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# A NOTE ON THOM CLASSES IN GENERAL COHOMOLOGY

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## § 1. Introduction

This note is dedicated to the second author's teacher, Professor Atsuo Komatsu, in celebration of his seventieth birthday.

It is well known [4], [7] that the theory of characteristic classes in general cohomology is essentially based upon one theorem, the Leray-Hirsch Theorem. We further claim [8] that the entire theory could be developed merely from the Künneth Formula if under suitable conditions a truly relative version of the Meyer-Vietoris sequence exists in the general cohomology theory. In his lectures delivered at Aarhus in 1968, Dold [6] set up the necessary machinery including the Leray-Hirsch Theorem to define Chern Classes with values in general cohomology. However, he then stated [6, p. 47] that he "found a difficulty here in choosing adequate orientations (Thom Classes) for the bundles involved", and proceeded differently, discarding the "classical" approach used in both ordinary cohomology [9] and $K$-theory [3]. Later, he [7] published a more categorical work, although the approach to Chern Classes was basically unchanged from his previous work. Consequently, the possibility of the classical direct approach to the theory has remained open.

The main purpose of this note is to overcome this difficulty. Thus the note starts with the definition of Chern Classes where Dold's work left off. Hopefully, the novelty of this note lies in the fact that the proof of the Stiefel-Whitney formula, and the selection of the appropriate Thom Class involve only the cohomology of the total space of the projective bundle associated with a complex vector bundle.

The numbers in the square brackets refer to the selected bibliography at the end of the paper.

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## § 2. Preliminaries

Let $W$ be the category of pairs of $CW$ complexes up to homotopy type with continuous functions as morphisms and let $A$ be the category of abelian groups. Let $h^* = \{h^n\}_{n\in Z}$ denote a general cohomology theory [1] from $W$ to $A$ satisfying the additive axiom. In addition, the cohomology theory is assumed to have either an external product "$\times$" or an internal product "$\cup$" and hence both [1], [7], [8]. For each pair of integers $p$, $q$ and pairs $(X, A)$ and $(Y, B)$ there is the external product,

$$\times : h^p(X, A) \otimes h^q(Y, B) \longrightarrow h^{p+q}(X\times Y, X\times B \cup A\times Y)\,,$$

satisfying naturality, associativity, commutativity, distributivity, and boundary properties. An internal product,

$$\cup : h^p(X, A) \otimes h^q(X, B) \longrightarrow h^{p+q}(X, A \cup B)\,,$$

satisfies the usual five properties mentioned above. Besides, $h$ has a unit $1 \in h^0(X)$. In this setting we may consider $h^*(X)$ for any $X$ in $W$ as a graded commutative ring with unit. In particular, $h^*(pt)$ is called the coefficient ring of the theory $h^*$ and will be denoted by $\Lambda$. For each pair $(X, A)$ in $W$ the external products,

$$h^*(pt) \otimes h^*(X, A) \longrightarrow h^*(pt\times X, pt\times A) = h^*(X, A)$$

and

$$h^*(X, A) \otimes h^*(pt) \longrightarrow h^*(X, A)$$

define both a left and a right $\Lambda$-scalar multiplication. By the associativity of the external product $h^*(X, A)$ becomes a $\Lambda$-$\Lambda$ bimodule. For the pairs $(X, A)$ and $(Y, B)$ we may define a graded abelian group denoted by $h^*(X, A) \otimes_\Lambda h^*(Y, B)$ and observe that the external product induces a graded abelian group homomorphism,

$$\overline{\times} : h^*(X, A) \underset{\Lambda}{\otimes} h^*(Y, B) \longrightarrow h^*(X\times Y, X\times B \cup A\times Y)\,.$$

Moreover, we may give $h^*(X, A) \otimes_\Lambda h^*(Y, B)$ a well defined $\Lambda$-$\Lambda$ bimodule structure such that the external product $\overline{\times}$ is a $\Lambda$-$\Lambda$ bimodule homomorphism. Similarly, the internal product induces $\Lambda$-$\Lambda$ bimodule homomorphisms, $h^*(X) \otimes_\Lambda h^*(X) \longrightarrow h^*(X)$ and $\Lambda \longrightarrow h^*(X)$. Consequently, $h^*(X)$ is indeed a graded algebra with unit over a graded commutative ring $\Lambda$ with unit.

