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SPECTRAL PROPERTIES OF FIRST ORDER ORDINARY DIFFERENTIAL OPERATORS WITH SHORT RANGE POTENTIALS

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§1. Introduction and main theorem

The purpose of the present paper is to give a complete proof of the theorem which will be used in a paper of the second author [4].

We will discuss certain spectral properties of selfadjoint ordinary differential operators of the form $iA(d/dx) + V$ acting in $L^2(\boldsymbol{R})_n = \sum \oplus L^2(\boldsymbol{R})$, where A is a real diagonal constant matrix and V an Hermitian matrix valued function on $\boldsymbol{R}$ which satisfies some conditions to be stated in the sequel.

According to [1, p. 156] a function v in $L^2_{\rm loc}(\boldsymbol{R})$ is said to belong to the class SR if, for some $\varepsilon > 0$, the multiplication map: $u(x) \to (1 + |x|)^{1+\varepsilon}v(x)u(x)$ is a compact operator from the Sobolev space $H_1(\boldsymbol{R})$ into $L^2(\boldsymbol{R})$ (the square integrable functions on $\boldsymbol{R}$). For a selfadjoint operator L in a Hilbert space $\mathfrak{H}$, let $L_{\rm p}$, $L_{\rm c}$ and $L_{\rm ac}$ stand, respectively, for the restriction of L to the subspace $\mathfrak{H}_{\rm p}$ spanned by all eigenvectors, $\mathfrak{H}_{\rm p}^{\perp}$ (the orthogonal complement of $\mathfrak{H}_{\rm p}$) and the absolutely continuous subspace $\mathfrak{H}_{\rm ac}$([5], p. 516). Thus we have $L = L_{\rm p} \oplus L_{\rm c}$ and $L_{\rm c} \supset L_{\rm ac}$. Let A be a real diagonal $n \times n$-matrix with (j, j)-component a_j and V an Hermitian $n \times n$-matrix valued function on $\boldsymbol{R}$ with the (j, k)-component V_{jk} in $L^2_{\rm loc}(\boldsymbol{R})$. As will be shown in Lemma 1, the symmetric operator $\dot{L} = iA(d/dx) + V$ with domain $C_0^\infty(\boldsymbol{R})_n = \sum \oplus C_0^\infty(\boldsymbol{R})$ is essentially selfadjoint in the Hilbert space $L^2(\boldsymbol{R})_n = \sum \oplus L^2(\boldsymbol{R})$, we denote the selfadjoint extension of $\dot{L}$ by L. Then our main result is the following.

THEOREM. (i) *Assume that* $a_1 \cdots a_n \neq 0$. *If each matrix element* V_{jk} *of* V *belongs to the class* SR, *then* $L = L_{\rm ac}$. *Under the additional assumption that for some* $\varepsilon > 0$ *and* $0 < \theta < 1/2$ *each* V_{jk} *satisfies*

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$$(*)\qquad \sup_{x\in R}\left[(1+|x|)^{2+2\varepsilon}\int_{|x-y|\leqslant 1}|V_{jk}(y)|^2|y-x|^{1-2\theta}dy\right]<\infty\,,$$

L_{ac} is unitarily equivalent to the selfadjoint multiplication operator M in $L^2(R)_n$ defined by $Mf(\lambda)=\lambda f(\lambda)$. Note that the condition (*) is satisfied if $V_{jk}(x)=O(|x|^{-1-\varepsilon})$ as $|x|\to\infty$.

