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ON THE LEAST POSITIVE EIGENVALUE OF THE LAPLACIAN FOR THE COMPACT QUOTIENT OF A CERTAIN RIEMANNIAN SYMMETRIC SPACE

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§1. Introduction and statement of results

Let $(\tilde{M}, g)$ be the standard Euclidean space or a Riemannian symmetric space of non-compact type of rank one. Let G be the identity component of the Lie group of all isometries of $(\tilde{M}, g)$. Let Γ be a discrete subgroup of G acting fixed point freely on $\tilde{M}$ whose quotient manifold M_Γ is compact. Let $-\Delta_\Gamma$ be the Laplace-Beltrami operator (cf. [4]) acting on smooth functions on M_Γ for the Riemannian metric g_Γ on M_Γ induced by g. The compactness of M_Γ implies that the spectrum of Δ_Γ forms a discrete subset of the set of non-negative real numbers. Let $\lambda_1(\Gamma)$ be the least positive eigenvalue of Δ_Γ. Let $\operatorname{vol}(M_\Gamma)$ be the volume of (M_Γ, g_Γ). Then we have

THEOREM A. *Let $(\tilde{M}, g)$ be the n-dimensional standard Euclidean space, so that (M_Γ, g_Γ) is a compact flat manifold. Then we have*

$$(1) \qquad \lambda_1(\Gamma)\operatorname{vol}(M_\Gamma)^{2/n} \leqq n^{-1}(2+n)^{1+2/n}\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}\right]^{2/n}[j_{n/2-1}]^{2-4/n},$$

where the number $j_{n/2-1}$ is the least positive zero point of the $(n/2-1)$-th Bessel function $J_{n/2-1}$.

Remark. Since $j_{n/2-1} \sim n/2$ as $n \to \infty$ (cf. [7] p. 153), the right hand side of (1) is $(\pi e/2)n = (4.2699\cdots)n$ asymptotically as $n \to \infty$.

Let $\mu_n = \max_\Gamma \lambda_1(\Gamma)\operatorname{vol}(M_\Gamma)^{2/n}$ where the maximum is taken over all lattices Γ of $\boldsymbol{R}^n$. For a lattice Γ of $\boldsymbol{R}^n$, the spectrum of the corresponding flat torus (M_Γ, g_Γ) is given by $\{4\pi^2|x|^2; x \in \Gamma^*\}$, where Γ^* is a dual lattice of Γ, $|x|^2 = (x, x)$, $x \in \boldsymbol{R}^n$ and $(\ ,\ )$ is the inner product of $\boldsymbol{R}^n$ which gives the standard Riemannian metric on $\boldsymbol{R}^n$ (cf. [1]). So we have $\lambda_1(\Gamma)$

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$= 4\pi^2 \min_{x\in\Gamma^*-(0)} |x|^2$. On the other hand, $\mathrm{vol}(M_\Gamma) = \det(\Gamma^*)^{-1/2}$ (cf. [6]). Here $\det(\Gamma^*)$ is the determinant of the matrix $((b_i, b_j))_{1\leq i,j\leq n}$, where $\{b_i\}_{i=1}^n$ is a basis of $\boldsymbol{R}^n$ generating the lattice Γ^*. Then the above μ_n coincides $4\pi^2$ times the largest possible value for the ratio

$$\mu(\Gamma^*) = \left(\min_{x\in\Gamma^*-(0)} |x|^2\right)(\det(\Gamma^*))^{-n}$$

where Γ^* varies over all lattices in $\boldsymbol{R}^n$. A problem to compute the value μ_n for every n is related to the following classical problem (cf. [6] p. 34): *What is the maximum possible density for a union of non-overlapping balls of fixed radius in* $\boldsymbol{R}^n$? But until now the value μ_n is unknown for $n \geqq 9$. In 1905, H. Minkowski has given (cf. [6]) a lower estimate for μ_n by

$$\mu_n > 4\pi^2\omega_n^{-2/n}$$

where ω_n is the volume of the unit disk in $\boldsymbol{R}^n$ and $4\pi^2\omega_n^{-2/n} \sim (2\pi/\mathrm{e})n = (2.3115\cdots)n$ as $n\to\infty$. On the other hand, in 1958, C. A. Rogers has given (cf. [6]) an upper estimate for μ_n by

$$\mu_n \leqq Q_n$$

where the constant Q_n is $(4\pi/\mathrm{e})n = (4.6229\cdots)n$ asymptotically as $n\to\infty$. The above remark implies that Theorem A improves the result of Rogers in the asymptotic sense.

THEOREM B. *Let* $(\tilde{M}, g)$ *be a Riemannian symmetric space of non-compact type of rank one. Let* G *be the connected component of the Lie group of all isometries of* $(\tilde{M}, g)$. *We normalize* g *in such a way that it is induced by the Killing form of the Lie algebra of* G. *Consider all discrete subgroups* Γ *of* G *acting fixed point freely on* $\tilde{M}$ *whose quotient manifold* M_Γ *is compact. Then we have*

$$\limsup_{\mathrm{vol}(M_\Gamma)\to\infty} \lambda_1(\Gamma) \leqq |\delta|^2\,, \tag{2}$$

for a positive constant $|\delta|^2$ *depending only on* $(\tilde{M}, g)$ (cf. § 2).

Notice that every real valued zonal spherical function φ_λ on $\tilde{M}$ corresponding to the principal series of G (cf. [10]) satisfies (cf. [3])

$$\Delta\varphi_\lambda = (|\lambda|^2 + |\delta|^2)\varphi_\lambda\,, \qquad |\lambda|^2 \geqq 0\,.$$

Here $-\Delta$ is the Laplace-Beltrami operator of $(\tilde{M}, g)$ and it satisfies (cf. § 4) $\Delta(f\circ\pi) = (\Delta_\Gamma f)\circ\pi$ for every smooth function f on M_Γ, where π is the natural projection of $\tilde{M}$ onto M_Γ. If $(\tilde{M}, g)$ is the unit disc with the Poincaré metric, then Theorem B has been obtained by H. Huber [5].

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§ 2. Preliminaries

In this section, following [2] and [3], we prepare some properties of the zonal spherical functions on the Euclidean space or a Riemannian symmetric space of non-compact type of rank one.