Let $S^n$ denote the $n$-sphere with the base point $*$ and let $s_n \in h^n(S^n, *)$ be the image of $1 \in h^0(pt)$ via the composition of the $n$-fold suspension isomorphism $\sigma^n$ and the excision isomorphism,

$$h^*(pt) \underset{\simeq}{\longleftarrow} h^*(S^0, *) \underset{\simeq}{\overset{\sigma^n}{\longrightarrow}} h^*(S^n, *) .$$

Then, $h^*(S^n, *)$ is a free $\Lambda$-module with a base $\{s_n\}$. Since $s_n^2 = 0$ for $n \geqq 1$, $h^*(S^n)$ is a truncated polynomial algebra in one variable $u$ of degree $n$ over $\Lambda$ such that $u^2 = 0$, where $u = j^*(s_n)$ for the inclusion $j: (S^n, \emptyset) \to (S^n, *)$. Dold [6], [7] and Connell [4] proved the following Leray-Hirsch type theorem.

THEOREM 2.1. *Let $h^*$ be an additive cohomology theory with a product and a unit and let $p: E \to X$ be a fibration over an arcwise connected CW complex $X$ with a fiber $F$ in $W$. Suppose that $h^*(F)$ is a free $\Lambda$-module with a finite basis $\{b_1, \cdots, b_m\}$ of homogeneous elements. In addition, suppose that there is a set $\{e_1, \cdots, e_m\}$ of homogeneous elements in $h^*(E)$ and a collection $\{i_x: F \to E\}_{x \in X}$ of maps from $F$ onto the fibre over $x$ such that the induced homomorphism $i_x^*: h^*(E) \to h^*(F)$ maps $e_j$ to $b_j$ for each $j \in \{1, \cdots, m\}$ and for each $x \in X$. Then,*

$$\varphi_X: h^*(X) \underset{\Lambda}{\otimes} h^*(F) \longrightarrow h^*(E)$$

*defined by $\varphi_X(\alpha \otimes b_j) = p^*(\alpha)e_j$ for each $\alpha \in h^*(X)$ and for each $j$, is a $\Lambda$-module isomorphism. Hence $h^*(E)$ is a free $h^*(X)$-module with a base $\{e_1, \cdots, e_m\}$.*

Let $\xi: E \overset{\pi}{\longrightarrow} X$ be an $n$-complex vector bundle and let $E^0$ be the subspace of $E$ consisting of all non-zero vectors in $E$. Then there exists a linear isomorphism,

$$i_x: (C^n, C^n \sim \{0\}) \longrightarrow (\pi^{-1}(x), \pi^{-1}(x) \cap E^0) \subset (E, E^0) .$$

DEFINITION 2.2. An element $U \in h^{2n}(E, E^0)$ is called a *Thom Class* for $\xi$ if and only if for each $x \in X$, $i_x^*: h^{2n}(E, E^0) \to h^{2n}(C^n, C^n \sim \{0\})$ maps $U$ to a $\Lambda$-basis element of $h^{2n}(C^n, C^n \sim \{0\})$ corresponding to $\pm s_{2n} \in h^{2n}(S^{2n}, *)$ via the $\Lambda$-isomorphisms,

$$h^{2n}(S^{2n}, *) \underset{\simeq}{\longrightarrow} h^{2n}(D^{2n}, S^{2n-1}) \underset{\simeq}{\longleftarrow} h^{2n}(C^n, C^n \sim \{0\}) .$$

If $k: (E, \emptyset) \to (E, E^0)$ denotes the inclusion and $i: X \to E$ denotes the zero section of $\xi$, the element $i^*k^*(U) \in h^{2n}(X)$ denoted by $e_U$ is called the *Euler Class* of $\xi$ associated with the Thom Class $U$.

