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# EINSTEIN HYPERSURFACES OF KÄHLERIAN $C$-SPACES

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## Introduction

A compact simply connected homogeneous complex manifold is called a $C$-space. A $C$-space is said to be kählerian if it carries a Kähler metric. It is known (Matsushima [7]) that a kählerian $C$-space has always an Einstein Kähler metric which is essentially unique.

Let $M$ be a kählerian $C$-space of dimension $n$ whose second Betti number equals 1. Denote by $h$ the positive generator of $H^2(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$. For a hypersurface $X$ of $M$, we define a positive integer $a(X)$, called the degree of $X$, by

$$c_1(\{X\}) = a(X)h ,$$

where $\{X\}$ denotes the holomorphic line bundle on $M$ associated with the non-singular divisor $X$. Take an Einstein Kähler metric $g$ on $M$ and fix it. Then it is known for $M = P_n(\boldsymbol{C})$ that $a(X) \leqq 2$ for any hypersurface $X$ which is Einstein with respect to the metric induced by $g$ (Smyth [9], Hano [3]). In this note we shall show that there exists also an upper bound for the degrees of Einstein hypersurfaces of general $M$.

Let $H$ be the holomorphic line bundle on $M$ with $c_1(H) = h$ and set

$$N_\ell = \dim \Gamma(H^\ell) \qquad \text{for } \ell \in \boldsymbol{Z} ,$$

where $\Gamma(H^\ell)$ denotes the space of holomorphic sections of $H^\ell$. The $N_\ell$'s are computed by Weyl's formula and monotone increasing with respect to $\ell \geq 0$. We define further a positive integer $\kappa$ by

$$c_1(M) = \kappa h ,$$

and set

$$\varepsilon(M) = \mathrm{Max}\left\{\text{positive integer } a ; N_{n-\kappa+a} \leq N_{n-\kappa} + \binom{N_1}{n}\right\} .$$

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For example, $\varepsilon(P_n(C)) = 2$ $(n \geqq 2)$, $\varepsilon(Q_n(C)) = 1$ $(n \geqq 3)$ and $\varepsilon(G_{p,q}(C)) \leqq \binom{p+q}{p} - pq$ $(2 \leqq p \leqq q)$ (Sakane [8]), where $Q_n(C)$ and $G_{p,q}(C)$ denote the complex quadric of dimension $n$ and the complex Grassmann manifold of $p$-subspaces of $C^{p+q}$ respectively. Then (Theorem 5.3) we have an inequality:

$$a(X) \leqq \varepsilon(M)$$

for any Einstein hypersurface $X$ of $M$.

The above inequality for $M = G_{p,q}(C)$ was proved by the first named author in [8]. Essentially the idea of our proof is the same as that of [8]. But we prove the rationality of the dual map for the canonical projective imbedding $M \hookrightarrow P_m(C)$ of $M$ without the use of explicit form of defining equations for $M \subset P_m(C)$.

## § 1. Preliminaries

Let $M$ be a complex manifold*) of dimension $m$. The (complex) tangent bundle and the cotangent bundle of $M$ are denoted by $T(M)$ and $T^*(M)$ respectively. Let $K(M) = \Lambda^m T^*(M)$ and $K^*(M)$ be the canonical line bundle of $M$ and its dual line bundle respectively. Then $K^*(M) = \Lambda^m T(M)$ and hence the first Chern class $c_1$ satisfies

$$c_1(K^*(M)) = c_1(M) . \tag{1.1}$$

If $M$ carries a Kähler metric $g$, then the Ricci form $\sigma$ defined by $\sigma(X, Y) = S(X, JY)$, where $S$ is the Ricci curvature for $g$ and $J$ is the complex structure tensor for $M$, is closed and satisfies (cf. Kobayashi-Nomizu [6])

$$c_1(K^*(M))_R = -\frac{1}{4\pi}[\sigma] . \tag{1.2}$$

Here $c_R$ means the image of $c \in H^2(M, Z)$ under the group extension $H^*(M, Z) \to H^*(M, R)$, and $[\eta]$ means the de Rham class of a closed form $\eta$ on $M$.

*Remark.* Hermitian fibre metrics $h$ on $K^*(M)$ correspond one to one to positive volume elements $v$ of $M$ by

$$\langle v, (-2)^m(\sqrt{-1})^{m^2} x \wedge \bar{x} \rangle = h(x, x) \qquad \text{for } x \in K^*(M) . \tag{1.3}$$

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*) In this note, a manifold is always assumed to be connected.

For a holomorphic line bundle $F$ on $M$, we write

$$F^{\ell} = \underbrace{F\otimes \cdots \otimes F}_{\ell}\,, \quad F^{-\ell} = \underbrace{F^{*}\otimes \cdots \otimes F^{*}}_{\ell} \qquad \text{for } \ell > 0\,,$$
$$F^{0} = 1\,,$$

where $F^*$ is the dual line bundle of $F$ and 1 is the trivial line bundle on $M$.

A real cohomology class $c \in H^2(M, \boldsymbol{R})$ is said to be *positive* if $c$ is the de Rham class of a closed form $\eta$ on $M$ of bi-degree (1, 1) which has a local expression: $\eta = \sqrt{-1}\sum \eta_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}$ such that the matrix $(\eta_{\alpha\bar{\beta}})$ is positive definite. For example, let $M$ have a Kähler metric $g$ and $\omega$ the Kähler form for $g$ defined by $\omega(X, Y) = g(X, JY)$. Then $-[\omega]$ is a positive real cohomology class. Moreover, an integral cohomology class $c \in H^2(M, \boldsymbol{Z})$ is said to be *positive* if $c_{\boldsymbol{R}} \in H^2(M, \boldsymbol{R})$ is positive in the above sense.

Let $V$ be a finite dimensional complex vector space. The set of non-zero vectors of $V$ will be denoted by $V_*$. Then the group $\boldsymbol{C}_*$ of non-zero complex numbers acts on $V_*$ from the right in natural manner. The quotient complex manifold $V_*/\boldsymbol{C}_*$ is denoted by $P(V)$. In particular, in case of $V = \boldsymbol{C}^{m+1}$ $(m \geqq 1)$ we write $P_m(\boldsymbol{C})$ for $P(V)$. For $z \in (\boldsymbol{C}^{m+1})_*$, the class of $z$ in $P_m(\boldsymbol{C})$ is denoted by $[z]$. Then the map $\pi\colon (\boldsymbol{C}^{m+1})_* \to P_m(\boldsymbol{C})$ defined by

$$\pi(z) = [z] \qquad \text{for } z \in (\boldsymbol{C}^{m+1})_*$$

is holomorphic and we get a holomorphic principal bundle $\boldsymbol{C}_* \longrightarrow (\boldsymbol{C}^{m+1})_* \xrightarrow{\pi} P_m(\boldsymbol{C})$. For each $\ell \in \boldsymbol{Z}$ we define a holomorphic character $\iota_\ell$ of $\boldsymbol{C}_*$ by

$$\iota_\ell(a) = a^{\ell} \qquad \text{for } a \in \boldsymbol{C}_*\,.$$

The holomorphic line bundle associated to the principal bundle $\boldsymbol{C}_* \longrightarrow (\boldsymbol{C}^{m+1})_* \xrightarrow{\pi} P_m(\boldsymbol{C})$ by $\iota_1$ is denoted by $E$ and called the *standard line bundle* on $P_m(\boldsymbol{C})$. Note that then for each $\ell \in \boldsymbol{Z}$ $E^{\ell}$ is associated to the same principal bundle by $\iota_\ell$. Let $S_\ell(\boldsymbol{C}^{m+1})$ denote the space of homogeneous polynomials on $\boldsymbol{C}^{m+1}$ of degree $\ell \geqq 0$. Then $S_\ell(\boldsymbol{C}^{m+1})$ is canonically identified with the space $H^0(P_m(\boldsymbol{C}), E^{-\ell})$ of holomorphic sections of $E^{-\ell}$. In fact, each $F \in S_\ell(\boldsymbol{C}^{m+1})$ restricted to $(\boldsymbol{C}^{m+1})_*$ is a tensorial form on $(\boldsymbol{C}^{m+1})_*$ of type $\iota_{-\ell}$, and hence it defines an element $\hat{F} \in H^0(P_m(\boldsymbol{C}), E^{-\ell})$. The correspondence $F \mapsto \hat{F}$ gives the required identification. The standard norm of $\boldsymbol{C}^{m+1}$ is denoted by

$$\|z\| = \sqrt{\sum_{\alpha=0}^{m} |z^\alpha|^2} \qquad \text{for } z = \begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^m \end{bmatrix} \in C^{m+1} .$$

Then the function $z \mapsto \|z\|^2$ on $(C^{m+1})_*$ is a tensorial form of type $a \mapsto |a|^{-2}$, and hence it defines a hermitian fibre metric $h_E$ on $E$. The Chern form $\omega$ of $E$ associated to $h_E$ is given by

$$\pi^*\omega = \frac{1}{2\pi\sqrt{-1}} d'd'' \log \|z\|^2 ,$$

and we have

$$c_1(E)_R = [\omega] . \tag{1.4}$$

The symmetric tensor $g$ on $P_m(C)$ defined by $g(X, Y) = \omega(JX, Y)$, $J$ being the complex structure tensor for $P_m(C)$, is a Kähler metric on $P_m(C)$. It is called the *Fubini-Study metric* on $P_m(C)$. Note that then $\omega$ is the Kähler form for $g$. It is known (cf. Kobayashi-Nomizu [6]) that the Kähler manifold $(P_m(C), g)$ has constant holomorphic sectional curvature $8\pi$.

