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EXTENDED f-ORBITS ARE APPROXIMATED BY ORBITS

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Introduction

Let f be a C^r-diffeomorphism, $r \leqq 1$, on a compact differentiable manifold M with $\dim M \geqq 2$. In [9] F. Takens introduced the concept of extended f-orbits and conjectured the following.

If f is an AS-diffeomorphism, then the set E_f^- of all extended f-orbits is equal to the set O_f of the closure of all f-orbits in $C(M)$, where $C(M)$ is the metric space of all non empty closed subsets of M.

In this paper we give an affirmative answer for this conjecture.

§ 1. Definitions and the main Theorem

We fix a metric d on M induced by a Riemannian metric, and we define a metric $\bar{d}$ on the set $C(M)$ of all non empty closed subsets of M as follows; for closed non empty subsets A and B of M,

$$\bar{d}(A, B) = \max(\max_{a \in A} d(a, B), \max_{b \in B} d(b, A))$$

where $d(a, B) = \min_{b \in B} d(a, b)$. We identify a closed subset of M with an element of $C(M)$. Here Z denotes the integers, N the natural numbers. For a diffeomorphism f and $x \in M$, we define *the f-orbit of x*, $O_f(x)$, to be the closure of $\{f^n(x) | n \in Z\}$. By definition, $O_f(x) \in C(M)$. Then we denote the closure of $\{O_f(x) | x \in M\}$ in $C(M)$ by O_f. O_f is a closed subset of $C(M)$. We say that a closed subset $A \subset M$ is *an ε-orbit of f*, $\varepsilon > 0$, if there is a sequence $\{x_j\}_{j \in Z}$ such that $d(f(x_j), x_{j+1}) < \varepsilon$ for any $j \in Z$ and $\{x_j\}_{j \in Z}$ is dense in A. We say that a closed subset $A \subset M$ is *an extended f-orbit* if for any $\varepsilon > 0$ and $\delta > 0$, there is an ε-orbit A_ε of f such that $\bar{d}(A, A_\varepsilon) < \delta$. Note that extended f-orbits are identified with elements of $C(M)$. Let E_f be the set of all extended f-orbits. By definition, E_f is a closed subset of $C(M)$ and $O_f \subset E_f$. See [9]. We recall that f is an AS-diffeo-

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morphism if f satisfies Axiom A and strong transversality condition. Then our main result is

THEOREM. *If f is an AS-diffeomorphism, then $E_f = O_f$.*

We shall prove Theorem in section 5.

§ 2. More definitions and a sketch of the proof

In this section we give some notations and definitions used throughout the paper and give a sketch of the proof of Theorem.

The nonwandering set of a diffeomorphism f is denoted by $\Omega(f)$ or Ω and the set of the periodic points of f is denoted by $\mathrm{Per}(f)$. For $x \in M$, define $\alpha(x) = \alpha(x, f) = \{y \in M$: there is a sequence of integers $n_i \to \infty$ such that $f^{-n_i}(x) \to y$ as $i \to \infty\}$. Let $\omega(x) = \omega(x, f) = \alpha(x, f^{-1})$. The nonwandering set of f satisfying Axiom A and no cycle property can be written as a disjoint union of closed subsets $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_m$ such that each Ω_i is invariant by f, and f is topologically transitive on each Ω_i. Then we call each Ω_i a basic set and may define an order on the set $\{\Omega_1, \cdots, \Omega_m\}$ as follows:

$$\Omega_i \leqq \Omega_j \qquad \text{if } W^u(\Omega_i) \cap W^s(\Omega_j) \neq \phi$$

where $W^u(\Omega_i)$ and $W^s(\Omega_j)$ are the unstable manifold and the stable manifold of Ω_i and Ω_j respectively. We may renumber Ω_i such that $\Omega_j \not\leqq \Omega_i$ if $i < j$. Henceforth we shall assume that Ω_i is numbered as above for any diffeomorphism f satisfying Axiom A and no cycle property.

We say that a sequence $\bar{x} = \{x_j\}_{j=a}^{b}$ ($a = -\infty$ or $b = +\infty$ is permitted) of points in M is an *ε-pseudo orbit* if

$$d(f(x_j), x_{j+1}) < \varepsilon \qquad \text{for any } j \in [a, b\text{-}1]\,.$$

A point $x \in M$ *δ-shadows* a sequence $\bar{x}$ if

$$d(f^j(x), x_{j+1}) < \delta \qquad \text{for any } j \in [a, b]\,.$$

See [1, Page 74].

We define a *relation $\prec$ on M*, induced by f, as follows: $x, y \in M$, then $x \prec y$ if and only if for any $\varepsilon > 0$, there is an ε-pseudo orbit $\{x_j\}_{j=0}^{n}$ with $x_0 = x$, $x_n = y$ and $n \geqq 1$. We define $N(f) = \{x \in M \mid x \prec x\}$. Note that $x \prec f^n(x)$ for any $n \geqq 1$ and $N(f) \supset \Omega(f)$. See [9] for details.

Now let f be an AS-diffeomorphism and let A be an extended

f-orbit with $A \not\subset \Omega$. Then there are k-points $x_i \in M$ such that $A - \Omega = \bigcup_{i=1}^{k} \bigcup_{n\in Z} f^n(x_i)$ and such that $\omega(x_i)$ and $\alpha(x_{i+1})$ belong to the same basic set Ω_{s_i} $(1 \leqq s_0 < \cdots < s_k \leqq m)$ by Proposition 3.6 in section 3. In section 4 we obtain that for $A_{s_i} = A \cap \Omega_{s_i}$, any $\delta > 0$ and small $\varepsilon > 0$, there is an ε-pseudo orbit $\bar{x} = \{x_j\}_{j=a}^{b}$ such that

$$\bar{d}(A_{s_i}, \text{ closure of } \{x_j\}_{j=a}^{b}) < \delta .$$

By [1, Proposition 3.6], $\bar{x}$ is δ-shadowed by some $z \in \Omega_{s_i}$. We shall select $x' \in M$ such that

$$\bar{d}(O_f(x'), A_{s_0} \cup O_f(x_1) \cup A_{s_1}) < \delta$$

so that we can select $x \in M$ such that $\bar{d}(O_f(x), A) < \delta$ by induction. Hence $A \in O_f$. Since we obtain in section 5 that if A is an extended f-orbit with $A \subset \Omega$, then $A \in O_f$, therefore $A \in O_f$ for any extended f-orbit A. Since $O_f \subset E_f$, $O_f = E_f$.