2.1. Let $(\tilde{M}, g)$ be the standard Euclidean space $(\boldsymbol{R}^n, g)$. Let $(x_1, \cdots, x_n)$ be the orthonormal coordinate of $\boldsymbol{R}^n$. Let $-\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ be the Laplace-Beltrami operator on $\boldsymbol{R}^n$. The zonal spherical functions on $\boldsymbol{R}^n$ (cf. [9]) are eigen-functions of Δ depending only on $r = |x|$, $x \in \boldsymbol{R}^n$, whose values at 0 are 1. For example (cf. [7]), for $p \in \boldsymbol{R}$ $(p > 0)$, consider the functions

$$\Phi_p(x) = \begin{cases} \Gamma\left(\dfrac{n}{2}\right)\left(\dfrac{pr}{2}\right)^{1-n/2} J_{n/2-1}(pr) & (x \neq 0)\,, \\ 1 & (x = 0)\,. \end{cases}$$

Then Φ_p is real analytic on $\boldsymbol{R}^n$ and written as $\Phi_p(x) = \Psi_p(r)$ where $\Psi_p(s) = \psi(ps)$ and ψ is an even function on $\boldsymbol{R}$ defined by

$$\psi(s) = \Gamma\left(\frac{n}{2}\right) \sum_{m=0}^{\infty} (-1)^m (m!)^{-1} \Gamma\left(\frac{n}{2} + m\right)^{-1} \left(\frac{s}{2}\right)^{2m} .$$

Then Ψ_p satisfies the equation

$$-\frac{d^2}{dr^2}\Psi_p - \frac{n-1}{r}\frac{d}{dr}\Psi_p = p^2 \Phi_p \,. \tag{2.1}$$

Recalling the general equality:

$$\Delta F = -\frac{\partial^2}{\partial r^2}F - \frac{n-1}{r}\frac{\partial F}{\partial r}$$

for a rotationary invariant function $F \in C^2(\boldsymbol{R}^n - (0))$, we get $\Delta\Phi_p = p^2\Phi_p$ (cf. [9]). Let $j_{n/2-1}$ be the least positive zero point of $J_{n/2-1}$. Let f be the even function on $\boldsymbol{R}$ by

$$f(s)=\begin{cases}\Psi_p(s)^{1+\varepsilon}\,, & |s|\leqq \dfrac{j_{n/2-1}}{p}\,,\\[2mm] 0\,, & |s|\geqq \dfrac{j_{n/2-1}}{p}\,,\end{cases}$$

where $0<\varepsilon<1$. Then f satisfies

LEMMA 2.1. *(1) f belongs to $C^1(\boldsymbol{R})$ and the support of f is contained in the set $\{|s|\leqq j_{n/2-1}/p\}$, (2) f'' is continuous except the point $|s|=j_{n/2-1}/p$, (3) $f''(s)=O(||s|-j_{n/2-1}/p|^{\varepsilon-1})$, so $f''\in L^1(\boldsymbol{R})$, and (4) $L_1(f)+(1+\varepsilon)p^2f\geqq 0$ ($|s|\neq 0, j_{n/2-1}/p$), for the differential operator L_1 on $\boldsymbol{R}-(0)$ defined by*

$$L_1=\frac{d^2}{ds^2}+\frac{n-1}{s}\frac{d}{ds}\,.$$

Proof. (1) and (2) are clear. (3) is due to the fact that the number $j_{n/2-1}$ is the zero point of $J_{n/2-1}$ of first order (cf. [7] p. 151). By (2.1), we have

$$L_1(f)+(1+\varepsilon)p^2f=(1+\varepsilon)\varepsilon\left(\frac{d\Psi_p}{ds}\right)^2\Psi_p^{\varepsilon-1}\geqq 0\,,$$

for $0<|s|<j_{n/2-1}/p$, so (4) holds. Q.E.D.

Let F be the function on $\boldsymbol{R}^n$ defined by $F(x)=f(|x|)$, $x\in\boldsymbol{R}^n$. Then we have

LEMMA 2.2. *F belongs to $C^1(\boldsymbol{R}^n)$ and $C^2(\boldsymbol{R}^n-\gamma_1)$ where $\gamma_1=\{x\in\boldsymbol{R}^n;\ |x|=0$ or $j_{n/2-1}/p\}$, and the support of F is contained in the set $\{x\in\boldsymbol{R}^n;\ |x|\leqq j_{n/2-1}/p\}$. Moreover*

$$(\Delta F)(x)=-(L_1f)(|x|)\qquad (x\neq 0)\ \textit{and}\ \Delta F\in L^1(\boldsymbol{R}^n)\,,\tag{2.2}$$

$$\Delta F\leqq(1+\varepsilon)p^2F\qquad \textit{on}\ \boldsymbol{R}^n-\gamma_1\,.\tag{2.3}$$

Proofs are immediate from Lemma 2.1.

Due to Lemma 2.1, there exists a sequence $\{f_m\}_{m=1}^\infty$ of smooth even functions on $\boldsymbol{R}$ such that (5) $f_m(s)=f(s)$ ($|s|\leqq j_{n/2-1}/2p$) and $f_m(s)=0$ ($|s|\leqq 2j_{n/2-1}/p$), (6) f_m (resp. f'_m) converges to f (resp. f') uniformly on $\boldsymbol{R}$ as $m\to\infty$ and (7) $\lim_{m\to\infty}\int_{-\infty}^{\infty}|f''_m(s)-f''(s)|\,ds=0$.

Define $F_m\in C^\infty(\boldsymbol{R}^n)$ by $F_m(x)=f_m(|x|)$, $x\in\boldsymbol{R}^n$. Then by (5), (6) and (7), the support of F_m is included in the set $\{x\in\boldsymbol{R}^n;\ |x|\leqq 2j_{n/2-1}/p\}$, F_m converges to F uniformly on $\boldsymbol{R}^n$ and

$$\lim_{m\to\infty}\int_{R^n}|\Delta F_m - \Delta F|\,dx = 0\,, \tag{2.4}$$

where dx is the Lebesgue measure on R^n.