Let

$$u_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} (i+1 \quad \in C^{m+1} \quad (0 \leqq i \leqq m)$$

be the standard unit vectors of $C^{m+1}$. A frame $(e_0, e_1, \cdots, e_m)$ of $C^{m+1}$ is said to be *unimodular* if $e_0 \wedge e_1 \wedge \cdots \wedge e_m = u_0 \wedge u_1 \wedge \cdots \wedge u_m$. We denote by $P(m+1)$ the set of unimodular frames of $C^{m+1}$. It is identified with the group $SL(m+1)$ of unimodular $(m+1) \times (m+1)$ complex matrices in natural manner. We define a holomorphic map $p: P(m+1) \to P_m(C)$ by

$$p(e_0, e_1, \cdots, e_m) = [e_0] \qquad \text{for } (e_0, e_1, \cdots, e_m) \in P(m+1) .$$

The subgroup of $SL(m+1)$ consisting of all unimodular matrices of the form

$$\begin{pmatrix} \lambda & * \\ 0 & \alpha \end{pmatrix} \begin{matrix} \}1 \\ \}m \end{matrix}$$

is denoted by $SL(1, m)$. Then we get a holomorphic principal bundle $SL(1, m) \longrightarrow P(m+1) \xrightarrow{p} P_m(C)$. We define further a holomorphic map $\varphi: P(m+1) \to (C^{m+1})_*$ with $\pi \circ \varphi = p$ by

$$\varphi(e_0, e_1, \cdots, e_m) = e_0 \qquad \text{for } (e_0, e_1, \cdots, e_m) \in P(m+1) .$$

The subgroup of $SL(1, m)$ consisting of all unimodular matrices of the form

$$\begin{pmatrix} 1 & * \\ 0 & \alpha \end{pmatrix} \begin{matrix} \}1 \\ \}m \end{matrix}$$

is denoted by $SL_0(1, m)$. Then we get also a principal bundle $SL_0(1, m) \longrightarrow P(m+1) \xrightarrow{\varphi} (C^{m+1})_*$. We define a holomorphic character $\chi_\ell$ of $SL(1, m)$ by

$$\chi_\ell(a) = \lambda^\ell \qquad \text{for } a = \begin{pmatrix} \lambda & * \\ 0 & \alpha \end{pmatrix} \in SL(1, m) .$$

LEMMA 1.1. *For each $\ell \in Z$ $E^\ell$ is associated to the principal bundle*

$$SL(1, m) \longrightarrow P(m+1) \xrightarrow{p} P_m(C)$$

*by the character $\chi_\ell$.*

*Proof.* The map $\varphi: P(m+1) \to (C^{m+1})_*$ satisfies

$$\begin{aligned} \varphi(ua) &= \varphi(u)\chi_1(a) && \text{for } u \in P(m+1),\ a \in SL(1, m) , \\ \chi_\ell(a) &= \iota_\ell(\chi_1(a)) && \text{for } a \in SL(1, m) . \end{aligned}$$

Thus $\varphi$ induces an isomorphism from the line bundle associated to $P(m+1)$ by $\chi_\ell$ to the line bundle $E^\ell$ associated to $(C^{m+1})_*$ by $\iota_\ell$. q.e.d.

Next we define a holomorphic representation $\rho: SL(1, m) \to GL(m)$, the group of non-singular $m \times m$ complex matrices, by

$$\rho(a) = \lambda^{-1}\alpha \qquad \text{for } a = \begin{pmatrix} \lambda & * \\ 0 & \alpha \end{pmatrix} \in SL(1, m) .$$

LEMMA 1.2. *The tangent bundle $T(P_m(C))$ of $P_m(C)$ is associated to the principal bundle*

$$SL(1, m) \longrightarrow P(m+1) \xrightarrow{p} P_m(C)$$

*by the representation $\rho$.*

*Proof.* Let $GL(m) \longrightarrow F(P_m(C)) \xrightarrow{q} P_m(C)$ be the bundle of frames of $P_m(C)$. We define a holomorphic map $\psi: P(m+1) \to F(P_m(C))$ by

$$\psi(e_0, e_1, \cdots, e_m) = ((\pi_*)_{e_0}e_1, \cdots, (\pi_*)_{e_0}e_m) \quad \text{for } (e_0, e_1, \cdots, e_m) \in P(m+1),$$

identifying the tangent space $T_{e_0}((C^{m+1})_*)$ with $C^{m+1}$. Then it satisfies $q \circ \psi = p$ and

$$\psi(ua) = \psi(u)\rho(a) \quad \text{for } u \in P(m+1),\ a \in SL(1, m).$$

Thus the lemma follows as in Lemma 1.1. q.e.d.

## § 2. Dual map for a complex submanifold of $P_m(C)$

In this section, $M$ is always assumed to be a complex submanifold of $P_m(C)$ with dimension $n \geqq 1$. Let $r = m - n \geqq 0$ be the codimension of $M$. Let $j: M \to P_m(C)$ denote the inclusion. The Kähler metric on $M$ induced by the Fubini-Study metric $g$ on $P_m(C)$ and its Kähler form will be also denoted by $g$ and $\omega$ respectively. We set

$$\hat{M} = \pi^{-1}(M).$$

Then, restricting the bundle $C_* \longrightarrow (C^{m+1})_* \xrightarrow{\pi} P_m(C)$ to $M$, we get a holomorphic principal bundle $C_* \longrightarrow \hat{M} \xrightarrow{\pi} M$. Note that for each $\ell \in Z$ the induced bundle $j^*E^\ell$ is associated to $C_* \longrightarrow \hat{M} \xrightarrow{\pi} M$ by $\iota_\ell$. We set

$$I_\ell(M) = \{F \in S_\ell(C^{m+1});\ F|\hat{M} = 0\}.$$

We denote by $P(M)$ the totality of $(e_0, e_1, \cdots, e_m) \in P(m+1)$ such that

(i) $e_0 \in \hat{M}$, and

(ii) $e_1, \cdots, e_n \in T_{e_0}(\hat{M})$,

identifying $T_{e_0}(\hat{M})$ with a subspace of $C^{m+1}$. The subgroup of $SL(1, m)$ consisting of all unimodular matrices of the form

$$a = \begin{pmatrix} \lambda & * & * \\ 0 & \alpha & * \\ 0 & 0 & \beta \end{pmatrix} \begin{matrix} \}1 \\ \}n \\ \}r \end{matrix} \tag{2.1}$$

is denoted by $SL(1, n, r)$. Then we get a holomorphic principal bundle $SL(1, n, r) \longrightarrow P(M) \xrightarrow{p} M$, which is a subbundle of $SL(1, m) \longrightarrow j^*P(m+1) \xrightarrow{p} M$. Now Lemma 1.1 implies the following lemma.

LEMMA 2.1. *For each $\ell \in Z$ $j^*E^\ell$ is associated to the principal bundle*

$$SL(1, n, r) \longrightarrow P(M) \xrightarrow{p} M$$

*by the character* $\chi_\ell$.

We define further

$$SL_0(1, n, r) = SL_0(1, m) \cap SL(1, n, r) ,$$

and denote the inclusion $\hat{M} \to (C^{m+1})_*$ by $\hat{j}$. Then we get a holomorphic principal bundle $SL_0(1, n, r) \longrightarrow P(M) \xrightarrow{\varphi} \hat{M}$, which is a subbundle of $SL_0(1, m) \longrightarrow \hat{j}^*P(m+1) \xrightarrow{\varphi} \hat{M}$.