§ 3. Nonwandering sets and extended f-orbits

In this section we give some results about $N(f)$ and extended f-orbits. We recall that f has *no C^0-Ω-explosion* if for each $\varepsilon > 0$, there is a neighborhood $U(f)$ of f in $\text{Diff}^r(M)$ with C^0-topology such that $\Omega(g) \subset U_\varepsilon(\Omega(f))$ for any $g \in U(f)$, where $\text{Diff}^r(M)$ is the set of C^r-diffeomorphisms with C^r-topology and $U_\varepsilon(.)$ is an ε-neighborhood of (.).

The following lemma is due to Z. Nitecki and M. Shub [6]. For the proof, the hypothesis $\dim M \geqq 2$ is needed.

LEMMA 3.1. *Suppose a finite collection $\{(p_i, q_i) \in M \times M : i = 1, \cdots, k\}$ of pairs of points on M is specified, together with a small positive constant $\delta > 0$ such that*:

(i) *For each i, $d(p_i, q_i) < \delta$*

(ii) *If $i \neq j$, then $p_i \neq p_j$ and $q_i \neq q_j$.*

Then there exists a diffeomorphism $\eta: M \to M$ such that

(a) $d(\eta(x), x) < 2\pi\delta$ *for every $x \in M$*

(b) $\eta(p_i) = q_i$ *for $i = 1, \cdots, k$.*

PROPOSITION 3.2. *If f has no C^0-Ω-explosion, then $N(f) = \Omega(f)$.*

Proof. It is sufficient to show that $N(f) \subset \Omega(f)$. Let $x \in N(f)$ and

$\varepsilon > 0$ be given. Since f has no C^0-Ω-explosion, there is a neighborhood $U(f)$ of f in $\mathrm{Diff}^r(M)$ with C^0-topology such that $\Omega(g) \subset U_\varepsilon(\Omega(f))$ for any $g \in U(f)$. Take $\delta > 0$ such that if $d(g(x), f(x)) < \delta$ for any $x \in M$, then $g \in U(f)$. From definition of $N(f)$, there is a $(\delta/2\pi)$-pseudo orbit $\{x_j\}_{j=0}^n$ with $x_0 = x$ and $x_n = x$. We may assume that $x_i \neq x_j$ if $i \neq j$. By Lemma 3.1, there is a diffeomorphism η on M such that $\eta(f(x_j)) = x_{j+1}$ and $d(\eta(x), x) < \delta$ for every $x \in M$. Then the composition $g = \eta \circ f$ is a diffeomorphism on M such that

(a) $d(g(x), f(x)) < \delta$ for any $x \in M$

(b) $g^n(x) = (\eta \circ f)^n(x_0) = x_n = x$.

Hence $g \in U(f)$ and $x \in \mathrm{Per}\,(g)$. Since $x \in \Omega(g) \subset U_\varepsilon(\Omega(f))$ and $\Omega(f)$ is closed, $x \in \Omega(f)$.

If f satisfies Axiom A and no cycle property, then f has no C^0-Ω-explosion [8]. Therefore we have

COROLLARY 3.3. *If f satisfies Axiom A and no cycle property, then $N(f) = \Omega(f)$.*

We shall assume throughout the remainder of this section that f satisfies Axiom A and no cycle property.

LEMMA 3.4.

(i) *If $f^n(x) \prec y$ for any $n \in N$, then $u \prec y$ for any $u \in \omega(x)$.*

(ii) *For any $x, y \in \Omega_i$, $x \prec y$ and $y \prec x$.*

Proof. Let $a \in \omega(x)$ and $\varepsilon > 0$ be given. Since $f(a) \in \omega(x)$, $d(f(a), f^m(x)) < \varepsilon$ for some $m \in N$. Then there is an ε-pseudo orbit $\{x'_j\}_{j=0}^n$ with $x'_0 = f^m(x)$ and $x'_n = y$. Define a sequence $\{x_j\}_{j=0}^{n+1}$ by

$$x_0 = a,\ x_j = x'_{j-1} \qquad \text{for any } 1 \leqq j \leqq n+1\,.$$

Then $\{x_j\}_{j=0}^{n+1}$ is an ε-pseudo orbit with $x_0 = u$ and $x_{n+1} = y$. As ε is arbitrary, $a \prec y$.

(ii) By [1, page 72], $\Omega_i = X_{1,i} \cup \cdots \cup X_{n_1,i}$ with $X_{j,i}$'s pairwise disjoint closed sets, $f(X_{j,i}) = X_{j+1,i}$ $(X_{n_i+1,i} = X_{1,i})$ and $f^{n_i}|X_{j,i}$ topological mixing i.e., for any open sets U, V of $X_{j,i}$ (i.e. in Ω), there is $k > 0$ such that $U \cap f^{k \times n_i}(V) \neq \phi$. Hence for any $x, y \in \Omega_i$, $x \prec y$ and $y \prec x$.

LEMMA 3.5. *If $x, y \in W^s(\Omega_i) - \Omega_i$ and $x \prec y$, then $f^n(x) = y$ for some $n \in N$.*

Proof. Suppose, on the contrary, that $f^n(x) \neq y$ for any $n \in N$. Clearly if $x \prec y$ and $f(x) \neq y$, then $f(x) \prec y$. Hence by induction, if $x \prec y$ and $f^n(x) \neq y$ for any $n \in N$, then $f^n(x) \prec y$. By Lemma 3.4 (i), we have

$$x \prec u \prec y \prec w \qquad \text{for any } u \in \omega(x) \text{ and any } w \in \omega(y)\,.$$

Since $u \in \omega(x) \subset \Omega_i$ and $w \in \omega(y) \subset \Omega_i$,

$$u \prec w\,, \quad w \prec u \qquad \text{by Lemma 3.4 (ii)}\,.$$

Hence $y \prec w \prec u \prec y$ and $y \in N(f) = \Omega(f)$, a contradiction.