DEFINITION 2.3. If $i_1: (S^2, *) \to (CP^\infty, *)$ denotes the inclusion of $CP^1 = S^2$ into the infinite dimensional complex projective space $CP^\infty$, an element $v \in h^2(CP^\infty, *)$ with the property $i_1^*(v) = s_2$ will be called an *$h^*$-orientation.*

Let $k: (CP^\infty, \emptyset) \to (CP^\infty, *)$ and $i_n: CP^n \to CP^\infty$ denote inclusions and let $u = k^*(v)$ and $u_n = i_n^*(u)$. It is shown in [4] and [7] that for each integer $n \geqq 1$ there exists a Thom Class $U_n$ for the Hopf line bundle $\gamma_n$ over the $n$-dimensional complex projective space $CP^n$ such that its associated Euler Class is $u_n$.

## § 3. $h^*$-Valued Chern class

From now on, $h^*$ is assumed to be an additive cohomology theory with a product and a unit. In addition, suppose that $h^*$ has an $h^*$-orientation $v \in h^2(CP^\infty, *)$.

Let $\xi: E \xrightarrow{\pi} X$ be an $n$-complex vector bundle over a $CW$ complex $X$ and let $P(\xi): P(E) \xrightarrow{\hat{\pi}} X$ be its associated projective bundle with $CP^{n-1}$ as a fibre. Then we may define the canonical line subbundle $L_\xi$ of $\hat{\pi}^\#(\xi)$ over $P(E)$. Let $f_\xi: P(E) \to CP^\infty$ be a classifying map of $L_\xi$ and let the element $f_\xi^*(u) \in h^2(P(E))$ be denoted by $x_\xi$. Since $P(\xi)$ is locally trivial, there exists a collection $\{i_y: CP^{n-1} \to P(E)\}_{y \in X}$ such that for each $y \in X$ the induced bundle $i_y^\#(L_\xi)$ is isomorphic to $\gamma_{n-1}$ over $CP^{n-1}$. Consequently the composition $f_\xi i_y$ is homotopic to the inclusion $i_{n-1}: CP^{n-1} \to CP^\infty$ by the universality of $\gamma$ over $CP^\infty$. Since for each $m \in \{0, \cdots, n-1\}$ $i_y^*(x_\xi^m) = u_{n-1}^m = i_{n-1}^*(u)^m$ and since $\{1, u_{n-1}, \cdots, u_{n-1}^{n-1}\}$ is a $\Lambda$-module base for $h^*(CP^{n-1})$ [7], Theorem 2.1 implies that $h^*(P(E))$ is a free $h^*(X)$-module with a base $\{1, x_\xi, \cdots, x_\xi^{n-1}\}$.

DEFINITION 3.1. Define a $h^*$-valued Chern Class $c(\xi) = \{c_m(\xi)\}_{m=0}^\infty$ for each $n$-complex vector bundle $\xi$ by the defining relation,

$$x_\xi^n - \hat{\pi}^* c_1(\xi) x_\xi^{n-1} + \cdots + (-1)^n \hat{\pi}^* c_n(\xi) = 0 \tag{3.1.1}$$

in $h^{2n}(P(E))$ with $c_0(\xi) = 1$ and $c_m(\xi) = 0$ for $m > n$.

Since the naturality and the Hopf bundle property for the $h^*$-valued Chern Class follow easily from the definition, the crux of the existence of the Chern Class is to prove the multiplicative property.

Let $(CP^\infty)^n$ denote the $n$-fold Cartesian product of $CP^\infty$ and let $q_m: (CP^\infty)^n \to CP^\infty$ denote the $m^{\text{th}}$ projection for each $m \in \{1, 2, \cdots, n\}$, then we have the following.