(ii) *Assume that $a_1=\cdots=a_m=0$ and $a_{m+1}\cdots a_n\neq 0$ for some $0<m<n$ and that*

$$(**)\qquad \begin{cases} V_{jk}=0 & \text{for } j,\ k=1,\cdots,m\,,\\ V_{jk} \text{ is bounded for } j=1,\cdots,m \text{ and } k=m+1,\cdots,n\,,\\ V_{jk} \text{ and } W_{jk}=\sum_{1\leqslant \ell\leqslant m}V_{j\ell}V_{\ell k} \text{ are of the class } SR \text{ for } j,k\\ \qquad =m+1,\cdots,n\,. \end{cases}$$

Then L has no eigenvalues differing from zero and $L_c=L_{ac}$. In addition, if each V_{jk} belongs to $C^1(R)$ and satisfies

$$V_{jk}(x)=O(|x|^{-1-\varepsilon})\qquad as\ |x|\longrightarrow\infty$$

for some $\varepsilon>0$, then L_c is unitarily equivalent to the selfadjoint multiplication operator M in $L^2(R)_{n-m}$ defined by $Mf(\lambda)=\lambda f(\lambda)$.

In § 3 a sufficient condition for L to have no eigenvalues will be found.

§ 2. Proof of the theorem

We proceed as Agmon [1]. To begin with, we explain our notations. The real and complex numbers will be denoted by R and C respectively. As usual, $C_\pm=\{z\in C:\pm\operatorname{Im}z>0\}$ and $R^*=R\backslash\{0\}$.

$$L^p_{loc}(R)=\left\{u(x):\int_K|u(x)|^pdx<\infty \text{ for any compact set } K \text{ in } R\right\}.$$

$L^2(R)=$ the square integrable functions with the usual norm $\|\ \ \|$. For real s,

$$L^{2,s}(R)=\{u(x):(1+x^2)^{s/2}u\in L^2(R)\} \text{ with the norm } \|\ \ \|_{0,s}:$$
$$\|u\|_{0,s}=\|(1+x^2)^{s/2}u\|\,.$$

For any integer $m\geq 0$ and real s, we define the weighted Sobolev space $H_{m,s}(R)$ by

$$H_{m,s}(R)=\{u(x):D^\alpha u\in L^{2,s}(R),\ 0\leqslant m\} \text{ with the norm } \|\ \ \|_{m,s}:$$
$$\|u\|_{m,s}=\Big(\sum_{0\leqslant\alpha\leqslant m}\|D^\alpha u\|^2_{0,s}\Big)^{1/2},\ \text{where } D=-i\frac{d}{dx}\,.$$

For real m, the Sobolev space $H_m(\boldsymbol{R})$ of order m is defined as the completion of $C_o^\infty(\boldsymbol{R})$ under the norm

$$\|u\|_m = \int |\hat{u}(\lambda)|^2(1 + \lambda^2)^m d\lambda .$$

Here $\hat{u}$ stands for the Fourier transform of u, namely,

$$\hat{u}(\lambda) = (2\pi)^{-1/2} \int u(x)e^{-ix\lambda}dx .$$

Thus $H_{m,0}(\boldsymbol{R}) = H_m(\boldsymbol{R})$ for non-negative integer m. The continuous functions and continuously differentiable functions on $\boldsymbol{R}$ will be denoted by $C(\boldsymbol{R})$ and $C^1(\boldsymbol{R})$ respectively. For any $0 < \theta < 1$ and real s we denote by $C^{\theta,s}(\boldsymbol{R})$ the continuous functions such that

$$|||u|||_{\theta,s} = \sup_{x\in \boldsymbol{R}} (1 + |x|)^s|u(x)| + \sup_{\substack{x,y \\ 0<|x-y|<1}} \left[(1 + |x|)^s \frac{|u(x) - u(y)|}{|x - y|^\theta}\right] < \infty .$$

C^n-valued functions on $\boldsymbol{R}$ whose components lie in $L^2(\boldsymbol{R})$, for example, will be denoted by $L^2(\boldsymbol{R})_n$.

Finally,

A: a real diagonal matrix with the (j, j)-component a_j.

V: an Hermitian matrix valued function on $\boldsymbol{R}$ whose (j, k)-component is V_{jk}.

$\tilde{V}$: an Hermitian matrix valued function on $\boldsymbol{R}$ whose (j, k)-component is $V_{jk}(m < j, k \leqslant n)$.

W: an Hermitian matrix valued function on $\boldsymbol{R}$ whose (j, k)-component is $W_{jk} = \sum_{1\leqslant \ell\leqslant m} V_{j\ell}V_{\ell k}(m < j, k \leqslant n)$.

