ is not a cusp form. So we see that $I_3(f_{i_0})$ is a modular form associated with the Eisenstein series of $\mathfrak{G}_2(2N, \chi^2)$. By virtue of Lemma 1.2, we have $\chi_1(p)(p+1) = \phi(p) + p\phi'(p)$ for all primes $p((p, \tilde{M}) = 1)$, where ϕ (resp. ϕ') is a Dirichlet character modulo M_1 (resp. M_2) and M_1M_2 is a divisor of $2N$. So we have $\chi_1(p) = \phi(p)$ for almost all primes p. On the other hand, the conductor of χ_1 is a multiple of 4 and that of ϕ is odd. This is a contradiction and we obtain the theorem.

REFERENCES

[1] Doi-Miyake, Automorphic forms and number theory (Japanese) Kinokuniya Shoten, Tokyo (1976).
[2] S. Niwa, Modular forms of half integral weight and the integral certain theta-functions, Nagoya Math. J., **56** (1975), 147–161.
[3] J.-P. Serre and H. M. Stark, Modular forms of weight 1/2, Springer Lecture Notes, **627** (1977), 29–66.
[4] G. Shimura, On modular forms of half integral weight, Ann. of Math., **97** (1973), 440–481.
[5] ——, The special values of the zeta functions associated with cusp forms, Comm. Pure Appl. Math., **29** (1976), 783–804.
[6] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J., **58** (1975), 83–126.

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