PROPOSITION 3.2. *The defining relation* (3.1.1) *for the vector bundle* $\xi = \gamma \times \gamma \times \cdots \times \gamma = \gamma^{(n)}: E_n \xrightarrow{\pi_n} (CP^\infty)^n$ *can be factored as*

$$(x_\xi - \hat{\pi}_n^* c_1(q_1^\# \gamma))(x_\xi - \hat{\pi}_n^* c_1(q_2^\# \gamma)) \cdots (x_\xi - \hat{\pi}_n^* c_1(q_n^\# \gamma)) = 0 .$$

*Proof.* We proceed by induction on $n$. The result is trivially true for $n = 1$. Since the $n$-fold bundle product $\gamma^{(n)}$ is isomorphic to the Whitney sum $q_1^\# \gamma \oplus q^\# \gamma^{(n-1)}$ with the obvious projection $q: (CP^\infty)^n \to (CP^\infty)^{n-1}$, it is sufficient to show that the proposition is true for the bundle $\xi = q_1^\# \gamma \oplus q^\# \gamma^{(n-1)}: \tilde{E}_n \xrightarrow{\tilde{\pi}} (CP^\infty)^n$. Since $q_1^\# \gamma: E_1 \xrightarrow{\tilde{\pi}_1} (CP^\infty)^n$ and $q^\# \gamma^{(n-1)}: \tilde{E}_{n-1} \xrightarrow{\tilde{\pi}_{n-1}} (CP^\infty)^n$ can be cannonically embedded into $\xi$, there exist embeddings $j_1: P(\tilde{E}_1) \to P(\tilde{E}_n)$ and $j_{n-1}: P(\tilde{E}_{n-1}) \to P(\tilde{E}_n)$ such that $f_\xi j_1$ and $f_\xi j_{n-1}$ classify the bundles $L_{q_1^\# \gamma}$ and $L_{q^\# \gamma^{(n-1)}}$ respectively. Let $U_1$ denote the space $P(\tilde{E}_n) \sim j_{n-1}(P(\tilde{E}_{n-1}))$ and let $U_{n-1}$ denote the space $P(\tilde{E}_n) \sim j_1(P(\tilde{E}_1))$. Then we may show that $j_1(P(\tilde{E}_1))$ and $j_{n-1}(P(\tilde{E}_{n-1}))$ are deformation retracts of $U_1$ and $U_{n-1}$ respectively, observing that $j_1 q_1^\# \gamma$ and $j_{n-1} q^\# \gamma^{(n-1)}$ are bundles having fibres a point $[1, 0, \cdots, 0]$ in $CP^{n-1} \sim CP^{n-2}$ and the subcomplex $CP^{n-2}$ of the fibre $CP^{n-1}$ of $\xi$. The fact that $U_1$ and $U_{n-1}$ are in the category $W$ under consideration, allows us to consider the long exact sequence relative to the pair $(P(\tilde{E}_n), U_{n-1})$,

$$\begin{array}{ccccccc}
\cdots \longrightarrow h^{2n-2}(P(\tilde{E}_n), U_{n-1}) & \longrightarrow & h^{2n-2}(P(\tilde{E}_n)) & \longrightarrow & h^{2n-2}(U_{n-1}) & \longrightarrow & \cdots \\
 & & & \searrow^{j_{n-1}^*} & \downarrow \simeq & & \\
 & & & & h^{2n-2}(P(\tilde{E}_{n-1})) & & \\
 & & & & \uparrow \bar{q}^* & & \\
 & & & & h^{2n-2}(P(E_{n-1})). & &
\end{array}$$

We may observe that $\alpha_{n-1} = (x_\xi - \hat{\pi}^* c_1(q_2^\# \gamma)) \cdots (x_\xi - \hat{\pi}^* c_1(q_n^\# \gamma))$ in $h^{2n-2}(P(\tilde{E}_n))$ is mapped by $j_{n-1}^*$ onto 0. To see this, we calculate,

$$\begin{aligned}
j_{n-1}^*(x_\xi - \hat{\pi}^* c_1(q_2^\# \gamma)) &= x_{q^\# \gamma^{(n-1)}} - \hat{\tilde{\pi}}_{n-1}^* q_2{}^* c_1(\gamma) \\
&= x_{q^\# \gamma^{(n-1)}} - \hat{\tilde{\pi}}_{n-1}^* q^* q_1^* c_1(\gamma) \\
&= x_{q^\# \gamma^{(n-1)}} - \bar{q}^* \hat{\pi}_{n-1}^* c_1(q_1^\# \gamma) \\
&= \bar{q}^*(x_{\gamma^{(n-1)}} - \hat{\pi}_{n-1}^* c_1(q_1^\# \gamma)).
\end{aligned}$$