D_L: the domain of the operator $L = iA(d/dx) + V$.

LEMMA 1. *The operator* $\dot{L} = iA(d/dx) + V$ *with domain* $C_o^\infty(\boldsymbol{R})_n$ *is essentially selfadjoint in* $L^2(\boldsymbol{R})_n$.

Proof. Recall that $V_{jk} \in L^2_{\text{loc}}(\boldsymbol{R})$. Obviously $\dot{L}$ is symmetric. Assume first that the diagonal matrix is non-degenerate. It remains only to show that the range of $\dot{L} - z$ is dense for any $z \in \boldsymbol{C}_\pm$. To this end, suppose that a $g \in L^2(\boldsymbol{R})_n$ satisfies

$$((\dot{L} - z)f, g) = 0 \qquad \text{for any } f \in C_o^\infty(\boldsymbol{R})_n . \tag{1}$$

Since $V_{jk} \in L^2_{\text{loc}}(\boldsymbol{R})$, (1) implies that g is absolutely continuous and that

$$iAg' + (V - z)g = 0 . \tag{2}$$

Thus it follows easily that

$$(3) \qquad (Ag(x), g(x))' = -2 \operatorname{Im} z(g(x), g(x)).$$

Since a monotone function in $L^1(\boldsymbol{R})$ is zero, the function $(Ag(x), g(x))$ is zero. Now from (3) it follows that $g = 0$. Next assume that $a_1 = \cdots = a_m = 0$ and $a_{m+1} \cdots a_n \neq 0$. Then (1) implies that components $g_j (m < j \leqslant n)$ are absolutely continuous. The rest of the proof is the same as that in the case where $\det A \neq 0$. Q.E.D.

Remark. The domain D_L is $H_1(\boldsymbol{R})_n$ in the case (i) and $L^2(\boldsymbol{R})_m \oplus H_1(\boldsymbol{R})_{n-m}$ in the case (ii) of our theorem. In order to verify this, recalling the theorem 4.3 of [5, p. 287], it suffices to show that there exist some constants $0 \leqslant a$ and $0 \leqslant b < 1$ such that

$$\|vf\|^2 \leqslant a^2\|f\|^2 + b^2\|f\|_1^2$$

for a function v belonging to class SR and for any $f \in H_1(\boldsymbol{R})$. To this end, note first that the following inequality holds for some constant c.

$$\|(1 + |x|)^{1+\varepsilon} vf\|^2 \leqslant c\|f\|_1^2 .$$

Hence there exists positive constant r such that

$$\int_{|x| \geqslant r} |vf|^2 dx \leqslant \|f\|_1^2/4 .$$

Since $\|f\|_\infty \leqslant c\|f\|_1^2$ for some constant, taking N large enough, we have

$$\int_{|x|<r} |vf|^2 dx = \left(\int_{|x|<r, |v| \leqslant N} + \int_{|x|<r, |v|>N}\right)|vf|^2 dx \leqslant N^2\|f\|^2 + \|f\|_1^2/4 .$$

2.1. Eigenvalues. The following lemma, together with Proposition 3 in § 3, implies that L has no eigenvalues in the case (i) and that L has no eigenvalues differing from zero in the case (ii).

LEMMA 2. *If v belongs to the class SR, then v is integrable.*

Proof. Assume that for a positive ε the map $u \to (1 + |x|)^{1+\varepsilon} vu$ is a compact operator from $H_1(\boldsymbol{R})$ into $L^2(\boldsymbol{R})$. Then $(1 + |x|)^{1+\varepsilon}|v||u|^2$ is integrable for any $u \in H_1(\boldsymbol{R})$, in particular, for $u = (1 + x^2)^{-(1+\varepsilon)/4}$. Q.E.D.