We define a holomorphic representation $\tau: SL(1, n, r) \to GL(n)$ by

$$\tau(a) = \lambda^{-1}\alpha \qquad \text{for } a = \begin{pmatrix} \lambda & * & * \\ 0 & \alpha & * \\ 0 & 0 & \beta \end{pmatrix} \in SL(1, n, r) .$$

Now Lemma 1.2 implies that $j^*T(P_m(C))$ is associated to $SL(1, n, r) \longrightarrow P(M) \xrightarrow{p} M$ by $\rho$. It follows that the subbundle $T(M)$ of $j^*T(P_m(C))$ is associated to the same principal bundle by $\tau$. Explicitly, the holomorphic map $\psi: P(M) \to F(M)$, the bundle of frames of $M$, defined by

$$\psi(e_0, e_1, \cdots, e_m) = ((\pi_*)_{e_0}e_1, \cdots, (\pi_*)_{e_0}e_n) \qquad \text{for } (e_0, e_1, \cdots, e_m) \in P(M)$$

provides an isomorphism from the vector bundle associated to $P(M)$ by $\tau$ to the tangent bundle $T(M)$. Since $\det \tau(a) = \lambda^{-n} \det \alpha$ for each $a \in SL(1, n, r)$ of (2.1), we have the following lemma.

LEMMA 2.2. *The line bundle $K^*(M)$ is associated to the principal bundle*

$$SL(1, n, r) \longrightarrow P(M) \xrightarrow{p} M$$

*by the holomorphic character of $SL(1, n, r)$ defined by*

$$a \mapsto \lambda^{-n} \det \alpha \qquad \textit{for } a = \begin{pmatrix} \lambda & * & * \\ 0 & \alpha & * \\ 0 & 0 & \beta \end{pmatrix} \in SL(1, n, r) .$$

Now we shall define the dual map for $M \subset P_m(C)$. Let $p$ be a point of $M$. Choose a vector $z \in \hat{M}$ such that $\pi(z) = p$. Then $T_z(\hat{M})$ is identified with a linear subspace of $C^{m+1}$ of codimension $r$, which is determined by $p$ and independent of the choice of $z$. The annihilator:

$$\vartheta(p) = \{\xi \in (C^{m+1})^*; \langle \xi, T_z(\hat{M}) \rangle = \{0\}\}$$

of $T_z(\hat{M})$ in the dual space $(C^{m+1})^*$ of $C^{m+1}$, is an $r$-dimensional linear subspace of $(C^{m+1})^*$, i.e., it is a point of the Grassmann manifold $Gr((C^{m+1})^*)$ of $r$-subspaces of $(C^{m+1})^*$. Regarding $Gr((C^{m+1})^*)$ as a submanifold of $P(\Lambda^r(C^{m+1})^*)$ by the Plücker imbedding, we get a map $\vartheta: M \to P(\Lambda^r(C^{m+1})^*)$, which is easily seen to be holomorphic. The map $\vartheta$ is called the *dual map* or *Gauss map* for $M \subset P_m(C)$.

The standard hermitian inner product on $C^{m+1}$ defines canonically a hermitian inner product on $\Lambda^r(C^{m+1})^*$. Identify $\Lambda^r(C^{m+1})^*$ with $C^{e+1}$, $e+1 = \binom{m+1}{r}$, by an orthonormal basis for $\Lambda^r(C^{m+1})^*$, and hence $P(\Lambda^r(C^{m+1})^*)$ with $P_e(C)$. Denote the Fubini-Study metric on $P_e(C)$ by $g'$.

The dual map $\vartheta$ is said to be a *rational map* of degree $d \geqq 0$ if there exists a homogeneous polynomial map $D: C^{m+1} \to \Lambda^r(C^{m+1})^*$ of degree $d$ such that (a) $D(\hat{M}) \subset (\Lambda^r(C^{m+1})^*)_*$ and (b) it induces the dual map $\vartheta: M \to P(\Lambda^r(C^{m+1})^*)$. If we identify $\Lambda^r(C^{m+1})^*$ with the dual space of $\Lambda^r(C^{m+1})$ by the pairing:

$$\langle \xi_1 \wedge \cdots \wedge \xi_r, e_1 \wedge \cdots \wedge e_r \rangle = \det(\langle \xi_i, e_j \rangle)_{1 \leqq i,j \leqq r}$$

for $\xi_i \in (C^{m+1})^*$ and $e_j \in C^{m+1}$, then the above conditions (a), (b) are equivalent to that

$$\langle D(e_0), e_{i_1} \wedge \cdots \wedge e_{i_r} \rangle = \begin{cases} \text{not zero} & \text{if } (i_1, \cdots, i_r) = (n+1, \cdots, m) \\ 0 & \text{otherwise} \end{cases}$$

for each frame $(e_0, e_1, \cdots, e_m)$ of $C^{m+1}$ with (i), (ii) and for each $0 \leqq i_1 < \cdots < i_r \leqq m$. Here, in case of $r = 0$, $e_{n+1} \wedge \cdots \wedge e_m$ will be understood to be $1 \in C$.

Assuming that the dual map $\vartheta: M \to P(\Lambda^r(C^{m+1})^*)$ is a rational map of degree $d \geqq 0$ induced by $D: C^{m+1} \to \Lambda^r(C^{m+1})^*$, we define

$$P_D(M) = \{(e_0, e_1, \cdots, e_m) \in P(M); \langle D(e_0), e_{n+1} \wedge \cdots \wedge e_m \rangle = 1\}.$$

For each $\ell \in Z$ the subgroup of $SL(1, n, r)$ consisting of all unimodular matrices $a$ of (2.1) such that

$$\lambda^{\ell-1} \det \alpha^{-1} = 1 ,$$

is denoted by $SL(1, n, r; \ell)$. Note that if for $(e_0, e_1, \cdots, e_m) \in P(M)$ and $a \in SL(1, n, r)$ of (2.1) we set $(e_0', e_1', \cdots, e_m') = (e_0, e_1, \cdots, e_m)a$, then

$$\begin{aligned} \langle D(e_0'), e_{m+1}' \wedge \cdots \wedge e_n' \rangle &= \lambda^d \det \beta \ \langle D(e_0), e_{m+1} \wedge \cdots \wedge e_n \rangle \\ &= \lambda^{d-1} \det \alpha^{-1} \langle D(e_0), e_{m+1} \wedge \cdots \wedge e_n \rangle . \end{aligned}$$

Here, in case of $r=0$, $\det\beta$ will be understood to be 1. It is not difficult to see from this that we have a holomorphic principal bundle $SL(1, n, r; d) \to P_D(M) \xrightarrow{p} M$, which is a subbundle of $SL(1, n, r) \to P(M) \xrightarrow{p} M$. Define $k \in \boldsymbol{Z}$ by

$$k = n + 1 - d \,. \tag{2.2}$$

Then, for each $a \in SL(1, n, r; d)$ of the form (2.1), we have

$$\lambda^{-n} \det \alpha = \lambda^{-n}\lambda^{d-1} = \lambda^{-(n+1-d)} = \lambda^{-k} = \chi_{-k}(a) \,.$$

It follows from Lemma 2.2 that $K^*(M)$ is associated to $SL(1, n, r; d) \to P_D(M) \xrightarrow{p} M$ by $\chi_{-k}$. Thus Lemma 2.1 implies that $K^*(M)$ is isomorphic to $j^*E^{-k}$. An explicit isomorphism is given as follows. The map $\varphi\colon P(m+1) \to (\boldsymbol{C}^{m+1})_*$ defined in § 1 by $\varphi(e_0, e_1, \cdots, e_m) = e_0$ induces a map $\varphi\colon P_D(M) \to \hat{M}$ with $\pi\circ\varphi = p$ satisfying

$$\begin{aligned} &\varphi(ua) = \varphi(u)\chi_1(a) && \text{for } u \in P_D(M),\ a \in SL(1, n, r; d)\,, \\ &\chi_{-k}(a) = \iota_{-k}(\chi_1(a)) && \text{for } a \in SL(1, n, r; d)\,. \end{aligned}$$

Therefore it induces a vector bundle isomorphism:

$$\varphi_D\colon K^*(M) \longrightarrow j^*E^{-k} \,. \tag{2.3}$$

In particular, by (1.4) we have

$$c_1(K^*(M))_{\boldsymbol{R}} = -k[\omega] \,. \tag{2.4}$$

The tensorial form $z \mapsto \|z\|^{-2k}$ on $\hat{M}$ of type $a \mapsto |a|^{2k}$ defines a hermitian fibre metric $h_k$ on $j^*E^{-k}$. Let $h_D$ be the hermitian fibre metric on $K^*(M)$ corresponding to $h_k$ under the isomorphism $\varphi_D$. Moreover, let $h$ be the hermitian fibre metric on $K^*(M)$ corresponding to the volume element $v = (-\frac{1}{2})^n\omega^n$ of $M$ (cf. Remark in § 1). With these notations we have the following theorem.