Thus, $j_{n-1}^*(\alpha_{n-1})$ is the image under $\bar{q}^*$ of the defining relation (3.1.1) for $\gamma^{(n-1)}$ over $(CP^\infty)^{n-1}$ and hence is zero by the induction hypothesis. Consequently, $\alpha_{n-1}$ is the image of some $\beta_{n-1} \in h^{2n-2}(P(\tilde{E}_n), U_{n-1})$. Similarly, there exists $\beta_1 \in h^2(P(\tilde{E}_1), U_1)$ such that $\alpha_1 = x_\xi - \hat{\pi}^* c_1(q_1^\# \gamma)$ is the image of $\beta_1$. Noting that $P(\tilde{E}_n) = U_1 \cup U_{n-1}$, we can see that $\alpha_1 \alpha_{n-1} \in h^{2n}(P(\tilde{E}_n)$ is the image of $\beta_1 \beta_{n-1} \in h^{2n}(P(E_n), P(E_n)) = 0$. Hence $(x_\xi - \hat{\pi}^* c^1(q_1^\# \gamma))(x_\xi - \hat{\pi}^* c_1(q_2^\# \gamma)) \cdots (x_\xi - \hat{\pi}^* c_1(q_n^\# \gamma)) = 0$.

Since it is routine to complete the proof of the multiplicativity, we simply state the following.

COROLLARY 3.3. *The defining relation* (3.1.1) *for the Whitney sum* $\xi$ *of n line bundles over a CW complex X factors into a product of linear factors.*

By using the splitting principle for vector bundles we further obtain:

PROPOSITION 3.4. *The* $h^*$*-valued Chern Class c in Definition* 3.1 *satisfies the multiplicative property,*

$$c(\xi \oplus \xi') = c(\xi)c(\xi'),$$

*for complex vector bundles* $\xi$ *and* $\xi'$.

## § 4. Thom classes for vector bundles

Consider the split cohomology sequence for the pair $(CP^n, CP^{n-1})$,

$$0 \longrightarrow h^*(CP^n, CP^{n-1}) \xrightarrow{k_n^*} h^*(CP^n) \xrightarrow{j_n^*} h^*(CP^{n-1}) \longrightarrow 0.$$

If $\sigma_n \in h^{2n}(CP^n, CP^{n-1})$ is a basis over $\Lambda$, and if $u_n$ is the Euler Class of the Hopf line bundle $\gamma_n$ over $CP^n$, then it is easy to see [6, p. 45] that $k_n^*(\sigma_n) = \lambda u_n^n$ where $\lambda \in \Lambda^0$ is a unit (invertible element) in the ring $\Lambda$. In the case when $h^*$ is $K$-theory, Bott [3, Lemma 2, p. 30] proved that $\lambda$ is $+1$ by applying the Chern Character and the Todd Class. In this section it will be shown that the crux of selecting an appropriate Thom Class is the fact that $\lambda = \pm 1$.

PROPOSITION 4.1. *If* $\sigma_n \in h^{2n}(CP^n, CP^{n-1})$ *and* $k_n: (CP^n, \emptyset) \to (CP^n, CP^{n-1})$ *is the inclusion, then* $k_n^*(\sigma_n) = \pm u_n^n$, *where* $u_n$ *is the Euler Class of* $\gamma_n$ *over* $CP^n$.