2.2. Let $(\tilde{M}, g)$ be a Riemannian symmetric space of non-compact type of rank one. Let G be the identity component of the Lie group of all isometries of $(\tilde{M}, g)$. Let K be the isotropy subgroup of G at some point o of $\tilde{M}$. The subgroup K is a maximal compact subgroup of G. Let $\mathfrak{g}$, (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to $\mathfrak{k}$. Then $\mathfrak{p}$ is identified with the tangent space of $\tilde{M}$ at o. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $\mathfrak{a}^*$ its dual and $\mathfrak{a}_C^*$ the complexification of $\mathfrak{a}^*$. Then rank $(\tilde{M}, g) = 1$ means $\dim\mathfrak{a} = 1$. Let B be the Killing form of $\mathfrak{g}$. We assume the Riemannian metric g on $\tilde{M} = G/K$ is induced by $g_o(X_o, Y_o) = B(X, Y)$, $X, Y \in \mathfrak{p}$, where X_o, Y_o are the tangent vectors of $\tilde{M}$ at $o = \{K\}$ corresponding to X, Y, respectively. For $\lambda \in \mathfrak{a}^*$, let $H_\lambda \in \mathfrak{a}$ be determined by $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{a}$. Put $(\lambda, \mu) = B(H_\lambda, H_\mu)$ for $\lambda, \mu \in \mathfrak{a}^*$. We fix an order on $\mathfrak{a}^*$ once and for all. Let Σ be the set of all non-zero restricted roots of $(\mathfrak{g}, \mathfrak{a})$ and Σ^+ the set of positive elements in Σ. For, $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}; [H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$. Let denote $m_\alpha = \dim\mathfrak{g}_\alpha$ for $\alpha \in \Sigma$, which is called the multiplicity of α. Let $\delta = 2^{-1}\sum_{\alpha\in\Sigma^+} m_\alpha\,\alpha$. Let $\mathfrak{n} = \sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ and N the connected subgroup of G corresponding to $\mathfrak{n}$. Each $g \in G$ can be uniquely written as $g = \kappa(g)\exp(H(g))n(g)$ where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. In case of rank one, the zonal spherical functions on $\tilde{M}$ mean the (complex valued) K-invariant eigen-functions of the Laplace-Beltrami operator $-\Delta$ of $(\tilde{M}, g)$ whose values at $o = \{K\}$ are 1. These functions are exhausted by $\varphi_\lambda(g) = \int_K e^{(\sqrt{-1}\lambda-\delta)H(gk)}\,dk$, $\lambda \in \mathfrak{a}_C^*$, $g \in G$, where dk is the Haar measure on K such that the total measure is 1. Here φ_λ satisfies $\varphi_\lambda(gk) = \varphi_\lambda(g)$ ($g \in G$, $k \in K$) and hence it is regarded as a function on $\tilde{M}$. Notice that $\Sigma^+ = \{\alpha, (2\alpha)\}$ and $\delta = 2^{-1}(m_\alpha + 2m_{2\alpha})\alpha$ since $\tilde{M}$ is of rank one. Let $H_0 \in \mathfrak{a}$ be the element such that $\alpha(H_0) = 1$ and hence $B(H_0, H_0) = 2(m_\alpha + 4m_{2\alpha})$. For $t \in R$, put $h_t = \exp(tH_0) \in A = \exp(\mathfrak{a})$. Then t can be regarded as the coordinate on the one dimensional Lie group A. Put $x = -(\sinh(t))^2$. Since $\varphi_\lambda(h_t)$ is an even function of t, it is written as $\varphi_\lambda(h_t) = g_\lambda(x)$. Then g_λ satisfies

$$x(x-1)\frac{d^2}{dx^2}g_\lambda + ((a+b+1)x - c)\frac{dg_\lambda}{dx} = -abg_\lambda\,, \tag{2.5}$$

where $a = 4^{-1}(m_\alpha + 2m_{2\alpha} + 2\sqrt{-1}\,\lambda(H_0))$, $b = 4^{-1}(m_\alpha + 2m_{2\alpha} - 2\sqrt{-1}\,\lambda(H_0))$ and $c = 2^{-1}(m_\alpha + m_{2\alpha} + 1)$ (cf. [3] p. 301). Notice that $a + b$, ab and $c > 0$. Thus $g_\lambda(x)$ is the hypergeometric function $F(a, b, c; x)$. Moreover, each K-invariant function $F \in C^2(M - (o))$ satisfies

$$\begin{aligned} &-2^{-1}(m_\alpha + 4m_{2\alpha})(\Delta F)(kh_t \cdot o) \\ &\quad = x(x-1)\frac{d^2G}{dx^2}(x) + ((a+b+1)x - c)\frac{dG}{dx}(x) \end{aligned}$$

$(t \neq 0, k \in K)$, where G is the function defined by $F(t) = G(x)$ (cf. [3] p. 302). Thus we have

$$2^{-1}(m_\alpha + 4m_{2\alpha})\Delta\varphi_\lambda = ab\varphi_\lambda \,.$$

If $\lambda \in \mathfrak{a}^*$, φ_λ is real valued and has the following asymptotic behavior:

$$\lim_{t\to\infty} |e^{t\delta(H_0)}\varphi_\lambda(h_t) - (c(\lambda)e^{t\sqrt{-1}\lambda(H_0)} + c(-\lambda)e^{-t\sqrt{-1}\lambda(H_0)})| = 0 \,, \tag{2.6}$$

where $c(\lambda) = \Gamma(c)\Gamma(\sqrt{-1}\lambda(H_0))\Gamma(a)^{-1}\Gamma(m_\alpha + 2 + 2\sqrt{-1}\lambda(H_0))^{-1}$ (cf. [3] p. 303).

Let dg_K be the volume element of $(\tilde{M}, g)$. Then it is known (cf. [4] p. 381) that

$$\int_{\tilde{M}} f(g\cdot o)dg_K = C\int_{-\infty}^{0} D(x)g(x)dx \tag{2.7}$$

for every integrable K-invariant function f on $\tilde{M}$ and $g(x) = f(h_t\cdot o)$, $x = -(\sinh(t))^2$. Here C is a positive constant which does not depend on f and $D(x) = (-x)^{2^{-1}(m_\alpha + m_{2\alpha} - 1)}(1 - x)^{2^{-1}(m_{2\alpha} - 1)}$.