PROPOSITION 3.6. *For each $A \in E_f$ such that $A \not\subset \Omega$, there are k-point $x_i \in M$ $(k \leq m-1)$ such that*

$$A - \Omega = \bigcup_{i=1}^{k} \bigcup_{n \in Z} f^n(x_i)$$

moreover there are $s_0, \cdots, s_k$ $(1 \leq s_i \leq m)$ such that $\alpha(x_1) \subset \Omega_{s_0}$, $\omega(x_k) \subset \Omega_{s_k}$ and both $\omega(x_i)$ and $\alpha(x_{i+1})$ are contained in Ω_{s_i} for any $1 \leq i \leq k-1$.

Proof. We define an equivalence relation on M before we prove. For x, $x' \in M$, we say that x is orbitally related or O-related to x' (write $x \sim x'$) if either $f^n(x) = x'$ or $f^{n'}(x') = x$ for some n, $n' \in N$. Let $A^i = W^s(\Omega_i) \cap (A - \Omega)$. Since $M = \bigcup_{i=1}^m W^s(\Omega_i)$, $A - \Omega = \bigcup_{i=1}^m A^i$. By definition of extended f-orbits, if x, $y \in A$, then either $x \prec y$ or $y \prec x$. If x, $y \in A^i$, then x, $y \in W^s(\Omega_i) - \Omega_i$. Hence by Lemma 3.5, if x, $y \in A^i$, then $x \sim y$. Hence either $A^i = \{f^n(x) | n \in Z\}$ for some $x \in A^i$ or $A^i = \phi$ so that there are k-points x_i of M $(k \leqq m-1)$ such that

$$A - \Omega = \bigcup_{i=1}^{k} \bigcup_{n \in Z} f^n(x_i)\,.$$

Let Ω_{s_i} be the basic set with $\omega(x_i) \subset \Omega_{s_i}$ and let Ω_{t_i} be the basic set with $\alpha(x_i) \subset \Omega_{t_i}$. We may assume that $s_1 < s_2 < \cdots < s_k$. If $\alpha(x_i)$ and $\alpha(x_j)$ are contained in the same basic set, then $x_i \sim x_j$ by Lemma 3.5 applied to f^{-1}. Hence $\Omega_{t_i} \neq \Omega_{t_j}$ $(i \neq j)$. By the ordering on the basic sets, $\Omega_{t_i} \neq \Omega_{s_j}$ for $i \leqq j$. Hence $\Omega_{t_1} \cap O_f(x_i) = \phi$ for $i = 2, \cdots, k$ and $\Omega_{t_2} \cap O_f(x_i) = \phi$ for $i = 3, \cdots, k$. Therefore there is $\delta > 0$ such that $O_f(x_i) \cap U_{2\delta}(\Omega_{t_1}) = \phi$ for $i = 2, \cdots, k$ and $O_f(x_i) \cap U_{2\delta}(\Omega_{t_2}) = \phi$ for $i = 3, \cdots, k$. We choose $\gamma > 0$ such that $U_{2\gamma}(\Omega_{t_i}) \subset f(U_\delta(\Omega_{t_i})) \cap U_\delta(\Omega_{t_i})$ for $i = 1, 2$. Then there is $N' \in N$ such that $f^{-n}(x_1) \in U_{\gamma/2}(\Omega_{t_1})$ and $f^{-n}(x_2) \in U_{\gamma/2}(\Omega_{t_2})$ for any $n \geqq N'$. Since $N(f) = \Omega(f)$ and $f^{-N'}(x_i) \notin \Omega(f)$ $(i = 1, 2)$, $f^{-N'}(x_i) \not\prec u_i$ for any $u_i \in$

Ω_{t_i}. Hence there is $\varepsilon' > 0$ such that there exists neither ε'-pseudo orbit $\{x_j\}_{j=0}^{n}$ with $x_0 = f^{-N'}(x_1)$ and $x_n = u_1$ nor ε'-pseudo orbit $\{x'_j\}_{j=0}^{n'}$ with $x'_0 = f^{-N'}(x_2)$ and $x'_{n'} = u_2$. Let $\varepsilon = \min\{\gamma/2, \varepsilon'/2\}$ and let $A_\varepsilon =$ closure of $\{y_j\}_{j\in Z}$ be an ε-orbit of f such that $\bar{d}(A_\varepsilon, A) < \varepsilon$. Then there is $n \in Z$ such that $y_n \in U_\varepsilon(A \cap \Omega_{t_1})$. Suppose that there is $\ell < n$ such that $y_\ell \in U_\gamma(\Omega_{t_1})$ and $y_{\ell-1} \notin U_\gamma(\Omega_{t_1})$. Then $y_{\ell-1} \in U_\delta(\Omega_{t_1})$ because $f(y_{\ell-1}) \in U_{2\gamma}(\Omega_{t_1})$. Since $\bar{d}(A_\varepsilon, A) < \varepsilon$, there is $z \in A \cap U_\varepsilon(y_{\ell-1})$. Clearly $z \notin U_{\gamma/2}(\Omega_{t_1})$ and $z \in U_{2\delta}(\Omega_{t_1})$. Since $O_f(x_i) \cap U_{2\delta}(\Omega_{t_1}) = \phi$ for $i = 2, \cdots, k$, $z = f^{-p}(x_1)$ for some $p < N'$. Since $y_n \in U_\varepsilon(A \cap \Omega_{t_1})$, there is $u_1 \in A \cap \Omega_{t_1}$ such that $d(u_1, y_n) < \varepsilon$. Now we define a sequence $\{z_j\}_{j=0}^{J}$ $(J = p - N' + n - \ell + 1)$ as follows;

$$(z_0, \cdots, z_J) = (f^{-N'}(x_1), \cdots, f^{-p-1}(x_1), y_\ell, \cdots, y_{n-1}, u_1)$$