Note that $u_n = i_n^*(u)$ and $u = k^*(v)$ where $v \in h^2(CP, *)$ is an $h^*$-orientation. Assuming Proposition 4.1, for the moment, we may prove:

THEOREM 4.2. *If $\xi$ is an n-dimensional complex vector bundle over X, there exists a Thom Class for $\xi$ whose Euler Class is the $n^{\text{th}}$ Chern Class $c_n(\xi)$.* [6, *p.* 52, *Theorem* 7.12]

*Proof.* Let $\xi: E \xrightarrow{\pi} X$ be the $n$-complex bundle and let $\xi': E' \xrightarrow{\pi'} X$ be the Whitney sum $\theta \oplus \xi$, where $\theta$ is the trivial line bundle over $X$. Denoting the associated projective bundles of $\xi$ and $\xi'$ by

$$P(\xi): P(E) \xrightarrow{\hat{\pi}} X \text{ and } P(\xi'): P(E') \xrightarrow{\hat{\pi}'} X,$$

we may consider $P(E)$ as a subspace of $P(E')$ such that the fibre $CP^{n-1}$ of $P(\xi)$ is the subcomplex of the fibre $CP^n$ of $P(\xi')$. If $f_{\xi'}: P(E') \to CP^\infty$ is a classifying map for the line bundle $L_{\xi'}$, then $f_{\xi'}j$ is a classifying map for $L_\xi$ where $j: P(E) \to P(E')$ is the inclusion. Consider the commutative diagram,

$$\text{(4.2.1)} \qquad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ h^{2n}(CP^n, CP^{n-1}) & \xleftarrow{i_x^*} & h^{2n}(P(E'), P(E)) \\ \downarrow{\scriptstyle k_n^*} & & \downarrow{\scriptstyle \hat{k}^*} \\ h^{2n}(CP^n) & \xleftarrow{i_x^*} & h^{2n}(P(E')) \\ \downarrow{\scriptstyle j_n^*} & & \downarrow{\scriptstyle j^*} \\ h^{2n}(CP^{n-1}) & \xleftarrow{i_x^*} & h^{2n}(P(E)) \\ \downarrow & & \downarrow \\ 0 & & 0, \end{array}$$

where $i_x: (CP^n, CP^{n-1}) \to (P(E'), P(E))$ is the map onto the fibre over $x \in X$. Define $a \in h^{2n}(P(E'))$ by

$$a = (-1)^n(x_{\xi'}^n - \hat{\pi}'^*c_1(\xi)x_{\xi'}^{n-1} + \cdots + (-1)^n\hat{\pi}'^*c_n(\xi) ,$$

then $j^*(a) = (-1)^n(x_\xi^n - \hat{\pi}^*c_1(\xi)x_\xi^{n-1} + \cdots + (-1)^n\hat{\pi}^*c_n(\xi)) = 0$. Therefore, there exists the unique element $V \in h^{2n}(P(E'), P(E))$ such that $\hat{k}^*(V) = a$. Since $i_x^*(x_{\xi'}) = u_n$, $j_n^*(u_n) = u_{n-1}$, and $u_{n-1}^n = 0$ we see that $j_n^{**}i_x^*(a)$ is $(-1)^n(-i_x^*\hat{\pi}^*c_1(\xi)u_{n-1}^{n-1} + i_x{}^*\hat{\pi}^*c_2(\xi)u_{n-1}^{n-2} - \cdots + (-1)^n i_x^*\hat{\pi}^*c_n(\xi)) = 0$. Since $\hat{\pi}i_x$ is a constant function, $i_x^*\hat{\pi}^*c_m(\xi) \in \Lambda^{2n-2m}$, and since $\{u_{n-1}^{n-1}, \cdots, 1\}$ is a $\Lambda$-base, $i_x^*\hat{\pi}^*c_m(\xi) = i_x^*\hat{\pi}'^*c_m(\xi) = 0$ for each $m \in \{1, \cdots, n\}$. Hence $i_x^*(a) = (-1)^n u_n^n$. By virtue of Proposition 4.1, there exists $\sigma_n \in h^{2n}(CP^n, CP^{n-1})$

such that $k_n^*(\sigma_n) = (-1)u_n^n$. Thus $i_x^*(V) = \sigma_n$. Since $\xi$ admits a Hermitian metric, we may define the associated disk and sphere bundles denoted by $D(\xi): D(E) \to X$ and $S(\xi): S(E) \to X$ respectively. It is easy to see [3, p. 36] that there exists a fibre preserving mapping $\bar{\eta}: (D(E), S(E)) \to (P(E'), P(E))$ such that for each $x \in X$ the diagram,