2.3. First, we notice that if $\lambda \in \mathfrak{a}^*$, $\lambda \neq 0$, then the function φ_λ has zero points. For, since $|c(\lambda)| = |c(-\lambda)|$, $\overline{c(\lambda)} = c(-\lambda)$, we have by (2.6)

$$\varphi_\lambda(h_t) \sim 2e^{-t\delta(H_0)}\,|c(\lambda)|\cos(t\lambda(H_0) + \arg(c(\lambda)))\,, \qquad \delta(H_0) > 0\,,$$

as $t \to \infty$. So let $-A_\lambda (0 < A_\lambda < \infty)$ be the first zero point of $g_\lambda(x)$, $x \leqq 0$. We consider also the function f_λ defined by

$$f_\lambda(x) = \begin{cases} g_\lambda(x)^{1+\varepsilon}\,, & -A_\lambda \leqq x \leqq 0\,, \\ 0\,, & -\infty < x < -A_\lambda\,, \end{cases}$$

where $0 < \varepsilon < 1$. The continuous function f_λ on $(-\infty, 0]$ has the following properties.

LEMMA 2.3. *(1′) f_λ belongs to $C^1(-\infty, 0]$ and the support of f_λ is contained in the set $\{-A_\lambda \leqq x \leqq 0\}$, (2′) f_λ'' is continuous on $-\infty < x < 0$ except*

$-A_\lambda$, (3') $f''_\lambda(x) = O(|x + A_\lambda|^{\varepsilon-1})$, *so* $f''_\lambda \in L^1(-\infty, 0]$, *and* (4') $L_2(f_\lambda)(x) + (1+\varepsilon)abf_\lambda(x) \geqq 0$, *except* $x = 0, -A_\lambda$ *for the differential operator* L_2 *on* $(-\infty, 0]$ *defined by*

$$L_2 = x(x-1)\frac{d^2}{dx^2} + ((a+b+1)x - c)\frac{d}{dx}\,.$$

Proof. (1') and (2') are clear. For (3'), we may show that $-A_\lambda$ is the zero point of g_λ of first order. By the properties of the hypergeometric function $g_\lambda(x) = F(a, b, c; x)$,

$$((-x)^c(1-x)^{a+b-c+1}g'_\lambda)' = -ab(-x)^{c-1}(1-x)^{a+b-c}g_\lambda\,.$$

Then $G(x) = (-x)^c(1-x)^{a+b-c+1}g'_\lambda$ satisfies $G'(x) < 0$ $(-A_\lambda < x < 0)$ and $G'(x) = 0$ $(x = 0, -A_\lambda)$. Hence $G(x) > G(0) = 0$ $(-A_\lambda \leqq x < 0)$, that is $g'_\lambda(x) > 0$ $(-A_\lambda \leqq x < 0)$. By (2.5),

$$L_2(f_\lambda)(x) + (1+\varepsilon)abf_\lambda(x) = (1+\varepsilon)\varepsilon x(x-1)(g'_\lambda)^2 g_\lambda^{\varepsilon-1} \geqq 0$$

$(-A_\lambda < x < 0)$, so (4') holds. Q.E.D.

Define a function F_λ on A by $F_\lambda(h_t) = f_\lambda(x)$, $x = -(\sinh(t))^2$. Then it belongs to $C^1(A)$ and is an even function, that is $F_\lambda(h_t) = F_\lambda(h_{-t})$. Hence it can be extended to $\tilde{M}$ uniquely as a K-invariant function, denoted by the same letter F_λ. It satisfies the following properties.

LEMMA 2.4. *F_λ belongs to $C^1(\tilde{M})$ and $C^2(\tilde{M} - \gamma_2)$ where $\gamma_2 = \{kh_t\cdot o;$ $k \in K, -(\sinh(t))^2 = 0, -A_\lambda\}$, and the support of F_λ is contained in the set $\{kh_t\cdot o; k \in K, -A_\lambda \leqq -(\sinh(t))^2\}$. Moreover*

$$2^{-1}(m_\alpha + 4m_{2\alpha})(\Delta F_\lambda)(kh_t\cdot o) = -(L_2 f_\lambda)(x) \tag{2.8}$$

$(t \neq 0, k \in K)$ *and* $\Delta F_\lambda \in L^1(\tilde{M})$,

$$2^{-1}(m_\alpha + 4m_{2\alpha})\Delta F_\lambda \leqq (1+\varepsilon)abF_\lambda \qquad \text{on } \tilde{M} - \gamma_2\,, \tag{2.9}$$

where $L^1(\tilde{M})$ is the space of integrable functions on $\tilde{M}$ with respect to the volume element dg_K in (2.7).

Proof. (2.8) follows from (2.7), (2.5) and Lemma 2.3. The remainds are immediate from Lemma 2.3. Q.E.D.

Due to Lemma 2.3, there exists a sequence $\{F_{\lambda,m}\}_{m=1}^\infty$ of smooth even functions on A such that (5') $F_{\lambda,m}(h_t) = F_\lambda(h_t)$ $(|t| \leqq t_o/2)$ and $F_{\lambda,m}(h_t) = 0$ $(|t| \geqq 2t_o)$, where $t_o > 0$ is given by $-(\sinh(t_o))^2 = -A_\lambda$, (6') $F_{\lambda,m}$ (resp. $F'_{\lambda,m}$)

converges to F_λ (resp. F'_λ) uniformly on A as $m \to \infty$, and (7')

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} |F''_{\lambda,m}(h_t) - F''_\lambda(h_t)|\, dt = 0 \,,$$

where F'_λ etc. means the differential of F_λ with respect to t. The functions $F_{\lambda,m}$ can be extended as K-invariant C^∞ functions on $\tilde{M}$, denoted by the same letter $F_{\lambda,m}$. Then the support of $F_{\lambda,m}$ is contained in the set $\{kh_t \cdot o;\ k \in K, |t| \leqq 2t_o\}$, $F_{\lambda,m}$ converges to F_λ uniformly on $\tilde{M}$ and

$$\lim_{m\to\infty} \int_{\tilde{M}} |\varDelta F_{\lambda,m} - \varDelta F_\lambda|\, dg_K = 0 \tag{2.10}$$

by (5'), (6'), (7'), (2.5) and (2.7).