2.2. The limiting absorbtion principle.

Case (i). Let $R_0(z)$ be the resolvent $(iA(d/dx) - z)^{-1}$ for $z \in \boldsymbol{C}_\pm$. We note that the theorem 4.1 of [1] holds for $R_0(z)$, hence the boundary value

$R_o^{\pm}(\lambda)$ is a well defined bounded operator in $B(L^{2,s}(\boldsymbol{R})_n, H_{1,-s}(\boldsymbol{R})_n)$ for any $s > 1/2$.

DEFINITION. A function $u \in H_1^{loc}(\boldsymbol{R})_n$ will be called a λ-outgoing function (resp. λ-incoming function) if for $\lambda \in \boldsymbol{R}$ the relation holds:

$$u = R_o^+(\lambda)f \quad (\text{resp. } u = R_o^-(\lambda)f) \text{ for some } f \in L^{2,s}(\boldsymbol{R})_n$$

with some $s > 1/2$. Among several steps to prove the limiting absorption principle (cf. Theorem 4.2, [1]), Lemma 4.2 of [1] is the only one whose proof needs new idea. A difficulty arises because A is not necessarily definite. Therefore, we confine ourselves to the proof of the following

LEMMA 3 (cf. Lemma 4.2, [1]). *Let $u \in H_1^{loc}(\boldsymbol{R})_n$ be a λ-outgoing (λ-incoming) function satisfying a differential equation in the distribution sense:*

$$\left(iA\frac{d}{dx} + V - \lambda\right)u = 0, \tag{5}$$

where the matrix element of V are of class SR. Then u belongs to $H_{1,s}(\boldsymbol{R})_n$ for all real s.

Proof. We shall prove the lemma for u outgoing, the proof for u incoming is similar. By the assumption, $u = R_o^+(\lambda)\, f$ for some $f \in L^{2,s_o}(\boldsymbol{R})_n$, $s_o > 1/2$. This implies

$$\begin{aligned} u_j(x) &= ia_j^{-1}\int_x^{\infty} e^{-ia_j^{-1}(x-y)\lambda} f_j(y)dy \qquad \text{for } j \in J_+ = \{j\colon a_j > 0\} \\ &= -ia_j^{-1}\int_{-\infty}^{x} e^{-ia_j^{-1}(x-y)\lambda} f_j(y)dy \qquad \text{for } j \in J_- = \{j\colon a_j < 0\}. \end{aligned} \tag{6}$$

Since f is integrable, it follows that $u_j(\infty) = 0$ (resp. $u_j(-\infty) = 0$) for $j \in J_+$(resp. $j \in J_-$) and that u is absolutely continuous. Thus (5) holds in the ordinary sense, which yields, setting $\operatorname{Im} z = 0$ in (3), the function $(Au(x), u(x))$ is constant. Thus we have

$$0 \geqslant \lim_{x\to-\infty} \sum_{j\in J_-} a_j|u_j(x)|^2 = \sum_{1\leqslant j\leqslant n} a_j u_j(x)^2 = \lim_{x\to\infty} \sum_{j\in J_+} a_j|u_j(x)|^2 \geqslant 0.$$

From this and (6) follows that $\hat{f}_j(-\lambda a_j^{-1}) = 0$. From now, the reasoning in the proof of Lemma 4.2 of [1] is applicable. Q.E.D.

Case (ii). Let $R(z)$ be the resolvent $(iA(d/dx) + V - z)^{-1}$ for $z \in \boldsymbol{C}_{\pm}$, I_+ the injection $(f_{m+1}, \cdots, f_n)^t \to (0, \cdots 0, f_{m+1}, \cdots, f_n)^t$ and P_+(resp. P_o) the projection $(f_1, \cdots, f_n)^t \to (f_{m+1}, \cdots, f_n)^t$ (resp. $(f_1, \cdots, f_m)^t$). For $z \in \boldsymbol{C}_\pm$ we consider an operator $\tilde{L}(z)$ with domain $H_1(\boldsymbol{R})_{n-m}$:

$$\tilde{L}(z) = i\tilde{A}\frac{d}{dx} + \tilde{V} + z^{-1}W - z\,. \tag{7}$$

First of all, note that the inverse $\tilde{R}(z)$ of $\tilde{L}(z)$ exists and that it satisfies

$$R(z) = z^{-1}(-P_o + P_o VI_+\tilde{R}(z)P_+) \oplus I_+\tilde{R}(z)P_+\,. \tag{8}$$