Then $\{z_j\}_{j=0}^{J}$ is an ε-pseudo orbit with $z_0 = f^{-N'}(x_1)$ and $z_J = u_1$. Since $\varepsilon < \varepsilon'$, $\{z_j\}_{j=0}^{J}$ is an ε'-pseudo orbit with $z_0 = f^{-N'}(x_1)$ and $z_J = u_1$. This contradicts to the choice of ε'. Hence $y_j \in U_\gamma(\Omega_{t_1})$ for any $j \leqq n$. Now if $\Omega_{t_2} \neq \Omega_{s_1}$, then $O_f(x_i) \cap \Omega_{t_2} = \phi$ for $i = 1, 3, \cdots, k$. We can assume that $O_f(x_i) \cap U_{2\delta}(\Omega_{t_2}) = \phi$ for $i = 1, 3, \cdots, k$. Then applying the same argument in case of Ω_{t_1}, we have that there is $n' \in Z$ such that $y_j \in U_\gamma(\Omega_{t_2})$ for any $j \leqq n'$. This contradicts to the fact that $y_j \in U_\gamma(\Omega_{t_1})$ for any $j \leqq n$. Hence $\Omega_{t_2} = \Omega_{s_1}$. Similarly $\Omega_{t_{i+1}} = \Omega_{s_i}$. We write s_0 for t_1. Then $\alpha(x_1) \subset \Omega_{s_0}$, $\omega(x_k) \subset \Omega_{s_k}$ and $\omega(x_i) \cup \alpha(x_{i+1}) \subset \Omega_{s_i}$ for any $1 \leqq i \leqq k-1$.

For simplicity, we write the Ω_i for the Ω_{s_i} in Proposition 3.6. Throughout the remainder of this paper we assume that there are k-points x_i of M $(k \leqq m-1)$ such that

$$A - \Omega = \bigcup_{i=1}^{k} \bigcup_{n\in Z} f^n(x_i)$$

moreover $\alpha(x_1) \subset \Omega_0$, $\omega(x_k) \subset \Omega_k$ and $\omega(x_i) \cup \alpha(x_{i+1}) \subset \Omega_i$ for any $1 \leqq i \leqq k-1$.

§ 4. Extended f-orbits in nonwandering set

Let A be an extended f-orbit. Then there are k-points x_i of M such that $A - \Omega = \bigcup_{i=1}^{k} \bigcup_{n\in Z} f^n(x_i)$ and $\omega(x_i) \cup \alpha(x_{i+1}) \subset \Omega_i$ and let $A_i = A \cap \Omega_i$

LEMMA 4.1. *For any $\delta > 0$ and $\varepsilon > 0$, there is $\gamma > 0$ with $0 < \gamma < \delta$ such that for any $0 < \gamma' < \gamma$, there is an ε-orbit A_ε of f; $A_\varepsilon =$ closure of $\{y_j\}_{j\in Z}$ satisfying the followings;*

(1) $\bar{d}(A, A_\varepsilon) < \gamma'$

(2) *if* $y_m, y_n \in U_{\gamma'}(A_i)$, *then* $y_j \in U_\delta(A_i) =$ *for any* $m < j < n$.

Proof. Let $\delta > 0$ and $\varepsilon > 0$ be given. There is $H \in N$ such that $f^n(x_i) \in U_{\delta/2}(\omega(x_i))$ and $f^{-n}(x_{i+1}) \in U_{\delta/2}(\alpha(x_{i+1}))$ for any $n \geqq H$. Then for any $u \in \Omega_i$, $f^H(x_i) \prec u$ and $u \prec f^{-H}(x_{i+1})$. Since $f^H(x_i)$ and $f^{-H}(x_{i+1})$ are not elements of Ω and $N(f) = \Omega(f)$, $u \not\prec f^H(x_i)$ and $f^{-H}(x_{i+1}) \not\prec u$. Therefore there is $\varepsilon_1 > 0$ such that there exists neither ε_1-pseudo orbit $\{x_j\}_{j=0}^n$ with $x_0 = u$ and $x_n = f^H(x_i)$ nor ε_1-pseudo orbit $\{x'_j\}_{j=0}^m$ with $x'_0 = f^{-H}(x_{i+1})$ and $x'_m = u$. We choose $\gamma_1 > 0$ such that for any pair (p, q) of points on M with $d(p, q) < \gamma_1$, $d(f(p), f(q)) < \varepsilon_1/2$. Let $\gamma = \min\{\delta/2, \varepsilon_1/2, \gamma_1\}$ and $\varepsilon' = \min\{\varepsilon, \varepsilon_1/2\}$. By definition of extended f-orbits, for any $0 < \gamma' < \gamma$, there is an ε'-orbit $A_{\varepsilon'}$ of f; $A_{\varepsilon'} =$ closure of $\{y_j\}_{j \in Z}$ such that $\bar{d}(A, A_{\varepsilon'}) < \gamma'$. Suppose that there are m, j and n with $m < j < n$ such that $y_m, y_n \in U_{\gamma'}(A_i)$ and $y_j \notin U_\delta(A_i)$. Since $\bar{d}(A, A_{\varepsilon'}) < \gamma'$, there is $z \in U_{\gamma'}(y_j) \cap A$. Clearly $z \notin U_{\delta/2}(A_i)$ because $U_{\gamma'}(y_j) \cap U_{\delta/2}(A_i) = \phi$. Then either $z \prec f^H(x_i)$ or $f^{-H}(x_{i+1}) \prec z$. We can assume that $z \prec f^H(x_i)$ without loss of generality. Then there is an ε'-pseudo orbit $\{x_j\}_{j=0}^s$ with $x_0 = z$ and $x_s = f^H(x_1)$. Since $y_m \in U_{\gamma'}(A_i)$, there is $u \in A_i$ such that $d(y_m, u) < \gamma'$. Since $\gamma' < \gamma_1$, $d(f(y_m), f(u)) < \varepsilon_1/2$. Hence

$$d(f(u), y_{m+1}) < d(f(u), f(y_m)) + d(f(y_m), y_{m+1}) < \varepsilon_1/2 + \varepsilon' < \varepsilon_1 .$$

Now we define a sequence $\{z_j\}_{j=0}^L$ $(L = j - m + s + 1)$ as follows;

$$(z_0, \cdots, z_L) = (u, y_{m+1}, \cdots, y_{j-1}, x_0, \cdots, x_s)$$

Then $\{z_j\}_{j=0}^L$ is an ε_1-pseudo orbit with $z_0 = u$ and $z_L = f^H(x_i)$. This is a contradiction.