§ 3. Proof of Theorem A

3.1. In this section, we preserve the notations in 2.1 and introduction. Let π denote the projection of $\boldsymbol{R}^n$ onto M_Γ. For $\gamma \in \Gamma$, let τ_γ be the action of γ on $\boldsymbol{R}^n$. The Laplace-Beltrami operator $-\varDelta_\Gamma$ on M_Γ satisfies $\varDelta(f \circ \pi) = (\varDelta_\Gamma f) \circ \pi$ for twice differentiable functions f on M_Γ. The volume element on M_Γ induced by dx is denoted by $d\omega$. Let $\mathscr{F}$ be the fundamental domain in $\boldsymbol{R}^n$ for Γ, that is $\mathscr{F} = \{x \in \boldsymbol{R}^n;\ |x| \leqq |x - \tau_\gamma \cdot 0| \text{ for all } \gamma \in \Gamma\}$. It is known (cf. [7]) that

$$\boldsymbol{R}^n = \bigcup_{\gamma\in\Gamma} \tau_\gamma \cdot \mathscr{F} \quad \text{and} \quad \tau_\gamma \cdot \mathscr{F} \cap \mathscr{F} \tag{3.1}$$

has measure 0 for every $\gamma \in \Gamma$, $\gamma \neq 1$.

Now since the functions F and F_m have the compact supports, we can define the Γ-invariant functions θ and θ_m on $\boldsymbol{R}^n$ by

$$\theta = \sum_{\gamma\in\Gamma} F \circ \tau_\gamma \,, \qquad \theta_m = \sum_{\gamma\in\Gamma} F_m \circ \tau_\gamma \,.$$

Then there exist functions φ and φ_m on M_Γ such that $\varphi \circ \pi = \theta$, $\varphi_m \circ \pi = \theta_m$. These functions have the following properties:

LEMMA 3.1. (1) *The function φ belongs to $C^1(M_\Gamma)$ and $C^2(M_\Gamma - \pi(\gamma_1))$, and $\varDelta_\Gamma \varphi$ belongs to $L^1(M_\Gamma)$,* (2) *$\varDelta_\Gamma \varphi \leqq (1+\varepsilon)p^2\varphi$ on $M_\Gamma - \pi(\gamma_1)$, and* (3) *$\varphi_m \in C^\infty(M_\Gamma)$ converges to φ uniformly on M_Γ as $m \to \infty$, and*

$$\lim_{m\to\infty} \int_{M_\Gamma} |\varDelta_\Gamma \varphi_m - \varDelta_\Gamma \varphi|\, d\omega = 0 \,.$$

Moreover (4)

$$\lim_{m\to\infty}\int_{M_\Gamma}\varphi_m(\Delta_\Gamma\varphi_m)d\omega=\int_{M_\Gamma}\varphi(\Delta_\Gamma\varphi)d\omega\ .$$

Proof. (1), (2) and (3) follow from Lemma 2.2 and (2.4). The inequality

$$\left|\int_{M_\Gamma}\varphi_m(\Delta_\Gamma\varphi_m)d\omega-\int_{M_\Gamma}\varphi(\Delta_\Gamma\varphi)\,d\omega\right|$$
$$\leqq\left\|\Delta_\Gamma\varphi_m-\Delta_\Gamma\varphi\right\|_{L^1(M_\Gamma)}\cdot\sup_{M_\Gamma}|\varphi_m|+\|\Delta_\Gamma\varphi\|_{L^1(M_\Gamma)}\cdot\sup_{M_\Gamma}|\varphi_m-\varphi|\ ,$$

together with (3), implies (4). Q.E.D.

Notice that

$$\int_{M_\Gamma}\varphi d\omega=V_{n-1}p^{-n}\int_0^{j_{n/2-1}}\psi(r)^{1+\varepsilon}r^{n-1}dr\ , \tag{3.2}$$

$$\int_{M_\Gamma}\varphi^2d\omega\geqq V_{n-1}p^{-n}\int_0^{j_{n/2-1}}\psi(r)^{2(1+\varepsilon)}r^{n-1}dr\ , \tag{3.3}$$

where $V_{n-1}=2\pi^{n/2}\Gamma(n/2)^{-1}$ is the total measure of the unit sphere S^{n-1} with respect to the measure induced by the volume element dx on $\boldsymbol{R}^n$. In fact, by (3.1) and the definitions of φ, θ and F, we have

$$\begin{aligned}\int_{M_\Gamma}\varphi d\omega&=\int_{\mathscr{F}}\theta dx\\&=\sum_{\gamma\in\Gamma}\int_{\mathscr{F}}(F\circ\tau_\gamma)dx\\&=\int_{\bigcup_{\gamma\in\Gamma}\tau_\gamma\cdot\mathscr{F}}Fdx\\&=\int_{\boldsymbol{R}^n}Fdx\\&=V_{n-1}p^{-n}\int_0^{j_{n/2-1}}\psi(r)^{1+\varepsilon}r^{n-1}dr\ .\end{aligned}$$

(3.2) follows from the inequality for the integrand:

$$\theta^2=\sum_{\gamma,\gamma'\in\Gamma}(F\circ\tau_\gamma)(F\circ\tau_{\gamma'})\geqq\sum_{\gamma\in\Gamma}(F\circ\tau_\gamma)^2\ ,$$

which follows from $F\geqq 0$.

3.2. It is known (cf. [1] p. 186) that the least positive eigenvalue $\lambda_1(\Gamma)$ of Δ_Γ satisfies the inequality

$$\int_{M_\Gamma}\eta(\Delta_\Gamma\eta)d\omega\geqq\lambda_1(\Gamma)\int_{M_\Gamma}\eta^2d\omega \tag{3.4}$$

for all $\eta \in C^\infty(M_\Gamma)$ such that

$$\int_{M_\Gamma} \eta d\omega = 0 .$$

We apply (3.4) for $\eta = \varphi_m - \alpha_m$, where

$$\alpha_m = \mathrm{vol}\,(M_\Gamma)^{-1} \int_{M_\Gamma} \varphi_m d\omega .$$

Then we have

$$\int_{M_\Gamma} \varphi_m(\Delta_\Gamma \varphi_m) d\omega \geqq \lambda_1(\Gamma)\Big[\int_{M_\Gamma} \varphi_m^2 d\omega - \mathrm{vol}\,(M_\Gamma)^{-1}\Big(\int_{M_\Gamma} \varphi_m d\omega\Big)^2\Big] .$$