$$\begin{array}{ccc} (D(E), S(E)) & \xrightarrow{\bar{\eta}} & (P(E'), P(E)) \\ \uparrow i_x & & \uparrow i_x \\ (D^{2n}, S^{2n-1}) & \xrightarrow{\eta} & (CP^n, CP^{n-1}), \end{array}$$

commutes, where $\eta$ may be defined by $\eta(\mu) = [1 - \|\mu\|^2, \mu]$. Consider the commutative diagram,

$$(4.2.2)\qquad \begin{array}{ccc} h^{2n}(C^n, C^n \sim \{0\}) & \xleftarrow{i_x^*} & h^{2n}(E, E^0) \\ \iota^* \downarrow \simeq & & \simeq \downarrow \iota^* \\ h^{2n}(D^{2n}, S^{2n-1}) & \xleftarrow{i_x^*} & h^{2n}(D(E), S(E)) \\ \eta^* \uparrow \simeq & & \simeq \uparrow \bar{\eta}^* \\ h^{2n}(CP^n, CP^{n-1}) & \xleftarrow{i_x^*} & h^{2n}(P(E'), P(E)). \end{array}$$

Define $U$ by $\iota^{*-1}\bar{\eta}^*(V)$, then $i_x^*(U) = \iota^{*-1}\eta^*(\sigma_n)$. Hence the proof is complete.

In order to prove Proposition 4.1 we need the following:

LEMMA 4.3. *If $\gamma: E \xrightarrow{\pi} CP^\infty$ is the universal line bundle, there exists a Thom Class $U \in h^2(E, E^0)$ such that its associated Euler Class is $u = c_1(\gamma)$.*

*Proof.* Since Proposition 4.1 is trivially true for $n = 1$, Proposition 4.2 is true for $\xi = \gamma$. Hence the proof is complete.

PROPOSITION 4.4. *The n-fold bundle product*

$$\gamma \times \cdots \times \gamma = \gamma^{(n)}: E^n \xrightarrow{\pi^n} (CP^\infty)^n$$

*has a Thom Class $U_n \in h^{2n}(E^n, (E^n)^0)$ whose associated Euler Class is $c_n(\gamma^{(n)}) = u \times u \times \cdots \times u \in h^{2n}((CP^\infty)^n)$, where x denotes the external product.*

*Proof.* From Lemma 4.3 the proposition is true for $n = 1$. We proceed by induction. Define $U_n$ by the $n$-fold product,

$$U \times U \times \cdots \times U = U \times U^{n-1} \in h^{2n}(E^n, (E^n)^0)$$
$$= h^{2n}(E \times E^{n-1}, E \times (E^{n-1})^0 \cup E^0 \times E^{n-1}),$$

then we may immediately see its associated element $e_{U^n}$ is $c^n(\gamma^{(n)}) = u \times u \times \cdots \times u \in h^{2n}((CP^\infty)^n)$. For each $x = (x_1, \cdots, x_n) \in (CP^\infty)^n$, define $i_x$ by $i_{x_1} \times i_{(x_2,\cdots,x_n)}: (C^n, C^n \sim \{0\}) \to (E, (E^n)^0)$. By the naturality of the external product, the diagram,

$$\begin{array}{ccc} h^2(E, E^0) \otimes h^{2n-2}(E^{n-1}, (E^{n-1})^0) & \xrightarrow{\times} & h^{2n}(E^n, (E^n)^0) \\ \downarrow i^*_{x_1} \otimes i^*_{(x_2,\cdots,k_n)} & & \downarrow i^*_x \\ h^2(C, C \sim \{0\}) \otimes h^{2n-2}(C^{n-1}, C^{n-1} \sim \{0\}) & \underset{\simeq}{\xrightarrow{\times}} & h^{2n}(C^n, C^n \sim \{0\}) \end{array},$$

commutes. Since $i^*_{x_1}(U) = \sigma_1$ and $i^*_{(x_2,\cdots,x_n)}(U_{n-1}) = \sigma_{n-1}$ are the canonical generators for $h^2(C, C \sim \{0\})$ and $h^{2n-2}(C^{n-1}, C^{n-1} \sim \{0\})$ and since $\sigma_1 \times \sigma_{n-1} = \pm\sigma_n$, we obtain, $i^*_x(U_n) = \pm\sigma_n$. This completes the proof.