By Lemma 4.1, for $\delta > 0$, small $\gamma' > 0$ and small $\varepsilon > 0$, there is an ε-orbit A_ε of f; $A_\varepsilon =$ closure of $\{y_j\}_{j \in Z}$ satisfying the followings;

(1) $\bar{d}(A_0$, closure of $\{y_j\}_{j=-\infty}^{n_0}) < \delta$

(2) $\bar{d}(A_i, \{y_j\}_{j=m_i}^{n_i}) < \delta$ for any $1 \leqq i \leqq k-1$

(3) $\bar{d}(A_k$, closure of $\{y_j\}_{j=m_k}^{+\infty}) < \delta$

where $m_i = \min\{j : y_j \in U_{\gamma'}(A_i)\}$ for any $1 \leqq i \leqq k$, and $n_i = \max\{j : y_j \in U_{\gamma'}(A_i)\}$ for any $0 \leqq i \leqq k-1$.

We denote y_{m_i} by $L_i^+(\gamma', \varepsilon)$ and y_{n_i} by $L_i^-(\gamma', \varepsilon)$.

LEMMA 4.2. *If* γ'_n *and* $\varepsilon_n \to 0$ *as* $n \to \infty$, *then the cluster points of the sequence* $L_i^+(\gamma'_n, \varepsilon_n)$ *are contained in* $\omega(x_i)$.

Proof. Let L_i^+ be the set of the cluster points of the sequence $L_i^+(\gamma'_n, \varepsilon_n)$,

$y^+ \in L_i^+$ and $\alpha > 0$ be given (α is sufficiently small). Now let $\|T_x f\| = \sup\{\|T_x f(v)\|: v \in T_x M$ and $\|v\| \leqq 1\}$ where $\|.\|$ is the Riemannian metric on M. Let $K = \max\{\|T_x f\|, \|T_x f^{-1}\|\}$. Then there is $\ell \in N$ such that $L_i^+(\gamma'_\ell, \varepsilon_\ell)$ is in $U_\alpha(y^+)$ and γ'_ℓ, $\varepsilon_\ell < \alpha/4K$. For $L_i^+(\gamma'_\ell, \varepsilon_\ell)$, there is $m_i \in Z$ such that $y_{m_i} \in A_{\varepsilon_\ell} \cap U_{\gamma'_\ell}(A_i)$ and $y_{m_i-1} \in A_{\varepsilon_\ell} - U_{\gamma'_\ell}(A_i)$. Since γ'_ℓ and ε_ℓ are small, there is $p \in N$ such that $f^p(x_i) \in U_{\gamma'_\ell}(y_{m_i-1})$. Then

$$d(y_{m_i}, f^{p+1}(x_i)) < d(y_{m_i}, f(y_{m_i-1})) + d(f(y_{m_i-1}), f^{p+1}(x_i)) < \varepsilon_\ell + K\gamma_\ell < \alpha/2\,.$$

Hence

$$d(y^+, f^{p+1}(x_i)) < d(y^+, y_{m_i}) + d(y_{m_i}, f^{p+1}(x_i)) < \alpha/2 + \alpha/2 < \alpha\,.$$

Since α is arbitrary $y^+ \in \omega(x_i)$. Hence $L_i^+ \subset \omega(x_i)$.

Similarly the cluster points of the sequence $L_i^-(\gamma_n, \varepsilon_n)$ are contained in $\alpha(x_{i+1})$.

LEMMA 4.3. *For any $\delta > 0$ and $\varepsilon > 0$, there is an ε-pseudo orbit $\{x_j^i\}_{j=a}^b$ of $f|\Omega_i$, a and b depend on i, such that*

(1) $\bar{d}(A_i$, *closure of* $\{x_j^i\}_{j=a}^b) < \delta$

(2) $x_a^i \in \omega(x_i)$ *for any* $1 \leq i \leq k$

(3) $x_b^i \in \alpha(x_{i+1})$ *for any* $0 \leq i \leq k-1$.

Proof. Let K be as in Lemma 4.2. For $\delta > 0$ and $\varepsilon > 0$, choose δ' and ε' such that $0 < \delta' < \delta/2$ and $0 < \varepsilon' < \varepsilon - (1+K)\delta'$. As stated above, there is ε'-pseudo orbit $\{y_j\}_{j=a}^b$ such that

(i) $\bar{d}(A_i$, closure of $\{y_j\}_{j=a}^b) < \delta'$.

(a and b are depend on i). By Lemma 4.2, we may assume that $y_a \in \omega(x_i)$ and $y_b \in \alpha(x_{i+1})$. By (i), there is $z_j \in A_i$ in $U_{\delta'}(y_j)$ for any $a < j < b$. Then we define a sequence $\{x_j^i\}_{j=a}^b$ as follows; $x_a^i = y_a$, $x_b^i = y_b$ and $x_j^i = z_j$ for any $a < j < b$. Since $d(f(x_j^i), f(y_j)) < K\delta'$,

$$\begin{aligned} d(f(x_j^i), x_{j+1}^i) &< d(f(x_j^i), f(y_j)) + d(f(y_j), y_{j+1}) \\ &\quad + d(y_{j+1}^i, x_{j+1}^i) < K\delta' + \varepsilon' + \delta' < \varepsilon\,. \end{aligned}$$

Since $U_{\delta'}(y_j) \subset U_\delta(x_j^i)$, $\{x_j^i\}_{j=a}^b$ is an ε-pseudo orbit of $f|\Omega_i$ satisfying (1), (2) and (3).