In fact, given an $f \in L^2(\boldsymbol{R})_n$, the equation $(L - z)u = f$ has a unique solution $u = R(z)f \in L^2(\boldsymbol{R})_m \oplus H_1(\boldsymbol{R})_{n-m}$. As one sees easily, $u = R(z)f$ if and only if

$$\begin{aligned} &\left(i\tilde{A}\frac{d}{dx} + \tilde{V} + z^{-1}W - z\right)P_+u = P_+f + z^{-1}P_+VP_of\,, \\ &P_ou = z^{-1}(-P_of + P_oVu)\,. \end{aligned} \tag{9}$$

Since V_{jk} $(m<j\leqslant n,\ 1\leqslant k\leqslant m)$ is bounded, the range $(P_+ + z^{-1}P_+VP_o)(L^2(\boldsymbol{R})_n)$ is equal to $L^2(\boldsymbol{R})_{n+m}$. Now assume that for a given $f_+ \in L^2(\boldsymbol{R})_{n-m}$ the equation $\tilde{L}(z)u_+ = f_+$ admits two different solutions $u_+^{(j)}$ $(j = 1, 2)$. Then, from the preceding observation, the equation $(L - z)u = I_+f_+$ has two distinct solutions, which is a contradiction. The existence of $\tilde{R}(z)$ has been proved. Now (8) follows from (9). We will show that $\tilde{R}(z)$ is a $\boldsymbol{B}((L^{2,s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})$-valued continuous function on $\boldsymbol{C}_\pm$ which has a continuous extension on $\boldsymbol{C}_\pm \cup \boldsymbol{R}^*(s > 1/2)$. To this end, note that

$$\tilde{R}(z) + \tilde{R}_o(Z)(\tilde{V} + z^{-1}W)\tilde{R}(z) = \tilde{R}_o(z) \qquad \text{for } z \in \boldsymbol{C}_\pm\,, \tag{10}$$

where $\tilde{R}_0(z)$ denotes the resolvent $(i\tilde{A}(d/dx) - z)^{-1}$. Since $\tilde{V}$ as well as W belongs to SR class by the assumption (**), repeating the argument in the proof of Theorem 4.2 [1], together with Lemma 3, we see that a $\boldsymbol{B}(H_{1,-s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})_{n-m})$-valued function $\tilde{T}(z) = \tilde{R}_o(z)$ $(\tilde{V} + z^{-1}W)$ has continuous extensions on $\boldsymbol{C}_\pm \cup \boldsymbol{R}^*$ and that $I + \tilde{T}^\pm(z)$ $(z \in \boldsymbol{C}_\pm \cup \boldsymbol{R}^*)$ is invertible if and only if z is not an eigenvalue of L. Since L has no non-zero eigenvaues, $\tilde{R}(z)$ has the boundary values $\tilde{R}^\pm(\lambda) = (I + T^\pm(\lambda))^{-1}R_o^\pm(\lambda)$, which is automatically continuous in $\lambda \in \boldsymbol{R}^*$:

$$\lim_{\substack{z\to\lambda \\ \pm \mathrm{Im}\, z>0}} \tilde{R}(z) = R^\pm(\lambda) \qquad \text{in } \boldsymbol{B}(L^{2,s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})_{n-m})\,.$$

In view of (8), $R(z)$ is a $\boldsymbol{B}(L^{2,s}(\boldsymbol{R})_m, L^{2,s}(\boldsymbol{R})_m) \oplus \boldsymbol{B}(L^{2,s}(\boldsymbol{R})_{n-m}, H_{1,-s}(\boldsymbol{R})_{n-m})$-valued function which admits continuous extensions $R^\pm(z)$ on $\boldsymbol{C}_\pm \cup \boldsymbol{R}^*$. Now the absolute continuity of the spectrum of L on $\boldsymbol{R}^*$ follows.