THEOREM 2.1. *Let the dual map* $\vartheta\colon M \to P(\Lambda^r(\boldsymbol{C}^{m+1})^*)$ *for* $M \subset P_m(\boldsymbol{C})$ *be a rational map of degree d induced by a polynomial map* $D\colon \boldsymbol{C}^{m+1} \to \Lambda^r(\boldsymbol{C}^{m+1})^*$. *Then we have*

$$h = \frac{n!}{(2\pi)^n}\frac{\|D(z)\|^2}{\|z\|^{2d}}h_D \,.$$

*Note here that the function* $z \mapsto \|D(z)\|^2/\|z\|^{2d}$ *on* $\hat{M}$ *can be regarded as a function on M.*

*Proof.* By Lemma 2.2, $K^*(M)$ is associated to $SL(1, n, r) \to P(M) \xrightarrow{p} M$ by the character $a \mapsto \lambda^{-n}\det\alpha$ of $SL(1, n, r)$. Therefore the tensorial form $F: P(M) \to \boldsymbol{R}^+$, the positive reals, corresponding to a hermitian fibre metric on $K^*(M)$ satisfies

$$F(ua) = |\lambda|^{-2n}|\det\alpha|^2 F(u) \qquad \text{for } u \in P(M),\ a \in SL(1, n, r)\,. \tag{2.5}$$

Let $F_h$ and $F_{h_D}$ be tensorial forms on $P(M)$ corresponding to $h$ and $h_D$ respectively. Then by (1.3)

$$\begin{aligned} F_h(e_0, e_1, \cdots, e_m) &= \langle v, (-2)^n(\sqrt{-1})^{n^2}(\pi_*)_{e_0}(e_1 \wedge \cdots \wedge e_n \wedge \bar{e}_1 \wedge \cdots \wedge \bar{e}_n)\rangle \\ &= \langle(\pi^*\omega^n)_{e_0}, (\sqrt{-1}e_1 \wedge \bar{e}_1) \wedge \cdots \wedge (\sqrt{-1}e_n \wedge \bar{e}_n)\rangle \end{aligned}$$

for each $(e_0, e_1, \cdots, e_m) \in P(M)$. In particular, if $(f_0, f_1, \cdots, f_m) \in P(M)$ is a unitary frame of $\boldsymbol{C}^{m+1}$, then

$$F_h(f_0, f_1, \cdots, f_m) = \frac{n!}{(2\pi)^n}\,, \tag{2.6}$$

since the Kähler form $\omega$ of $P_m(C)$ is $SU(m+1)$-invariant.

Now take an arbitrary $(e_0, e_1, \cdots, e_m) \in P_D(M)$. Then

$$F_{h_D}(e_0, e_1, \cdots, e_m) = \|e_0\|^{-2k}\,.$$

Choose a unitary frame $(f_0, f_1, \cdots, f_m) \in P(M)$ and $a \in SL(1, n, r)$ of the form (2.1) such that $(e_0, e_1, \cdots, e_m) = (f_0, f_1, \cdots, f_m)a$. Note here that then $\|e_0\| = |\lambda|$. Now (2.5) and (2.6) imply

$$\begin{aligned} F_h(e_0, e_1, \cdots, e_m) &= |\lambda|^{-2n}|\det\alpha|^2 F_h(f_0, f_1, \cdots, f_m) \\ &= \frac{n!}{(2\pi)^n}|\lambda|^{-2n}|\det\alpha|^2\,. \end{aligned} \tag{2.7}$$

On the other hand,

$$\begin{aligned} 1 = \langle D(e_0), e_{n+1} \wedge \cdots \wedge e_m\rangle &= \det\beta\,\langle D(e_0), f_{n+1} \wedge \cdots \wedge f_m\rangle \\ &= \lambda^{-1}\det\alpha^{-1}\langle D(e_0), f_{n+1} \wedge \cdots \wedge f_m\rangle \end{aligned}$$

implies

$$\langle D(e_0), f_{i_1} \wedge \cdots \wedge f_{i_r}\rangle = \begin{cases} \lambda\det\alpha & \text{if } (i_1, \cdots, i_r) = (n+1, \cdots, m) \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leqq i_1 < \cdots < i_r \leqq m$. Since the set $\{f_{i_1} \wedge \cdots \wedge f_{i_r};\ 0 \leqq i_1 < \cdots < i_r \leqq m\}$ is an orthonormal basis for $\Lambda^r(\boldsymbol{C}^{m+1})$, we have $\|D(e_0)\|^2 = |\lambda|^2|\det\alpha|^2$, and hence $|\det\alpha|^2 = |\lambda|^{-2}\|D(e_0)\|^2$. Substituting this into (2.7), we have

$$F_h(e_0, e_1, \cdots, e_m) = \frac{n!}{(2\pi)^n} |\lambda|^{-2(n+1)} \|D(e_0)\|^2$$
$$= \frac{n!}{(2\pi)^n} \|e_0\|^{-2(n+1)} \|D(e_0)\|^2 ,$$

and hence

$$\frac{F_h(e_0, e_1, \cdots, e_m)}{F_{h_D}(e_0, e_1, \cdots, e_m)} = \frac{n!}{(2\pi)^n} \frac{\|D(e_0)\|^2}{\|e_0\|^{2d}} .$$

This proves the theorem. q.e.d.

*Remark.* Hano [3] proved this theorem in case where $M$ is a complete intersection. Note that in this case the dual map is always a rational map.

THEOREM 2.2 (Hano [3]). *Let $M$ be a compact complex submanifold of $P_m(C)$ and let the dual map $\vartheta: M \to P(\Lambda^r(C^{m+1})^*)$ be a rational map of degree $d$ induced by a polynomial map $D: C^{m+1} \to \Lambda^r(C^{m+1})^*$. Then the following conditions are mutually equivalent*:

1) *The induced metric $g$ on $M$ is Einstein.*
2) *$\|D(z)\|^2/\|z\|^{2d}$ is a constant function on $M$.*
3) *$\vartheta^* g' = d \cdot g$.*

*In this case, we have an inequality*:

$$\dim (S_d(C^{m+1})/I_d(M)) \leqq \binom{m+1}{r} .$$

*Proof.* This was proved by Hano [3] in case where $M$ is a complete intersection. We can apply his proof to our case, since he used only the property of Theorem 2.1 in his proof. q.e.d.

## § 3. Kählerian $C$-spaces

A compact simply connected homogeneous complex manifold is called a *C-space.* A $C$-space is said to be *kählerian* if it has a Kähler metric. In this section we summarize some known results on kählerian $C$-spaces (cf. Borel-Hirzebruch [1], Takeuchi [10]).

(I) *A kählerian C-space M has always an Einstein Kähler metric which is essentially unique in the following sense*; *For any Einstein Kähler metrics $g, g'$ on $M$, there exist a holomorphism $\varphi$ of $M$ and a constant $c > 0$ such that $\varphi^* g' = cg$* (Matsushima [7]).

In what follows in this section, let $M$ be a kählerian $C$-space. Let $G$ denote the identity component $\mathrm{Aut}^0(M)$ of the group $\mathrm{Aut}(M)$ of holomorphisms of $M$. It is a connected complex semi-simple Lie group without the center. Fix a point $o \in M$ and set

$$U = \{\varphi \in G; \varphi(o) = o\} .$$

It is a closed connected complex Lie subgroup of $G$, and we have an identification: $M = G/U$. Let $\mathfrak{g} = \mathrm{Lie}\, G$, the Lie algebra of $G$, and denote the Killing form of $\mathfrak{g}$ by $(\ ,\ )$. Now $\mathfrak{u} = \mathrm{Lie}\, U$ is a parabolic Lie subalgebra of $\mathfrak{g}$ and described as follows. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{u}$ and denote the real part of $\mathfrak{h}$ by $\mathfrak{h}_{\boldsymbol{R}}$. The root system $\Sigma$ of $\mathfrak{g}$ relative to $\mathfrak{h}$ is identified with a subset of $\mathfrak{h}_{\boldsymbol{R}}$ by means of the duality defined by $(\ ,\ )$. Then there exist a lexicographic order $>$ on $\mathfrak{h}_{\boldsymbol{R}}$ and a subset $\Pi_0$ of the fundamental root system $\Pi$ with the following property; If we set $\Sigma_0 = \Sigma \cap \boldsymbol{Z}\Pi_0$ and $\Sigma_{\mathfrak{m}}^+ = \{\alpha \in \Sigma - \Sigma_0; \alpha > 0\}$, then $\mathfrak{u}$ is given by

$$\mathfrak{u} = \mathfrak{h} + \sum_{\alpha \in \Sigma_0 \cup \Sigma_{\mathfrak{m}}^+} \mathfrak{g}_\alpha ,$$

where $\mathfrak{g}_\alpha$ stands for the root space for $\alpha$.