For any $1 \leqq i \leqq k-1$, a and b are finite. If i is equal to 0, then $a = -\infty$. If i is equal to k, then $b = +\infty$.

§ 5. Proof of Theorem

Throughout it is assumed that f is an AS-diffeomorphism and let $\Omega(f)$

$= \Omega_1 \cup \cdots \cup \Omega_m$ such that if $i < j$, then $\Omega_j \nleq \Omega_i$. The stable manifold of x is the set $W^s(x, f) = W^s(x) = \{y \in M\colon d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$ for any $x \in M$. Let $W^s_\delta(x) = \{y \in M\colon d(f^n(x), f^n(y)) < \delta \text{ for any } n \geqq 0\}$. The unstable manifold of x is the set $W^u(x, f) = W^S(x, f^{-1})$ and $W^u_\delta(x) = W^s_\delta(x, f^{-1})$. For small $\delta > 0$ and $x \in \Omega$,

$$W^s_\delta(x) = \{y \in M\colon d(f^n(x), f^n(y)) < \lambda^n \delta \text{ for any } n \geqq 0\}$$

where λ is a positive constant with $\lambda \in (0, 1)$. For small $\delta > 0$ there is a u-disc family $\tilde{W}^u_\delta$ through a compact neighborhood U_i of Ω_i in M which reduces to W^u_δ at Ω_i and semi-invariant in the sense that

$$\tilde{W}^u_\delta(f(x)) \subset f(\tilde{W}^u_\delta(x)) \qquad \text{for } x \in U_i \cap f^{-1}(U_i)\,.$$

See [2]. For $x \in M$, let $O^+_f(x) =$ closure of $\{f^n(x)\colon n \geqq 0\}$ and let $O^-_f(x) =$ closure of $\{f^n(x)\colon n \leqq 0\}$.

The following proposition is due to R. Bowen [1].

PROPOSITION 5.1. *For any $\delta > 0$, there is an $\varepsilon > 0$ so that every ε-pseudo orbit of $f \mid \Omega$ is δ-shadowed by some $z \in \Omega$.*

COROLLARY 5.2. *Let A be an extended f-orbit with $A \subset \Omega$. Then $A \in O_f$.*

Proof. It is clear that $A \subset \Omega$ implies $A \subset \Omega_i$ for some $1 \leqq i \leqq m$. By Lemma 4.3, for any $\delta > 0$ and any $\varepsilon > 0$, there is an ε-pseudo orbit $\bar{x}$ of $f \mid \Omega$ such that

$$\bar{d}(A, \text{closure of } \bar{x}) < \delta/2\,.$$

By Proposition 5.1, taking sufficiently small $\varepsilon > 0$, $\bar{x}$ is $(\delta/2)$-shadowed by $z \in \Omega_i$. Hence

$$\begin{aligned}\bar{d}(A, O_f(z)) &< \bar{d}(A, \text{closure of } \bar{x}) + \bar{d}(O_f(z), \text{closure of } \bar{x}) \\ &< \delta/2 + \delta/2 < \delta\,.\end{aligned}$$

Since δ is arbitrary and O_f is closed, $A \in O_f$.

Remark 5.3. Let $z \in \Omega$ δ-shadows ε-pseudo orbit $\{x_j\}_{j=a}^b$ of $f \mid \Omega$. Then we may assume that

(1) if a and b are finite, then $z \in W^u_\alpha(x_a)$ and $f^{b-a}(z) \in W^s_\alpha(x_b)$ for small $\alpha > 0$

(2) if $b = +\infty$, then $z \in W^u_\alpha(x_a)$

(3) if $a = -\infty$, then $z \in W^s_\alpha(x_b)$. See [1].

We shall need the following lemma before we prove Theorem.

LEMMA 5.4. *Let* $y \in \Omega_i$, $t \in W^s_\delta(y)$ $(\alpha(t) \subset \Omega_j, j \neq i)$ *and let* $y' \in \omega(y)$, $z \in W^u_\delta(y') \cap \Omega_i$ *for small* $\delta > 0$. *Then for any* $r > 0$, *any u-disc D which is* C^1*-close to* $W^u(t) \cap B_r(t)$ *and any s-disc* D' *which is* C^1*-close to* $W^s_\delta(z) \cap B_r(z)$, *there is* $v \in D$ *such that* $f^n(v) \in D'$ *for some* $n \in N$. *Moreover*

$$d(f^j(v), f^j(t)) < 2\delta \qquad \text{for any } 0 \leq j \leq n$$

where $B_r(.)$ *is an r-ball of* $(.)$, $u = \dim T_t(W^u(t))$ *and* $s = \dim T_z(W^s_\delta(z))$.

Proof. We shall first prove that for any $r > 0$, there is $v' \in W^u(t) \cap B_r(t)$ such that $f^n(v') \in W^s_\delta(z) \cap B_r(z)$ for some $n \in N$. By generalized λ-lemma [5, Proposition 2.3], there is u-disc $\bar{D}$ in $W^u(t) \cap B_r(t)$ such that $f^n(\bar{D})$ is C^1-close to $W^u_\delta(f^n(y))$ for large $n \in N$. Since $f^n(y)$ is near to y' $(y' \in \omega(y))$, $W^u_\delta(f^n(y))$ is C^1-close to $W^u_\delta(y')$. Hence $f^n(\bar{D})$ is C^1-close to $W^u_\delta(y')$ so that $f^n(\bar{D}) \cap (W^s_\delta(z) \cap B_r(z)) \neq \phi$. Taking sufficiently large $n \in N$, there is σ, $0 < \sigma < \lambda^n\delta$ such that $\tilde{W}^u_\delta(a) \cap f^n(\bar{D}) = \phi$ for any $a \in W^s_{\lambda^n\delta}(f^n(y)) - W^s_\sigma(f^n(y))$ because $f^n(\bar{D})$ is C^1-close to $W^u_\delta(f^n(y))$. And there is $q \in W^s_\sigma(f^n(y))$ such that

$$\tilde{W}^u_\delta(q) \cap f^n(\bar{D}) \cap (W^s_\delta(z) \cap B_r(z)) \neq \phi .$$