2.3. The multiplicity of L_{ac}.

Case (i). We assume the condition (*). In our case Theorem 5.1 of [1] runs as follows.

PROPOSITION 1. *There exist two families $\varphi_\pm(x, \lambda)$ of generalized eigenfunctions of L defined for any $\lambda \in \boldsymbol{R}$ having the following properties (recall that L has no eigenvalues).*

(i) *As a function of x and λ, $\varphi_\pm(x, \lambda)$ is a measurable matrix valued function of class $L^2_{loc}(\boldsymbol{R}\times\boldsymbol{R})$.*

(ii) *For every fixed λ the function $\varphi_\pm(x, \lambda)$ belongs to $C(\boldsymbol{R}) \cap H_1^{loc}(\boldsymbol{R})$ and satisfies the differential equation $(iA(d/dx) + V - \lambda)\varphi_\pm(x, \lambda) = 0$.*

(iii) *For any vector g in $\boldsymbol{C}^m$, put $\varphi_o(x, \lambda) = e^{(iA)^{-1}x\lambda}|A|^{-1/2}$ and*

$$\varphi^g_\pm(x, \lambda) = \varphi_\pm(x, \lambda)g,\ \varphi^g_o(x, \lambda) = \varphi_o(x, \lambda)g\,.$$

Here $|A|^{-1/2}$ denotes the diagonal matrix with (j, j) component $|a_j|^{-1/2}$. Then for a fixed $\lambda \in \boldsymbol{R}$, the function $\varphi^g_\pm(x, \lambda)$ has the representation

$$\varphi^g_\pm(x, \lambda) = \varphi^g_o(x, \lambda) - R^\mp(\lambda)[V(.)\varphi^g_o(.\,, \lambda)](x)\,,$$

where $R^\mp(\lambda)$ are boundary values of the resolvent $R(z)$ of L. In particular $\varphi^g_\pm(x, \lambda)$ lies in $C^{0,-s}(\boldsymbol{R})_n \cap H_{1,-s}(\boldsymbol{R})_n$ for any $s > 1/2$ and satisfies the differential equation (5).

Therefore we can verify the eigenfunction expansion theorem for L along the line of the proof of Theorem 6.2 [1]. Namely, define bounded linear maps $F_\pm : L^2(\boldsymbol{R})_n \to L^2(\boldsymbol{R})_n$ by

$$F_\pm f(\lambda) = (2\pi)^{-1/2} \lim_{N\to\infty} \int_{|x|<N} \varphi^*_\pm(x, \lambda) f(x) dx \qquad \text{in } L^2(\boldsymbol{R})_n\,,$$

Then $F_\pm$ unitarily transforms L into the selfadjoint multiplication operator M defined by $Mf(\lambda) = \lambda f(\lambda)$.

Case (ii). We first note

PROPOSITION 2. *Let the potential V be of class $C^1(\boldsymbol{R})$ and satisfy the first condition of the conditions (**). Then the multiplicity of $L^\perp$ (the restriction of L to the orthogonal complement of the space $\mathfrak{H}_0$ spanned by eigenvectors for eigenvalue zero) is at most $n - m$.*

Proof. We shall show that L has an $(n - m)\times(n - m)$-matrix valued spectral matrix ρ. The proof follows the same development as that of Theorem 3.1 in Chapter 10 of [3]. However, in connection with the proof of Parseval equality we should note that the image $L(C_0^\infty(R)_n)$ is dense in the orthogonal complement $\mathfrak{H}_0^\perp$, that it is a subset of D_L because V is smooth and that, making use of notations in Chapter 10 [3], we have

$$\int_{\varepsilon<|\lambda|<1} \lambda^2 |g(\lambda)|^2 d\rho_\delta \leqslant \int_{\varepsilon<|\lambda|<1} |g|^2 d\rho_\delta \leqslant \int_R |Lf(x)|^2 dx ,$$
$$\int_{1<|\lambda|<\mu} \lambda^2 |g(\lambda)|^2 d\rho_\delta \leqslant \mu^{-2} \int_{1<|\lambda|<\mu} \lambda^4 |g(\lambda)|^2 d\rho_\delta \leqslant \mu^{-2} \int_R |L^2 f(x)|^2 dx .$$

The lemma below completes the proof of our theorem.