Let $\{\Lambda_\alpha; \alpha \in \Pi\} \subset \mathfrak{h}_{\boldsymbol{R}}$ be the fundamental weights corresponding to $\Pi$. We set

$$\mathfrak{c} = \{H \in \mathfrak{h}_{\boldsymbol{R}}; (H, \Pi_0) = \{0\}\}$$

and

$$Z_{\mathfrak{c}} = \left\{\Lambda \in \mathfrak{c}; \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \in \boldsymbol{Z} \text{ for each } \alpha \in \Sigma\right\} ,$$

which is a lattice of $\mathfrak{c}$ generated the $\Lambda_\alpha$'s for $\alpha \in \Pi - \Pi_0$. Let $\tilde{G}$ be the universal covering group of $G$ and $\tilde{U}$ the (closed) connected complex Lie group of $\tilde{G}$ generated by $\mathfrak{u}$. Then we have also an identification: $M = \tilde{G}/\tilde{U}$. For each $\Lambda \in Z_{\mathfrak{c}}$ there exists a unique holomorphic character $\chi_\Lambda$ of $\tilde{U}$ such that $\chi_\Lambda(\exp H) = \exp(\Lambda, H)$ for each $H \in \mathfrak{h}$. Then the correspondence $\Lambda \mapsto \chi_\Lambda$ gives an isomorphism of $Z_{\mathfrak{c}}$ to the group of holomorphic characters of $\tilde{U}$. Let $F_\Lambda$ denote the holomorphic line bundle on $M$ associated to the principal bundle $\tilde{U} \to \tilde{G} \to M$ by $\chi_\Lambda$. The correspondence $\Lambda \to F_\Lambda$ induces a homiomorphism of $Z_{\mathfrak{c}}$ to the group $H^1(M, \mathcal{O}^*)$ of isomorphism classes of holomorphic line bundles on $M$. Also the correspondence $F \mapsto c_1(F)$ defines a homomorphism of $H^1(M, \mathcal{O}^*)$ to $H^2(M, \boldsymbol{Z})$.

(II) *Both of these homomorphisms*:

$$Z_{\mathfrak{c}} \xrightarrow{F} H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, Z)$$

*are isomorphisms* (Ise [5]).

Thus the second Betti number $b_2(M)$ is given by

$$b_2(M) = \dim \mathfrak{c} = \text{the cardinality of } \Pi - \Pi_0. \tag{3.1}$$

We define positive integers $k_\alpha$ by

$$k_\alpha = \sum_{\beta \in \Sigma^+_{\mathfrak{m}}} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \qquad \text{for } \alpha \in \Pi - \Pi_0 .$$

Let $\kappa$ be the greatest common divisor of $\{k_\alpha\}_{\alpha \in \Pi - \Pi_0}$ and set

$$\kappa_\alpha = \frac{k_\alpha}{\kappa} \qquad \text{for } \alpha \in \Pi - \Pi_0$$

and

$$\Lambda_0 = \sum_{\alpha \in \Pi - \Pi_0} \kappa_\alpha \Lambda_\alpha .$$

We define

$$Z_{\mathfrak{c}}^+ = \{\Lambda \in Z_{\mathfrak{c}}; (\Lambda, \alpha) > 0 \text{ for each } \alpha \in \Sigma^+_{\mathfrak{m}}\} .$$

Then we have

$$Z_{\mathfrak{c}}^+ = \sum_{\alpha \in \Pi - \Pi_0} Z^+ \Lambda_\alpha ,$$

where $Z^+$ denotes the set of positive integers. Thus we have $\Lambda_0 \in Z_{\mathfrak{c}}^+$. The set $Z_{\mathfrak{c}}^+$ is invariant under the action of the group $\mathrm{Aut}(\Pi, \Pi_0)$ defined by

$$\mathrm{Aut}(\Pi, \Pi_0) = \{\sigma \in GL(\mathfrak{h}_R); \sigma\Sigma = \Sigma, \sigma\Pi = \Pi, \sigma\Pi_0 = \Pi_0\} .$$

Let $\mathrm{Aut}(\Pi, \Pi_0)\backslash Z_{\mathfrak{c}}^+$ denote the quotient of $Z_{\mathfrak{c}}^+$ modulo $\mathrm{Aut}(\Pi, \Pi_0)$.

A holomorphic immersion $j: M \to P_m(C)$ is said to be $\mathrm{Aut}^0(M)$-*equivariant* or simply *equivariant*, if for each $\varphi \in G$ there exists an element $\Phi$ of $PL(m+1)$, the group of projective transformations of $P_m(C)$, such that $j \circ \varphi = \Phi \circ j$. Holomorphic immersions $j: M \to P_m(C)$ and $j': M \to P_{m'}(C)$ are said to be *equivalent* if $m = m'$ and there exist $\varphi \in \mathrm{Aut}(M)$ and $\Phi \in PL(m+1)$ such that $j \circ \varphi = \Phi \circ j'$. A Kähler metric $g$ on $M$ is called a *homogeneous Kähler metric* if the group $\mathrm{Aut}(M, g)$ of isometric holomorphisms of $(M, g)$ is transitive on $M$. A holomorphic immersion $j: M \to P_m(C)$ is called a *homogeneous Kähler immersion* or an *Einstein Kähler immersion* if the Kähler metric on $M$ induced by the Fubini-Study metric on $P_m(C)$ is homogeneous or Einstein. Homogeneous or Einstein Kähler immersions $j: M \to P_m(C)$ and $j': M \to P_{m'}(C)$ are said to be *equivalent* if $m = m'$ and there exist $\varphi \in \mathrm{Aut}(M)$ and an element $\Phi$ of $PU(m+1)$, the group of unitary projective transformations of $P_m(C)$, such that $j \circ \varphi = \Phi \circ j'$. Let $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{E}$ denote the set of equivalence classes of full equivariant holomorphic immersions, homogeneous Kähler immersions and Einstein Kähler immersions of $M$ respectively.

These immersions are constructed in the following way. Let $\mathfrak{g}_u$ be a compact real form of $\mathfrak{g}$ such that the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_u$ leaves $\mathfrak{h}$ invariant, and $G_u$ the (compact) connected Lie subgroup of $G$ generated by $\mathfrak{g}_u$. Take $\Lambda \in Z_c^+$ and let $\rho_\Lambda: \mathfrak{g}_u \to \mathfrak{su}(m+1)$ be an irreducible unitary representation of $\mathfrak{g}_u$ such that its $C$-linear extension $\rho_\Lambda: \mathfrak{g} \to \mathfrak{sl}(m+1)$ has the highest weight $\Lambda$. The extension of $\rho_\Lambda$ to $\tilde{G}$ will be also denoted by $\rho_\Lambda: \tilde{G} \to SL(m+1)$. Taking a highest weight vector $z_0 \in C^{m+1}$, we can define a full equivariant holomorphic imbedding $j_\Lambda: M = \tilde{G}/\tilde{U} \to P_m(C)$ by

$$j_\Lambda(x\tilde{U}) = [\rho_\Lambda(x)z_0] \qquad \text{for } x \in \tilde{G} .$$

The Kähler metric on $M$ induced by the Fubini-Study metric on $P_m(C)$ is denoted by $g_\Lambda$. Then $j_\Lambda$ is further a full homogeneous Kähler imbedding, and the identity component $\mathrm{Aut}^0(M, g_\Lambda)$ of $\mathrm{Aut}(M, g_\Lambda)$ coincides with $G_u$. Moreover we have:

(III) *The space of* $\mathrm{Aut}^0(M, g_\Lambda)$*-invariant closed 2-forms on $M$ coincides with the space of harmonic 2-forms on* $(M, g_\Lambda)$ (Takeuchi [10]).