Let $v' \in f^{-n}(\tilde{W}^u_\delta(q)) \cap \bar{D} \cap f^{-n}(W^s_\delta(z) \cap B_r(z))$. Then $f^j(v') \in f^j(f^{-n}(\tilde{W}^u_\delta(q))$ for any $0 \leqq j \leqq n$. By semi-invariance of u-disc family $\tilde{W}^u_\delta$, $f^j(v') \in \tilde{W}^u_\delta(f^{j-n}(q))$. Since t and $f^{-n}(q)$ are in $W^s_\delta(y)$, $d(f^j(t), f^{j-n}(q)) < \delta$ for any $0 \leqq j \leqq n$. Hence $d(f^j(v'), f^j(t)) < 2\delta$ for any $0 \leqq j \leqq n$.

Secondly by strong transversality, there is $v \in D$ and $n \in N$ such that $f^n(v) \in D'$ for any u-disc D which is C^1-close to $W^u(t) \cap B_r(t)$ and any s-disc D' which is C^1-close to $W^s_\delta(z) \cap B_r(z)$. Moreover $d(f^j(v), f^j(t)) < 2\delta$ for any $0 \leqq j \leqq n$.

Proof of Theorem. Since $O_f \subset E_f$, it is sufficient to show that $E_f \subset O_f$. If A is an extended f-orbit with $A \subset \Omega$, then $A \in O_f$ by Corollary 5.2. Therefore we may assume that A is not contained in Ω. Then since AS-diffeomorphisms satisfy Axiom A and no cycle property, by Proposition 3.6 there are k-points $x_i \in M$ such that

$$A - \Omega = \bigcup_{i=1}^{k} \bigcup_{n \in Z} f^n(x_i)$$

moreover $\alpha(x_1) \subset \Omega_0$, $\omega(x_k) \subset \Omega_k$ and $\omega(x_i) \cup \alpha(x_{i+1}) \subset \Omega_i$ for any $1 \leqq i \leqq$

$k-1$. For small $\delta > 0$, we choose a compact neighborhood U_i of Ω_i such that there is u-disc family $\tilde{W}^u_\delta$ through U_i. Let $A_i = A \cap \Omega_i$.

By Lemma 4.3 for any $\delta > 0$ and small $\varepsilon > 0$, there is an ε-pseudo orbit $\{x^i_j\}^b_{j=a}$ of $f|\Omega_i$ ($1 \leqq i \leqq k-1$, a and b depend on i, a and b are finite) such that $x^i_a \in \omega(x_i)$, $x^i_b \in (x_{i+1})$ and $\bar{d}(A_i, \{x^i_j\}^b_{j=a}) < \delta/2$. We denote x^i_a by y'_i and x^i_b by y''_i. By Proposition 5.1, taking sufficiently small $\varepsilon > 0$, $\{x^i_j\}^b_{j=a}$ is $\delta/2$-shadowed by $z_i \in \Omega_i$ with $z_i \in W^u_\delta(y'_i)$, $f^{b-a}(z_i) \in W^s_\delta(y''_i)$. Hence

$$\bar{d}(A_i, \{f^j(z_i): 0 \leqq j \leqq b-a\}) < \delta\,.$$

Similarly for A_0 and A_k, there are $z_0 \in \Omega_0$ with $z_0 \in W^s_\delta(y''_0)$ ($y''_0 \in \alpha(x_1)$) and $z_k \in \Omega_k$ with $z_k \in W^u_\delta(y'_k)$ ($y'_k \in \omega(x_k)$) such that

$$\bar{d}(A_0, \text{closure of } \{f^j(z_0): j \in (-\infty, 0]\}) < \delta$$
$$\bar{d}(A_k, \text{closure of } \{f^j(z_k): j \in [0, +\infty)\}) < \delta\,.$$

And there is $M_i \in N$ such that

(i) $f^n(x_i) \in U_{\delta/4}(\omega(x_i))$ for any $n \geqq M_i$

(ii) $f^{-n}(x_{i+1}) \in U_{\delta/4}(\alpha(x_{i+1}))$ for any $n \geqq M_i$.

Similarly for $\alpha(x_1)$ and $\omega(x_k)$, there are M_0, $M_k \in N$ such that

(i)′ $f^{-n}(x_1) \in U_{\delta/4}(\alpha(x_1))$ for any $n \geqq M_0$

(ii)′ $f^n(x_k) \in U_{\delta/4}(\omega(x_k))$ for any $n \geqq M_k$.

Then let $t_i = f^{M_i}(x_i)$ ($1 \leqq i \leqq k$), and let $w_i = f^{-M_i}(x_{i+1})$ ($0 \leqq i \leqq k-1$) By [3], there are y^+_i and $y^-_i \in \Omega_i$ such that $t_i \in W^s_\delta(y^+_i)$ and $w_i \in W^u_\delta(y^-_i)$. Since $\omega(t_i) = \omega(y^+_i)$ and $\alpha(x_{i+1}) = \alpha(y^-_i)$, $y'_i \in \omega(y^+_i)$ and $y''_i \in \alpha(y^-_i)$. Hence by Lemma 5.4, for any $r > 0$, there is $v \in W^u(t_i) \cap B_r(t_i)$ such that $f^{n_i}(v) \in W^s_\delta(z_i) \cap B_r(z_i)$ for some $n_i \in N$. Since $f^{n_i}(v) \in W^s_\delta(z_i) \cap B_r(z_i)$, $f^{n_i+b-a}(v)$ is near to $f^{b-a}(z_i)$ for sufficient small $r > 0$. Let $u_{i-1} = \dim T_{t_i}(W^u(t_i))$, $s_i = \dim T_{z_i}(W^s_\delta(z_i))$ and $u_i = \dim T_{z_i}(W^u_\delta(z_i))$. Since $u_{i-1} + s_i \geqq \dim M$ by strong transversality condition and $u_i + s_i = \dim M$ by the hyperbolicity of Ω, $u_{i-1} \geqq u_i$. By generalized λ-lemma, we know that there is a u_i-disc D in $W^u(t_i) \cap B_r(t_i)$ such that