*Proof of Proposition* 4.1. To save space, the commutative diagrams (4.2.1) and (4.2.2) are used here in the case when $\xi = \gamma^{(n)}$. By Proposition 4.4 there exists the Thom Class $U_n \in h^{2n}(E, E^0)$ for $\gamma^{(n)}$. Let $V \in h^{2n}(P(E'), P(E))$ be $\bar{\eta}^{*-1}\iota^*(U_n)$, then $i^*_x(V) = \sigma_n$ and $\hat{k}^*(V) \in h^{2n}(P(E'))$ can be written as $\hat{k}^*(V) = \hat{\pi}'^*\alpha_0 x^n_{\xi'} + \hat{\pi}'^*\alpha_1 x^{n-1}_{\xi'} + \cdots + \hat{\pi}'^*\alpha_n$ with $\alpha_m \in h^{2m}((CP^\infty)^n)$ for each $m \in \{0, 1, \cdots, n\}$. If $i: X \to E$ is the trivial cross section of $\xi$ and if $i': X \to P(E')$ is the embedding onto $P(X \times C) \simeq X \subset P(E')$, then $i'^*\hat{k}^*(V) = i^*k^*(U_n) = e_{U_n} = c_n(\xi)$ where $k^*$ is induced by the inclusion $k: (E, \emptyset) \to (E, E^0)$. On the other hand, $i'^*\hat{k}^*(V) + \alpha_n$, because $f_\xi i'$ is a classifying map for the trivial bundle over $X = (CP^\infty)^n$ and $\hat{\pi}'i' = 1$. Thus $\alpha_n = c_n(\xi)$. Since $j^*\hat{k}^*(V) = \hat{\pi}^*\alpha_0 x^n_\xi + \cdots + \hat{\pi}^*c_n(\xi) = 0$ and the defining relation for $\xi = \gamma^{(n)}$ allows us to substitute $\hat{\pi}^*c_1(\xi)x^{n-1}_\xi - \cdots + (-1)^{n+1}\hat{\pi}^*c_n(\xi)$ for $x^n_\xi$, we obtain $\alpha_1 = \alpha_0(-1)c_1(\xi), \cdots, \alpha_{n-1} = \alpha_0(-1)^{n-1}c_{n-1}(\xi)$, and $(\alpha_0(-1)^{n+1} + 1)c_n(\xi) = 0$. Considering Corollary 8.9 in [7], or [8], $\alpha_0 \in h^0((CP^\infty)^n)$ is a unique infinite sum,

$$a_{(0,\cdots,0)} + \sum_{(i_1,\cdots,i_n)} a_{(i_1,\cdots,i_n)} u^{i_1} \times \cdots \times u^{i_n}.$$

Since $c_n(\gamma^{(n)}) = u \times u \times \cdots \times u$, we may easily calculate that $\alpha_0 = a_{(0,\cdots,0)} = (-1)^n$ and furthermore $\alpha_m = (-1)^{n+m}c_m(\gamma^{(n)})$ for each $m \in \{1, \cdots, n\}$. We may then write

$$\hat{k}^*(V) = (-1)^n(x^n_{\xi'} - \hat{\pi}'^*c_1(\xi)x^{n-1}_{\xi'} + \cdots + (-1)^n\hat{\pi}'^*c_n(\xi)).$$

As was shown in Proposition 4.2, we obtain $i^*_x\hat{k}^*(V) = (-1)^n u^n_u$. Hence

$k_n^*(\sigma_n) = k_n^* i_x^*(V) = i_x^* \hat{k}^*(V) = (-1)^n u_u^n$. This completes the proof.

## A Selected Biblicgraphy


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