For each $p \in Z^+$ we write $j_p$ and $g_p$ for $j_{p\Lambda_0}$ and $g_{p\Lambda_0}$ respectively. Then $j_p$ is a full Einstein Kähler imbedding, and the Ricci curvature $S_p$ for $g_p$ is given by

$$S_p = \frac{4\pi\kappa}{p} g_p . \tag{3.2}$$

Thus (1.1) and (1.2) imply

$$c_1(M)_R = -\frac{\kappa}{p}[\omega_p] , \tag{3.3}$$

where $\omega_p$ denotes the Kähler form for $g_p$. The imbedding $j_p$ is called the *p-th full Einstein Kähler imbedding* of $M$.

(IV) *Any Einstein Kähler immersion is a homogeneous Kähler immersion* (*by* (*I*)), *and any homogeneous Kähler immersion is an equivariant holomorphic immersion* (Takeuchi [10]). *Thus we have natural maps*:

$$\mathscr{E} \xrightarrow{\alpha} \mathscr{K} \xrightarrow{\beta} \mathscr{H} .$$

*The map* $\alpha$ *is injective and the map* $\beta$ *is bijective* (Takeuchi [10]).

(V) *The correspondence* $p \mapsto j_p$ *induces a bijection* $Z^+ \xrightarrow{\gamma} \mathscr{E}$, *and the correspondence* $\Lambda \mapsto j_\Lambda$ *induces a bijection* $\mathrm{Aut}(\Pi, \Pi_0)\backslash Z_c^+ \xrightarrow{\delta} \mathscr{K}$ (Takeuchi [10]).

Let $\Lambda \in Z_c^+$. We set

$$N_{\ell\Lambda} = \dim H^0(M, j_\Lambda^* E^{-\ell}) \qquad \text{for } \ell \in \boldsymbol{Z} .$$

For the imbedding $j_\Lambda\colon M \to P_m(C)$ and the standard line bundle $E$ on $P_m(C)$, we have

$$j_\Lambda^* E = F_\Lambda . \tag{3.4}$$

Thus, applying Borel-Weil-Bott theorem (Bott [2]) to the $F_\Lambda$'s we have the following:

(VI) *Let* $\Lambda \in Z_c^+$.

(i) *For each* $\ell \geqq 0$, $H^0(M, j_\Lambda^* E^{-\ell})$ *is an irreducible* $\tilde{G}$*-module with the lowest weight* $-\ell\Lambda$, *and* $H^p(M, j_\Lambda^* E^{-\ell}) = \{0\}$ *for* $p \geqq 1$. *Therefore* $N_{\ell\Lambda}$ $(\ell \geqq 0)$ *is given by Weyl's degree formula*:

$$N_{\ell\Lambda} = \prod_{\alpha>0} \frac{(\ell\Lambda + \delta, \alpha)}{(\delta, \alpha)} , \qquad \text{where } \delta = \frac{1}{2} \sum_{\alpha>0} \alpha .$$

(ii) *For each* $\ell > 0$, $H^0(M, j_\Lambda^* E^{\ell}) = \{0\}$ *and hence* $N_{-\ell\Lambda} = 0$.

COROLLARY. *For each* $\ell \geqq 0$, *we have an exact sequence*:

$$0 \longrightarrow I_\ell(M) \longrightarrow H^0(P_m(C), E^{-\ell}) \xrightarrow{j_\Lambda^*} H^0(M, j_\Lambda^* E^{-\ell}) \longrightarrow 0 .$$

*Proof.* The map $j_\Lambda^*$ is a non-trivial $\tilde{G}$-homomorphism and $H^0(M, j_\Lambda^* E^{-\ell})$ is an irreducible $\tilde{G}$-module by (VI). These imply the surjectivity of $j_\Lambda^*$. Moreover, since $H^0(P_m(C), E^{-\ell})$ is canonically identified with $S_\ell(C^{m+1})$, the kernel of $j_\Lambda^*$ is identified with $I_\ell(M)$. q.e.d.

*Remark* 1. Weyl's formula implies that $N_{\ell\Lambda} < N_{(\ell+1)\Lambda}$ for $\ell \geqq 0$, and hence the $N_{\ell\Lambda}$'s are monotone increasing with respect to $\ell \geqq 0$.

*Remark* 2. The above corollary for $\ell = 1$, the fullness of $j_\Lambda$ and (3.4) imply that $j_\Lambda^*: (C^{m+1})^* = H^0(P_m(C), E^{-1}) \to H^0(M, F_\Lambda^{-1})$ is a $\tilde{G}$-isomorphism. It follows that for each $\Lambda \in Z_c^+$ the holomorphic line bundle $F_\Lambda^{-1}$ is very ample and the associated Kodaira imbedding is equivalent to the holomorphic imbedding $j_\Lambda$. Conversely let $F_\Lambda$ for $\Lambda \in Z_c$ be very ample and let $j: M \to P_m(C)$ be the associated Kodaira imbedding. Then $F_\Lambda = j^*E^{-1}$ and hence $c_1(F_\Lambda)$ is positive. An explicit description (cf. Borel-Hirzebruch [1]) of the Chern form of $F_\Lambda$ shows that $\Lambda \in -Z_c^+$. Thus the set $\mathscr{H}$ corresponds one to one to the set of equivalence classes of Kodaira imbeddings of $M$.

## §4. Dual map for a kählerian $C$-space in $P_m(C)$

THEOREM 4.1. *Let $M$ be a kählerian $C$-space of dimension $n$ and $j: M \hookrightarrow P_m(C)$ a full equivariant holomorphic imbedding of codimension $r$. Then the dual map $\vartheta: M \to P(\Lambda^r(C^{m+1})^*)$ for $M \subset P_m(C)$ is a rational map if and only if*

1) *$j$ is equivalent to an Einstein Kähler imbedding, say $j_p$, and*
2) *$\kappa$ is divisible by $p$.*

*In this case, the degree $d$ of $\vartheta$ and the positive integer $k = \kappa/p$ is related as:*

$$d = n + 1 - k .$$

*Proof.* By (IV), (V) we may assume that $j = j_\Lambda$ for some $\Lambda \in Z_c^+$. The induced Kähler metric on $M$ is denoted by $g$, and the Kähler form, Ricci curvature, Ricci form for $g$ are denoted by $\omega, S, \sigma$ respectively.

Assume that $\vartheta$ is a rational map of degree $d$. Set $k = n + 1 - d$. Then by (1.2) and (2.4) we have

$$c_1(K^*(M))_R = -\frac{1}{4\pi}[\sigma] = -k[\omega] .$$

Since both $-(1/4\pi)\sigma$ and $-k\omega$ are $\mathrm{Aut}^0(M, g)$-invariant closed 2-forms, we have $-(1/4\pi)\sigma = -k\omega$ by (III). Thus $\sigma = 4\pi k\omega$, and hence $S = 4\pi kg$. This proves that $j = j_p$ for some $p \in Z^+$. In this case, by (3.2) we have $S = (4\pi\kappa/p)g$, and hence $k = \kappa/p$. This proves the assertion 2).

Assume conversely that $j = j_p$ for some $p \in Z^+$ and $k = \kappa/p$ is an integer. By (3.2), $S = 4\pi kg$ and hence $\sigma = 4\pi k\omega$. On the other hand, by (1.2) and (1.4) we have

$$c_1(K^*(M))_R = -\frac{1}{4\pi}[\sigma] = -k[\omega] = c_1(j^*E^{-k})_R ,$$

and hence $c_1(K^*(M)) = c_1(j^*E^{-k})$. Now (II) implies

$$K^*(M) \cong j^*E^{-k} . \tag{4.1}$$

Set $d = n + 1 - k$. We choose an orthonormal basis $\{u_0, u_1, \cdots, u_m\}$ of the representation space $C^{m+1}$ of $\rho_{pA_0}: \tilde{G} \to SL(m + 1)$ in such a way that $u_0$ is a highest weight vector and $\{u_0, u_1, \cdots, u_n\}$ span $\rho_{pA_0}(\mathfrak{g})u_0$. We may assume that $\rho_{pA_0}$ is a matrix representation with respect to this basis. We denote by $\hat{G}$ the quotient group of $\tilde{G}$ modulo the kernel of $\rho_{pA_0}$. Then it is identified with a closed subgroup of $SL(m + 1) = P(m + 1)$. We define

$$\hat{U} = \hat{G} \cap SL(1, m) \subset SL(1, n, r) .$$

Then we have an identification: $M = \hat{G}/\hat{U}$ and the principal bundle $\hat{U} \to \hat{G} \xrightarrow{p} M$ may be identified with a subbundle of $SL(1, n, r) \to P(M) \xrightarrow{p} M$. We define further