LEMMA 4. *Let L_0 be the selfadjoint operator $iA(d/dx)$ in $L(R)_n$. For any $f \in C_0^\infty(R)_n$ of the form $f=(0,\cdots,0,f_{m+1},\cdots,f_n)^t$, $e^{itL}e^{-itL_0}f$ converges strongly as $t\to\infty$.*

Proof. As is well known ([5], Theorem 3.7 in Chapter X), the convergence follows from the fact that $\|Ve^{-itL_0}f\|$ is integrable on some interval (t_0, ∞). By the assumption (***) there exist positive constants ε, K and r (>1) such that $|V_{jk}(x)| \leqslant K|x|^{-1-\varepsilon}$ for $|x|>r$. Since $(e^{-itL_0}f)_j(x) = f_j(x+a_jt)$, assuming that a finite interval $(-c, c)$ includes the support of f and denoting $\min_{m<j}|a_j|$ (resp. $\sup_{j,x}|f_j(x)|$) by a (resp. s), we have the following inequality:

$$\|Ve^{-itL_0}f\|^2 \leqslant 2cK^2s^2n^3|c+at|^{-2-2\varepsilon} ,$$

which yields the desired integrability of $\|Ve^{-itL_0}f\|$. Q.E.D.

Proposition 2 and Lemma 4 imply that L_{ac} is unitary equivalent to the multiplication operator in $L^2(R)_{n-m}$. Since we have shown that $L_c = L_{ac}$ (see 2.2), the last assertion of our theorem has been proved.

§ 3. Sufficient condition for L to have no eigenvalues

As stated in § 1, A denotes a real diagonal matrix, while V stands for an Hermitian matrix valued function of class $L^2_{loc}(R)$.

PROPOSITION 3. (i) *Assume that* $\det A \neq 0$. *If A is positive (or negative) definite or if V is integrable on a half line, then L has no eigenvalue.*

(ii) *Assume that $a_1=\cdots=a_m=0$ and $a_{m+1}\cdots a_n\neq 0$ for some $0<m<n$ and that*

$V_{jk} = 0 \quad$ *for* $1 \leqslant j,\ k \leqslant m$,

V_{jk} *and* $W_{jk} = \sum_{1 \leq l \leq m} V_{jl} V_{lk}$ *are integrable on a common half line*

for $m < j,\ k \leq n$,

then L has no eigenvalues differing from zero.

Proof. Suppose $u \in D_L$ satisfies the following equation for a real λ.

$$\left(iA\frac{d}{dx} + V - \lambda\right)u = 0\,. \tag{11}$$

We shall show that $u=0$. Note that u_j are absolutely continuous in the case (i) and that u_j $(j > m)$ are also absolutely continuous in the case (ii) (cf. the proof of Lemma 1). If A is definite, (3) implies that the function $(Au(x), u(x))$ is constant, thus $u=0$. If V is integrable, say on $(0, \infty)$, define $v \in L^2(\boldsymbol{R})_n$ by the formula $u=e^{(iA)^{-1}x\lambda}v$. Then v satisfies

$$v'(x)=e^{-(iA)^{-1}x\lambda}V(x)e^{(iA)^{-1}x\lambda}v(x).$$

Since v has a non-zero limit as $x \to \infty$, provided $v \neq 0$ ([3], problem 6 in Chapter 3), we conclude that $u=0$. In the case (ii) we must show $u=0$, assuming that $\lambda \neq 0$. We rewrite (11) in the form (9) with $f=0$ and $z=\lambda$. Since the Hermitian matrix valued function $\tilde{V} + \lambda^{-1}W$ is integrable on a half line, it follows that $P_+u=0$ via the same reasoning for the case (i). From the second equality of (9), $P_0u=0$. Thus $u=0$. Q.E.D.

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