As $m \to \infty$, we have

$$\text{(3.5)} \qquad \int_{M_\Gamma} \varphi(\Delta_\Gamma \varphi) d\omega \geqq \lambda_1(\Gamma)\Big[\int_{M_\Gamma} \varphi^2 d\omega - \mathrm{vol}\,(M_\Gamma)^{-1}\Big(\int_{M_\Gamma} \varphi d\omega\Big)^2\Big]$$

by (3) and (4) in Lemma 3.1. Since $\pi(\gamma_1)$ has measure 0, we have

$$\text{(3.6)} \qquad \int_{M_\Gamma} \varphi(\Delta_\Gamma \varphi) d\omega \leqq (1+\varepsilon)p^2 \int_{M_\Gamma} \varphi^2 d\omega$$

by (2) in Lemma 3.1. Then, by (3.5) and (3.6),

$$\lambda_1(\Gamma)\Big[1 - \mathrm{vol}\,(M_\Gamma)^{-1}\Big(\int_{M_\Gamma} \varphi d\omega\Big)^2\Big(\int_{M_\Gamma} \varphi^2 d\omega\Big)^{-1}\Big] \leqq (1+\varepsilon)p^2 .$$

Hence, together with (3.2) and (3.3), we have

$$\lambda_1(\Gamma)\Big[1 - p^{-n} V_{n-1}\,\mathrm{vol}\,(M_\Gamma)^{-1}\Big(\int_0^{j_{n/2-1}} \psi(r)^{1+\varepsilon} r^{n-1} dr\Big)^2 \times \Big(\int_0^{j_{n/2-1}} \psi(r)^{2(1+\varepsilon)} r^{n-1} dr\Big)^{-1}\Big] \leqq (1+\varepsilon)p^2 .$$

Letting $\varepsilon \to 0$, we obtain

PROPOSITION 3.1. *Under the above situation, we have*

$$\text{(3.7)} \qquad \lambda_1(\Gamma) \leqq \inf_{p>0} \{p^2[1 - V_{n-1}K_n\,\mathrm{vol}\,(M_\Gamma)^{-1}p^{-n}]^{-1};\ 1 - V_{n-1}K_n\,\mathrm{vol}\,(M_\Gamma)^{-1}p^{-n} > 0\}$$

where

$$K_n = \Big(\int_0^{j_{n/2-1}} \psi(r) r^{n-1} dr\Big)^2 \Big(\int_0^{j_{n/2-1}} \psi(r)^2 r^{n-1} dr\Big)^{-1}$$

and

$$V_{n-1}=2\pi^{n/2}\Gamma\left(\frac{n}{2}\right)^{-1}.$$

3.3. We calculate the right hand side of (3.7). Since

$$\psi(r)=\Gamma\left(\frac{n}{2}\right)\left(\frac{r}{2}\right)^{1-n/2}J_{n/2-1}(r),$$

we have

$$K_n=\left(\int_0^{j_{n/2-1}}J_{n/2-1}(r)r^{n/2}dr\right)^2\left(\int_0^{j_{n/2-1}}J_{n/2-1}(r)^2rdr\right)^{-1}.$$

Since the derivative of $J_{n/2}(r)r^{n/2}$ (resp. $(r^2/2)(J_{n/2-1}(r)^2-J_{n/2-2}(r)J_{n/2}(r))$) is $J_{n/2-1}(r)r^{n/2}$ (resp. $J_{n/2-1}(r)^2r$) (cf. [7] p. 189), we have

$$\begin{aligned}K_n&=(J_{n/2}(j_{n/2-1})(j_{n/2-1})^{n/2})^2(\tfrac{1}{2}(j_{n/2-1})^2(-1)J_{n/2-2}(j_{n/2-1})J_{n/2}(j_{n/2-1}))^{-1},\\&=(J_{n/2}(j_{n/2-1})(j_{n/2-1})^{n/2})^2(\tfrac{1}{2}(j_{n/2-1})^2J_{n/2}(j_{n/2-1})^2)^{-1}\\&\qquad\text{(by } J_{n/2}(j_{n/2-1})+J_{n/2-2}(j_{n/2-1})=0 \text{ (cf. [7] p. 158))},\\&=2(j_{n/2-1})^{n-2}.\end{aligned}$$

Put

$$G(p)=p^2(1-V_{n-1}K_n\operatorname{vol}(M_\Gamma)^{-1}p^{-n})^{-1}$$

and

$$p_0=(2^{-1}(2+n)V_{n-1}K_n\operatorname{vol}(M_\Gamma)^{-1})^{1/n}.$$

If $1-V_{n-1}K_n\operatorname{vol}(M_\Gamma)^{-1}p^{-n}>0$, then

$$G'(p)<0\ (p<p_0),\qquad G'(p)>0\ (p>p_0)\ \text{and}\ G'(p_0)=0.$$

So we have

$$\begin{aligned}\inf_{p>0}\{G(p);1-V_{n-1}K_n\operatorname{vol}(M_\Gamma)^{-1}p^{-n}>0\}&=G(p_0)\\&=2^{-2/n}n^{-1}(2+n)^{2/n+1}(V_{n-1}K_n\operatorname{vol}(M_\Gamma)^{-1})^{2/n}\\&=n^{-1}(2+n)^{2/n+1}\left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)^{2/n}(j_{n/2-1})^{2-4/n}\operatorname{vol}(M_\Gamma)^{-2/n}.\end{aligned}$$

Thus Theorem A is proved.

§4. Proof of Theorem B

4.1. In this section, we preserve the notations in 2.2, 2.3 and introduction. Theorem B will be proved by the same way as Theorem A. Let

π denote the projection of $\tilde{M}$ onto M_Γ. For $\gamma \in \Gamma$, let τ_γ be the action of γ on $\tilde{M}$. The Laplace-Beltrami operator $-\Delta_\Gamma$ on M satisfies $\Delta(f \circ \pi) = (\Delta_\Gamma f) \circ \pi$ for all twice differentiable functions f on M_Γ. The volume element on M_Γ induced by dg_K is denoted by $d\omega$. Let $\mathscr{F}$ be the fundamental domain in $\tilde{M}$ for Γ, that is $\mathscr{F} = \{g \cdot o \in \tilde{M}; r(g \cdot o, o) \leqq r(g \cdot o. \tau_\gamma \cdot o)$ for all $\gamma \in \Gamma\}$ where $r(\cdot, \cdot)$ is the distance function on $(\tilde{M}, g)$. It is known (for example, cf. [2]) that

$$\tilde{M} = \bigcup_{\gamma \in \Gamma} \tau_\gamma \mathscr{F} \quad \text{and} \quad \tau_\gamma \mathscr{F} \cap \mathscr{F} \tag{4.1}$$

has measure 0 for all $\gamma \in \Gamma$, $\gamma \neq 1$.