$$f^{n_i+b-a}(D) \text{ is } C^1\text{-close to } W^u_\delta(f^{b-a}(z_i))\,.$$

The stable manifold and the unstable manifold of f are the unstable manifold and the stable manifold of f^{-1} respectively. Hence by Lemma 5.4 applied to f^{-1}, there is $v' \in f^{n_i+b-a}(D)$ such that $f^{n'_i}(v') \in W^s(w_i) \cap B_r(w_i)$ ($W^s(w_i) \subset W^s(\Omega_{i+1})$) for some $n'_i \in N$. Hence there is a u_i-disc in $W^u(t_i) \cap B_r(t_i)$ such that $f^{m'}(\bar{D})$ is C^1-close to $W^u(w_i) \cap B_r(w_i)$, where $m' = n_i +$

$b-a+n_i'$. Therefore

(1) $f^{m'}(\bar{\bar{D}})$ is C^1-close to $W^u(w_i)\cap B_r(w_i)$ for any u_i-disc $\bar{\bar{D}}$ which is C^1-close to $\bar{D}$.

And if r is small, then

(2) $\bar{d}(O_f^+(t_i)\cup A_i\cup O_f^-(w_i), \{f^j(p): 0\leqq j\leqq m'\})<2\delta$ for any $p\in\bar{D}$.

We shall choose a point $x\in M$ such that $\bar{d}(A, O_f(x))<2\delta$. For any $1\leqq i\leqq k$, let

$$Q_\delta(x_i)=\{y\in M: d(f^j(x_i), f^j(y))<\delta \text{ for any } -M_i\leqq j\leqq M_i\}.$$

Then there is $r_1>0$ such that

$$B_{r_1}(t_i)\subset f^{M_i}(Q_\delta(x_i)),\ B_{r_1}(w_i)\subset f^{-M_i}(Q_\delta(x_{i+1})).$$

By Lemma 5.4 applied to f^{-1}, there is $\bar{v}\in W_\delta^u(z_0)\cap B_r(z_0)$ $(r<r_1)$ such that $f^{n_0}(\bar{v})\in W^s(w_0)\cap B_r(w_0)$ for some $n_0\in N$. Hence there is a u_0-disc D_0' in $W_\delta^u(z_0)\cap B_r(z_0)$ such that $f^{n_0}(D_0')$ is C^1-close to $W^u(w_0)\cap B_r(w_0)$. Since $D_0'\subset W_\delta^u(z_0)$,

$$\bar{d}(A_0, \text{closure of } \{f^j(p'): -\infty<j\leqq 0)<2\delta \qquad \text{for any } p'\in D_0'.$$

Hence if r is small, then

(3) $\bar{d}(A_0\cup O_f^-(w_0)$, closure of $\{f^j(p''): -\infty<j\leqq n_0\})<2\delta$ for any $p''\in D_0'$.

If $f^{n_0}(D_0')$ is sufficiently C^1-close to $W^u(w_0)\cap B_r(w_0)$, then

$$f^{n_0+M_0+M_1}(D_0') \text{ is } C^1\text{-close to } W^u(t_1)\cap B_r(t_1).$$

Then by (1), there is a u_1-disc D_1 in $f^{n_0+M_0+M_1}(D_0')$ such that

$$f^{m(1)}(D_1) \text{ is } C^1\text{-close to } W^u(w_1)\cap B_r(w_1)$$

$(m(i)=n_i+|I_i|+n_i'$ where $|I_i|=b-a$ as $I_i=[a,b])$. Hence there is a u_1-disc D_1 in D_1' such that

$$f^{n_0+M_0+M_1+m(1)}(D_1) \text{ is } C^1\text{-close to } W(w_1)\cap B_r(w_1).$$

Therefore

$$f^{M(2)}(D_1) \text{ is } C^1\text{-close to } W^u(t_2)\cap B_r(t_2)$$

where $M(j)=n_0+M_0+2\sum_{i=1}^{j-1}M_i+\sum_{i=1}^{j-1}m(i)+M_j$. By induction, there is a u_{k-1}-disc D_{k-1} in $W_\delta^u(z_0)\cap B_r(z_0)$ such that

$$f^{M(k)}(D_{k-1}) \text{ is } C^1\text{-close to } W^u(t_k)\cap B_r(t_k).$$

By Lemma 5.4, there is $y \in f^{M(k)}(D_{k-1})$ such that $f^{n_k}(y) \in W^s_\delta(z_k) \cap B_r(z_k)$. Hence

$$\bar{d}(A_k, \text{closure of } \{f^j(y): 0 \leqq j < +\infty\}) < 2\delta .$$

Let $x = f^{-M(k)}(y)$. Since $x \in W^u_\delta(z_0) \cap B_r(z_0)$,

$$\bar{d}(A_0, \text{closure of } \{f^j(x): -\infty < j \leqq n_0\}) < 2\delta$$

by (3). Since $f^{M(i)-M_i}(x) \in Q_\delta(x_i)$ for any i by the choice of r_1 and $r < r$,

$$\bar{d}(f^j(x_i), f^j(f^{M(i)-M_i}(x))) < \delta \qquad \text{for any } -M_{i-1} \leqq j \leqq M_i .$$

By (2), for any $1 \leqq i \leqq k-1$,

$$\bar{d}(O_f^+(t_i) \cup A_i \cup O_f^-(w_i), \{f^j(f^{M(i)}(x)): 0 \leqq j \leqq m(i)\}) < 2\delta .$$

Hence $d(A, O_f(x)) < 2\delta$. Since δ is arbitrary and O_f is closed in $C(M)$, $A \in O_f$. Hence $E_f \subset O_f$.

During the preparation of this paper, we heard that A. Morimoto gave a proof of Theorem [4] but our proof is a different from his.

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