$$\hat{U}_0 = \hat{U} \cap SL_0(1, m) \subset SL_0(1, n, r) .$$

Then we have an identification: $\hat{M} = \hat{G}/\hat{U}_0$ and the principal bundle $\hat{U}_0 \to \hat{G} \xrightarrow{\varphi} \hat{M}$ may be identified with a subbundle of $SL_0(1, n, r) \to P(M) \xrightarrow{\varphi} \hat{M}$. Now Lemmas 2.1 and 2.2 imply that $j^*E^{-k}$ and $K^*(M)$ are associated to $\hat{U} \to \hat{G} \xrightarrow{p} M$ by the characters

$$a \mapsto \lambda^{-k} \quad \text{and} \quad a \mapsto \lambda^{-n} \det \alpha \quad \text{for } a = \begin{pmatrix} \lambda & * & * \\ 0 & \alpha & * \\ 0 & 0 & \beta \end{pmatrix} \in \hat{U}$$

of $\hat{U}$ respectively. It follows from (4.1) and (II) that $\lambda^{-k} = \lambda^{-n} \det \alpha$, and hence $\lambda^{d-1} \det \alpha = \lambda^{n-k} \det \alpha = 1$ for each $a \in \hat{U}$. This means

$$\hat{U} \subset SL(1, n, r; d) . \tag{4.2}$$

Now we shall define a map $D: \hat{M} \to (\Lambda^r(C^{m+1})^*)_*$ such that

$$\langle D(e_0), e_{i_1} \wedge \cdots \wedge e_{i_r} \rangle = \begin{cases} 1 & \text{if } (i_1, \cdots, i_r) = (n + 1, \cdots, m) \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

for each $(e_0, e_1, \cdots, e_m) \in \hat{G}$ and for each $0 \leqq i_1 < \cdots < i_r \leqq m$. Let $z \in \hat{M}$. Choose $(e_0, e_1, \cdots, e_m) \in \hat{G}$ with $e_0 = z$ and define $D(z) \in (\Lambda^r(C^{m+1})^*)_*$ by

$$\langle D(z), e_{i_1} \wedge \cdots \wedge e_{i_r} \rangle = \begin{cases} 1 & \text{if } (i_1, \cdots, i_r) = (n+1, \cdots, m) \\ 0 & \text{otherwise} . \end{cases}$$

Another $(e'_0, e'_1, \cdots, e'_m) \in \hat{G}$ with $e'_0 = z$ can be written as

$$(e'_0, e'_1, \cdots, e'_m) = (e_0, e_1, \cdots, e_m) \begin{pmatrix} 1 & * & * \\ 0 & \alpha & * \\ 0 & 0 & \beta \end{pmatrix}$$

with $\det \alpha = \det \beta = 1$ by (4.2). Thus we have

$$\begin{aligned} \langle D(z), e'_{i_1} \wedge \cdots \wedge e'_{i_r} \rangle &= \langle D(z), e_{i_1} \wedge \cdots \wedge e_{i_r} \rangle \\ &= \begin{cases} 1 & \text{if } (i_1, \cdots, i_r) = (n+1, \cdots, m) \\ 0 & \text{otherwise} . \end{cases} \end{aligned}$$

This shows that $D$ is well defined and satisfies (4.3). The map $D$ is holomorphic. In fact, choose a local holomorphic section $s(z) = (z, e_1(z), \cdots, e_m(z))$ of the bundle $\hat{U}_0 \to \hat{G} \xrightarrow{\varphi} \hat{M}$. Then we have

$$\langle D(z), e_{i_1}(z) \wedge \cdots \wedge e_{i_r}(z) \rangle = \begin{cases} 1 & \text{if } (i_1, \cdots, i_r) = (n+1, \cdots, m) \\ 0 & \text{otherwise} , \end{cases}$$

and hence $D(z)$ is holomorphic in $z$. We shall next show that $D$ is homogeneous of degree $d$. Let $z \in \hat{M}$ and $\lambda \in C_*$ be arbitrary. Choose $(e_0, e_1, \cdots, e_m) \in \hat{G}$ with $e_0 = z$ and an element $a \in \hat{U}$ of the form (2.1), and set $(e'_0, e'_1, \cdots, e'_m) = (e_0, e_1, \cdots, e_m)a$. Then we have

$$\begin{aligned} \langle D(e_0), e'_{i_1} \wedge \cdots \wedge e'_{i_r} \rangle &= \det \beta \, \langle D(e_0), e_{i_1} \wedge \cdots \wedge e_{i_r} \rangle \\ &= \det \beta \, \langle D(e'_0), e'_{i_1} \wedge \cdots \wedge e'_{i_r} \rangle \end{aligned}$$

for each $0 \leqq i_1 < \cdots < i_r \leqq m$, and hence

$$D(e'_0) = \det \beta^{-1} D(e_0) = \lambda^d D(e_0)$$

by (4.2). Thus we get the required property:

$$D(\lambda z) = \lambda^d D(z) \qquad \text{for each } \lambda \in C_*, \ z \in \hat{M} .$$

Therefore, if we define

$$D_{i_1 \cdots i_r}(z) = \langle D(z), u_{i_1} \wedge \cdots \wedge u_{i_r} \rangle \qquad \text{for } z \in \hat{M}$$

for $0 \leqq i_1 < \cdots < i_r \leqq m$, then $D_{i_1 \cdots i_r}$ may be identified with an element of $H^0(M, j^* E^{-d})$. Since $D_{i_1 \cdots i_r} \neq 0$ for some $(i_1, \cdots, i_r)$, we have $d \geqq 0$ by (VI) (ii). It follows from Corollary of (VI) that each $D_{i_1 \cdots i_r}$ is extended to a homogeneous polynomial on $C^{m+1}$ of degree $d$, and hence $D$ is extended to a homogeneous polynomial map $\tilde{D}: C^{m+1} \to \Lambda^r(C^{m+1})^*$ of degree $d$. It is clear from (4.3) that $\tilde{D}$ induces the dual map $\vartheta$ for $M \subset P_m(C)$. q.e.d.

COROLLARY. *We have* $\kappa \leqq n+1$. *The equality holds if and only if* $M = P_n(C)$.

*Proof.* Consider the first full Einstein Kähler imbedding $j_1: M \to P_m(C)$. It follows from the above theorem that the dual map $\vartheta$ for $j_1$ is a rational map of degree $d = n + 1 - \kappa$, where $d \geqq 0$. This implies the required inequality. The equality holds if and only if $d = 0 \Leftrightarrow D: \hat{M} \to (\Lambda^r(C^{m+1})^*)_*$ is a constant map $\Leftrightarrow r = 0$ (since $j_1$ is full) $\Leftrightarrow M = P_n(C)$. q.e.d.

## §5. Einstein hypersurfaces of kählerian $C$-spaces

We assume in this section that $M$ is a kählerian $C$-space with $b_2(M) = 1$. Then by (3.1) $\Pi - \Pi_0$ consists of only one root, say $\alpha_0$. Thus we have $\mathfrak{c} = \boldsymbol{R}\Lambda_{\alpha_0}$, $Z_{\mathfrak{c}} = \boldsymbol{Z}\Lambda_{\alpha_0}$, $\kappa = k_{\alpha_0}$, $\kappa_{\alpha_0} = 1$, $\Lambda_0 = \Lambda_{\alpha_0}$, $Z_{\mathfrak{c}}^+ = \boldsymbol{Z}^+\Lambda_{\alpha_0}$ and $\mathrm{Aut}(\Pi, \Pi_0)\backslash Z_{\mathfrak{c}}^+$ is identified with $\boldsymbol{Z}^+\Lambda_0$. We write $N_\ell$ for $N_{\ell\Lambda_0}$. Now (IV) and (V) imply the following theorem.

THEOREM 5.1. *For a kählerian C-space M with* $b_2(M) = 1$, *the maps*:

$$\boldsymbol{Z}^+ \xrightarrow{\gamma} \mathscr{E} \xrightarrow{\alpha} \mathscr{K} \xrightarrow{\beta} \mathscr{H}$$

*are all bijections.*

The full equivariant holomorphic imbedding of $M$ corresponding to $1 \in \boldsymbol{Z}^+$ under the above bijection, will be called the *canonical projective imbedding* of $M$.