Since F_λ and $F_{\lambda,m}$ have the compact supports, we define the Γ-invariant functions θ_λ and $\theta_{\lambda,m}$ on $\tilde{M}$ by

$$\theta_\lambda = \sum_{\gamma \in \Gamma} F_\lambda \circ \tau_\gamma\,, \qquad \theta_{\lambda,m} = \sum_{\gamma \in \Gamma} F_{\lambda,m} \circ \tau_\gamma\,.$$

Then there exist functions φ_λ and $\varphi_{\lambda,m}$ on M_Γ such that $\varphi_\lambda \circ \pi = \theta_\lambda$ and $\varphi_{\lambda,m} \circ \pi = \theta_{\lambda,m}$. These functions have the following properties.

LEMMA 4.1. (1) *The function φ_λ belongs to $C^1(M_\Gamma)$ and $C^2(M_\Gamma - \pi(\gamma_2))$, and $\Delta_\Gamma \varphi_\lambda$ belongs to $L^1(M_\Gamma)$,* (2) *$2^{-1}(m_\alpha + 4m_{2\alpha})\Delta_\Gamma \varphi_\lambda \leqq (1+\varepsilon)ab\varphi_\lambda$ on $M_\Gamma - \pi(\gamma_2)$, and* (3) *$\varphi_{\lambda,m} \in C^\infty(M_\Gamma)$ converges to φ_λ uniformly on M_Γ as $m \to \infty$ and*

$$\lim_{m \to \infty} \int_{M_\Gamma} |\Delta_\Gamma \varphi_{\lambda,m} - \Delta_\Gamma \varphi_\lambda|\, d\omega = 0\,.$$

Moreover (4)

$$\lim_{m \to \infty} \int_{M_\Gamma} \varphi_{\lambda,m}(\Delta_\Gamma \varphi_{\lambda,m}) d\omega = \int_{M_\Gamma} \varphi_\lambda(\Delta_\Gamma \varphi_\lambda) d\omega\,.$$

Proofs are similar to Lemma 3.1.

Notice that

$$\int_{M_\Gamma} \varphi_\lambda d\omega = \int_{\tilde{M}} F_\lambda dg_K = C \int_{-A_\lambda}^0 D(x) g_\lambda(x)^{1+\varepsilon} dx\,, \tag{4.2}$$

$$\int_{M_\Gamma} \varphi_\lambda^2 d\omega \geqq C \int_{-A_\lambda}^0 D(x) g_\lambda(x)^{2(1+\varepsilon)} dx\,. \tag{4.3}$$

Then, due to (4.2), (4.3) and Lemma 4.1, we have the following proposition by the similar manner to Proposition 3.1.

PROPOSITION 4.1. *Under the above assumption, we obtain*

$$\lambda_1(\Gamma)[1 - C\,\mathrm{vol}\,(M_\Gamma)^{-1}K_\lambda] \leq 2(m_\alpha + 4m_{2\alpha})^{-1}ab\ , \tag{4.4}$$

where

$$K_\lambda = \left(\int_{-A\lambda}^{0} D(x)g_\lambda(x)dx\right)^2\left(\int_{-A\lambda}^{0} D(x)g_\lambda(x)^2dx\right)^{-1},$$

the constant C and the function $D(x)$ are the ones in (2.7).

4.2. We prove Theorem B due to Proposition 4.1. We fix any $\lambda \in \mathfrak{a}^*$, $\lambda \neq 0$. For a discrete subgroup Γ of G with sufficiently large $\mathrm{vol}\,(M_\Gamma)$ such that $\mathrm{vol}\,(M_\Gamma) > CK_\lambda$, we have, by Proposition 4.1,

$$\lambda_1(\Gamma) \leqq 2(m_\alpha + 4m_{2\alpha})^{-1}ab[1 - C\,\mathrm{vol}\,(M_\Gamma)^{-1}K_\lambda]^{-1}\ .$$

Hence, by the definition of a and b, we have

$$\begin{aligned}\limsup_{\mathrm{vol}\,(M_\Gamma)\to\infty} \lambda_1(\Gamma) &\leqq 2(m_\alpha + 4m_{2\alpha})^{-1}ab \\ &= \tfrac{1}{8}(m_\alpha + 4m_{2\alpha})^{-1}((m_\alpha + 4m_{2\alpha})^2 + 4\lambda(H_0)^2)\end{aligned}$$

for every $\lambda \in \mathfrak{a}^*$, $\lambda \neq 0$. So we have

$$\limsup_{\mathrm{vol}\,(M_\Gamma)\to\infty} \lambda_1(\Gamma) \leqq \tfrac{1}{8}(m_\alpha + 4m_{2\alpha})^{-1}(m_\alpha + 2m_{2\alpha})^2\ .$$

Here $B(H_0, H_0) = 2(m_\alpha + 4m_{2\alpha})$ implies that the right hand side of the above inequality coincides with $|\delta|^2 = (\delta, \delta)$. Thus Theorem B is proved.

§ 5. Supremum of L^2 spectrum

For a complete orientable Riemannian manifold (M, g) (not necessarily compact), consider

$$\sigma(M, g) = \inf_{\varphi \in C_0^\infty(M)} \frac{\int_M (\Delta_g\varphi)\varphi dv_g}{\int_M \varphi^2 dv_g}\ ,$$

where $C_0^\infty(M)$ is the space of all real valued C^∞ functions on M with compact support, $-\Delta_g$ is the Laplace-Beltrami operator of (M, g) acting on smooth functions on M, and dv_g is the volume element of (M, g) (cf. [11]). Then $\sigma(M, g) \geqq 0$ and it is called the supremum of the L^2 spectrum of (M, g) (cf. [11]). Since the operator Δ_g is a real symmetric operator, we notice that

$$\sigma(M, g) = \inf_{\varphi \in C_0^\infty(M)^C} \frac{\int_M (\Delta_g\varphi)\bar{\varphi} dv_g}{\int_M \varphi\bar{\varphi} dv_g}\ ,$$

where $C_o^\infty(M)^C$ is the space of all complex valued C^∞ functions on M with compact support, and $\bar{\varphi}(x)$, $x \in M$, is the complex conjugate of $\varphi(x)$.