Let $j_1: M \to P_m(C)$ be the first full Einstein Kähler imbedding of $M$. The induced Kähler form on $M$ is denoted by $\omega$. Recall that we have isomorphisms:

$$(5.1) \qquad \boldsymbol{Z}\Lambda_0 \xrightarrow{F} H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \boldsymbol{Z}) .$$

We set

$$H = F_{\Lambda_0}^{-1}, \qquad h = c_1(H) .$$

Then, by (3.4) we have $H = j_1^* E^{-1}$. It follows $c_1(H) = -j_1^* c_1(E)$, and hence $c_1(H)_{\boldsymbol{R}} = -[\omega]$ by (1.4). Thus $h$ is the positive generator of $H^2(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$. Note that (3.3) implies

$$c_1(M) = \kappa h .$$

Note also that $N_\ell$ is given by

$$N_\ell = \dim H^0(M, H^\ell) .$$

For a divisor $D$ on $M$, $\{D\}$ denotes the holomorphic line bundle on $M$ associated to $D$. Then for a positive divisor $D$ on $M$, there exists a positive integer $a(D)$ such that

$$c_1(\{D\}) = a(D)h .$$

The integer $a(D)$ is called the *degree* of $D$. For a hypersurface $X$ of $M$, the degree of the positive divisor defined by $X$ is called the *degree* of $X$ and denoted by $a(X)$.

LEMMA 5.1. *Let $X$ be a compact hypersurface of $M$ with degree $a$ and regard it as a complex submanifold of $P_m(C)$ through $j_1: M \to P_m(C)$. Then*

$$\dim (S_\ell(C^{m+1})/I_\ell(X)) = N_\ell - N_{\ell-a} \qquad \text{for } \ell \geqq a .$$

*Proof.* In general, for a complex manifold $M$, a non-singular divisor $S$ on $M$ and a holomorphic vector bundle $W$ on $M$, we have an exact sequence:

$$0 \longrightarrow \mathcal{O}(W) \longrightarrow \mathcal{O}(W \otimes \{S\}) \longrightarrow \mathcal{O}((W \otimes \{S\}) \,|\, S) \longrightarrow 0 ,$$

where $\mathcal{O}$ means the sheaf of germs of holomorphic sections (cf. Hirzebruch [4]). We apply this to the divisor $S$ defined by $X$ and $W = j_1^* E^{-\ell+a}$. Since $c_1(\{S\}) = ah = ac_1(j_1^* E^{-1}) = c_1(j_1^* E^{-a})$, we have $\{S\} = j_1^* E^{-a}$ by (5.1). Therefore we have an exact sequence:

$$0 \longrightarrow \mathcal{O}(j_1^* E^{-\ell+a}) \longrightarrow \mathcal{O}(j_1^* E^{-\ell}) \longrightarrow \mathcal{O}(i^* E^{-\ell}) \longrightarrow 0 ,$$

where $i: X \to P_m(C)$ denotes the inclusion. In the cohomology exact sequence:

$$0 \longrightarrow H^0(M, j_1^* E^{-\ell+a}) \longrightarrow H^0(M, j_1^* E^{-\ell}) \longrightarrow H^0(X, i^* E^{-\ell}) \longrightarrow H^1(M, j_1^* E^{-\ell+a}) ,$$

the last term vanishes for $\ell \geqq a$ by (VI) (i), and hence

$$\dim H^0(X, i^* E^{-\ell}) = N_\ell - N_{\ell-a} .$$

On the other hand, $H^0(P_m(C), E^{-\ell}) \to H^0(M, j_1^* E^{-\ell})$ is surjective by Corollary of (VI). Together with the surjectivity of $H^0(M, j_1^* E^{-\ell}) \to H^0(X, i^* E^{-\ell})$, we get the surjectivity of $H^0(P_m(C), E^{-\ell}) \to H^0(X, i^* E^{-\ell})$. This implies

$$H^0(X, i^*E^{-\ell}) \cong S_\ell(C^{m+1})/I_\ell(X) .$$

Thus we get our assertion. q.e.d.

THEOREM 5.2 (Ise [5]). *Let $M$ be a kählerian C-space with $b_2(M) = 1$ and $j: M \to P_m(C)$ the canonical projective imbedding of $M$. Then, for each positive divisor $D$ on $M$ of degree $a$, there exists a homogeneous polynomial $F$ on $C^{m+1}$ of degree $a$ such that $D$ is the pull back by $j$ of the divisor on $P_m(C)$ defined by $F$.*

*Remark.* In case where $D$ is the divisor defined by a hypersurface $X$ of $M$, we have

$$\hat{X} = \{z \in \hat{M}; F(z) = 0\} , \quad \text{and} \quad (\hat{j}^* dF)(z) \neq 0 \quad \text{for each } z \in \hat{X} ,$$

where $\hat{j}: \hat{M} \to C^{m+1}$ denotes the inclusion.

For a kählerian $C$-space $M$ of dimension $n$ with $b_2(M) = 1$, we define

$$\varepsilon(M) = \text{Max}\left\{a \in Z^+; N_{n-\kappa+a} \leqq N_{n-\kappa} + \binom{N_1}{n}\right\} .$$

Note that $\varepsilon(M)$ is finite since the $N_\ell$'s are monotone increasing with respect to $\ell \geqq 0$ (Remark 1 in § 3).

THEOREM 5.3. *Let $M$ be a kählerian C-space of dimension $n \geqq 2$ with $b_2(M) = 1$, and $g$ an Einstein Kähler metric on $M$. Then, for any compact hypersurface $X$ of $M$ which is Einstein with respect to the metric induced by $g$, we have an inequality:*

$$a(X) \leqq \varepsilon(M) .$$

*Proof.* Since an Einstein Kähler metric on $M$ is essentially unique by (I), we may assume that $g$ is induced from the Fubini-Study metric by the first full Einstein Kähler imbedding $j_1: M \to P_m(C)$. Here $m + 1 = N_1$ by (VI). Let $r$ be the codimension of $M$ in $P_m(C)$. We regard $X$ as a complex submanifold of $P_m(C)$ through $j_1$ and denote the inclusion by $i: X \to P_m(C)$. Then the metric on $X$ induced by the Fubini-Study metric on $P_m(C)$ is Einstein from the assumption.

By Theorem 4.1, the dual map $\vartheta'$ for $j_1$ is a rational map of degree $n + 1 - \kappa$. Let $\vartheta'$ be induced by a polynomial map $D': C^{m+1} \to \Lambda^r(C^{m+1})^*$. Take a homogeneous polynomial $F$ on $C^{m+1}$ of degree $a(X)$ which has the property in Theorem 5.2 for the divisor on $M$ defined by $X$. We define a map $D: C^{m+1} \to \Lambda^{r+1}(C^{m+1})^*$ by

$$D = D' \wedge dF\,.$$

It is clearly a homogeneous polynomial map of degree

$$d = n + 1 - \kappa + a(X) - 1 = n - \kappa + a(X)\,.$$

Recalling Remark following Theorem 5.2, we see that $D(\hat{X}) \subset (\Lambda^{r+1}(C^{m+1})^*)_*$ and $D$ induces the dual map $\vartheta: X \to P(\Lambda^{r+1}(C^{m+1})^*)$ for $i: X \to P_m(C)$. Then, by Theorem 2.2 we have an inequality:

$$\dim (S_{n-\kappa+a(X)}(C^{m+1})/I_{n-\kappa+a(X)}(X)) \leqq \binom{m+1}{r+1} = \binom{m+1}{n} = \binom{N_1}{n}\,.$$

Assume first $M \neq P_n(C)$. Then $n - \kappa + a(X) \geqq a(X)$ by Corollary of Theorem 4.1, and hence by Lemma 5.1

$$\dim (S_{n-\kappa+a(X)}(C^{m+1})/I_{n-\kappa+a(X)}(X)) = N_{n-\kappa+a(X)} - N_{n-\kappa}\,.$$

Thus we get

$$N_{n-\kappa+a(X)} \leqq N_{n-\kappa} + \binom{N_1}{n}\,.$$

This implies the required inequality in this case.

Assume next $M = P_n(C)$. Then $\kappa = n + 1$, $m = n$ and $X$ is a hypersurface of $P_n(C)$ of degree $a(X)$. Therefore $n - \kappa + a(X) < a(X)$ and $n - \kappa < 0$, and hence $I_{n-\kappa+a(X)}(X) = \{0\}$ and $N_{n-\kappa} = 0$. Thus we have also

$$\begin{aligned}\dim (S_{n-\kappa+a(X)}(C^{m+1})/I_{n-\kappa+a(X)}(X)) &= \dim S_{n-\kappa+a(X)}(C^{n+1}) \\ &= N_{n-\kappa+a(X)} - N_{n-\kappa}\,.\end{aligned}$$

This implies the required inequality for $M = P_n(C)$. q.e.d.

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