In this section, we calculate $\sigma(M, g)$ when (M, g) is a Riemannian symmetric space of non-compact type of rank one. We preserve the notations in § 2.

PROPOSITION 5.1. *Let $(\tilde{M}, g)$ be a Riemannian symmetric space of non-compact type (not necessarily of rank one). We normalize g in such a way that it is induced by the Killing form of the Lie algebra $\mathfrak{g}$ of the connected component G of the Lie group of all isometries of $(\tilde{M}, g)$. Then we have*

$$\sigma(\tilde{M}, g) \geqq |\delta|^2 ,$$

where $|\delta|$ is the norm of $\delta = \frac{1}{2}\sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ by the inner product induced from the Killing form as in § 2.

Proof. It holds (cf. [10]) that

$$\int_{\tilde{M}} (\Delta_g f)\bar{f}\, dg_K = C\int_{\mathfrak{a}^* \times K/Z_K(\mathfrak{a})} (\Delta_g f)^\sim(\lambda, \dot{k})\overline{\tilde{f}(\lambda, \dot{k})}\,|c(\lambda)|^{-2}\, d\lambda d\dot{k} , \tag{5.1}$$

for each $f \in C_o^\infty(\tilde{M})$. Here C is a positive constant, not depending on f, $Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K, $\tilde{f}(\lambda, \dot{k})$ ($\lambda \in \mathfrak{a}^*$, $\dot{k} \in K/Z_K(\mathfrak{a})$) is the Fourier transform of f defined by

$$\tilde{f}(\lambda, \dot{k}) = \int_{\tilde{M}} f(g\cdot o)e^{-(\sqrt{-1}\lambda+\delta)H(g^{-1}k)}dg_K ,$$

$d\lambda$ is the Euclidean measure on $\mathfrak{a}^*$, and $d\dot{k}$ is the measure on $K/Z_K(\mathfrak{a})$ induced by the Haar measure dk on K (cf. [10]). Since $(\Delta_g f)^\sim(\lambda, \dot{k}) = (|\lambda|^2 + |\delta|^2)\tilde{f}(\lambda, \dot{k})$ (cf. [12] p. 92, [13] p. 458),

$$\begin{aligned} &\text{the right hand side of (5.1)} \\ &\geqq |\delta|^2 C\int_{\mathfrak{a}^* \times K/Z_K(\mathfrak{a})} |\tilde{f}(\lambda, \dot{k})|^2\,|c(\lambda)|^{-2}\, d\lambda\, d\dot{k} \\ &= |\delta|^2 \int_{\tilde{M}} f\bar{f}\, dg_K . \end{aligned}$$

Thus we have $\sigma(\tilde{M}, g) \geqq |\delta|^2$. Q.E.D.

In particular, when $(\tilde{M}, g)$ is of rank one, the following theorem holds.

THEOREM C. *Let $(\tilde{M}, g)$ be a Riemannian symmetric space of non-compact type of rank one. We normalize g as in Proposition* 5.1. *Then we have*

$$\sigma(\tilde{M}, g) = |\delta|^2 .$$

Proof. We may prove $\sigma(\tilde{M}, g) \leqq |\delta|^2$. We use the notations in 2.3. Since the supports of $\dot{F}_\lambda$ and $F_{\lambda,m}$, $m = 1, 2, \cdots$ are contained in the set $\{kh_t \cdot o; k \in K, 0 \leqq t \leqq 2t_o\}$, and $F_{\lambda,m}$ converges to F_λ uniformly on $\tilde{M}$,

$$\lim_{m\to\infty} \int_{\tilde{M}} F_{\lambda,m}^2 dg_K = \int_{\tilde{M}} F_\lambda^2 dg_K . \tag{5.2}$$

Moreover we have

$$\lim_{m\to\infty} \int_{\tilde{M}} (\Delta_g F_{\lambda,m}) F_{\lambda,m} dg_K = \int_{\tilde{M}} (\Delta_g F_\lambda) F_\lambda dg_K . \tag{5.3}$$

In fact, it follows from the inequality

$$\begin{aligned} &\left| \int_{\tilde{M}} (\Delta_g F_{\lambda,m}) F_{\lambda,m} dg_K - \int_{\tilde{M}} (\Delta_g F_\lambda) F_\lambda dg_K \right| \\ &\qquad \leqq \|\Delta_g F_{\lambda,m} - \Delta_g F_\lambda\|_{L^1(\tilde{M})} \sup_{\tilde{M}} |F_{\lambda,m}| + \|\Delta_g F_\lambda\|_{L^1(\tilde{M})} \sup_{\tilde{M}} |F_{\lambda,m} - F_\lambda| , \end{aligned}$$

(2.10), (2.8) and Lemma 2.4.

Thus we have

$$\sigma(\tilde{M}, g) \int_{\tilde{M}} F_\lambda^2 dg_K \leqq \int_{\tilde{M}} (\Delta_g F_\lambda) F_\lambda dg_K , \tag{5.4}$$

by (5.2), (5.3) and the definition of $\sigma(\tilde{M}, g)$. Moreover we estimate

$$\begin{aligned} &\text{the right hand side of (5.4)} \\ &\quad \leqq 2(m_\alpha + 4m_{2\alpha})^{-1}(1+\varepsilon)ab \int_{\tilde{M}} F_\lambda^2 dg_K , \end{aligned}$$

due to (2.9). Then

$$\sigma(\tilde{M}, g) \leqq 2(m_\alpha + 4m_{2\alpha})^{-1}(1+\varepsilon)ab ,$$

for every $0 < \varepsilon < 1$ and $0 \neq \lambda \in \mathfrak{a}^*$. Thus we have $\sigma(\tilde{M}, g) \leqq |\delta|^2$. Q.E.D.

Remark. Due to 2.1, it is proved by the similar way to Theorem C, that

$$\sigma(\boldsymbol{R}^n, g) = 0 ,$$

where $(\boldsymbol{R}^n, g)$ is the standard Euclidean space.

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