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DECOMPOSITION OF REPRESENTATIONS OF $SL(2, C)$ INDUCED BY THE CONTINUOUS SERIES OF $E(2)$

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§ 1. Introduction

Since the representations of $SL(2, C)$ induced by irreducible unitary representations of $E(2) = \left\{\begin{pmatrix} e^{i\psi} & 0 \\ \zeta & e^{-i\psi}\end{pmatrix} : \zeta \in C\right\}$ appear as the restriction to the Lorentz group of some irreducible unitary representations of the inhomogeneous Lorentz group, the decomposition of the induced representations deserves our investigation. For the representations of $SL(2, C)$ induced by irreducible unitary representations with discrete spin of $E(2)$, the decomposition has been obtained by Mukunda [9]. We hope that our analysis will justify the calculations by Chakrabarti [1], [2] and [3].

As is known (see, for example, § 3 of [6]), the problem to decompose a unitary representation of $SL(2, C)$ into irreducible ones can be reduced to the problem to specify the spectral type of certain selfadjoint operators. In our case we must deal with ordinary differential operators L_0^ρ and L_k^ρ ($\rho > 0$ and $k = 1/2, 1, 3/2, \cdots$):

$$(1) \qquad L_0^\rho = -\partial_\tau^2 + \rho^2 e^{-2\tau} \qquad \text{in } L^2(R)\,,$$

$$(2) \qquad L_k^\rho = -2i(\nu)\partial_\tau + 2i\rho e^{-\tau} V_k \qquad \text{in } L^2(R)_{2k+1} = \sum_{\nu=-k}^{k} \oplus L^2(R)\,,$$

where (ν) stands for a diagonal matrix $[k, k-1, \cdots, \quad, \cdots, -k]$ and V_k is a skew symmetric constant matrix whose $(\nu, \nu+1)$ component and $(\nu, \nu-1)$ component are equal to

$$-\sqrt{(k-\nu)(k+\nu+1)}/2 \quad \text{and} \quad \sqrt{(k-\nu+1)(k+\nu)}/2$$

respectively, remaining components being equal to zero. To study the spectral type of these operators is itself of our interest.

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§ 2. Definitions and main theorems

For a positive number ρ denote by π_ρ^+ (resp. π_ρ^-) a unitary representation of $E(2)$ on the Hilbert space

$$L^+ = \left\{f \in L^2(0, 4\pi): f(\psi) = \sum_{\nu \in Z} a_\nu e^{i\nu\psi}\right\}$$

$$\left(\text{resp. } L^- = \left\{f \in L^2(0, 4\pi): f(\psi) = \sum_{\nu+1/2 \in Z} a_\nu e^{i\nu\psi}\right\}\right)$$

defined by

$$\pi_\rho^\pm\left(\begin{pmatrix} e^{i\psi/2} & 0 \\ \zeta e^{i\psi/2} & e^{-i\psi/2}\end{pmatrix}\right) f(\psi') = e^{i\rho \operatorname{Re}(e^{-i\psi'}\zeta)} f(\psi' + \psi) . \tag{3}$$

In the following, G and G_o stands for $SL(2, C)$ and $E(2)$ respectively. One realization of the induced representation $\mathrm{Ind}_{G_o \uparrow G}\, \pi_\rho^\pm$ is to be defined. As is known,

$$V_o = \left\{y = \begin{pmatrix} y_0 - y_3 & y_2 - iy_1 \\ y_2 + iy_1 & y_0 + y_3 \end{pmatrix}: \det y = 0,\ y_0 > 0\right\}$$

is a G-homogeneous space with the G-action $y\cdot g = g^* y g$, whose G-invariant measure $d\mu$ and the little group at $\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}$ are $dy_1 dy_2 dy_3/y_0$ and G_o respectively. Defining the projection $p: G \to V_o$ by $p(g) = g^*\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} g$, we attach to an element u of $SU(2)$ a section $s_u: V_o \to G$ such that $s_u \circ p(\langle\tau, \theta, \varphi\rangle u) = \langle\tau, \theta, \varphi\rangle u$, where

$$\langle\tau, \theta, \varphi\rangle = \begin{pmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2}\end{pmatrix}\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2\end{pmatrix}\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2}\end{pmatrix}$$

with $(\tau, \theta, \varphi) \in R \times (0, \pi) \times (0, 2\pi)$. We denote by $U_\rho^{+,u}$ (resp. $U_\rho^{-,u}$) a unitary representation of G on the Hilbert space $H^+ = L^2(L^+, V_o, \mu)$ (resp. $H^- = L^2(L^-, V_o, \mu)$) defined by

$$U_\rho^{\pm,u}(g) f(y) = \pi_\rho^\pm(g_o) f(y\cdot g) , \tag{4}$$

$$s_u(y) = g_o s_u(y\cdot g) . \tag{5}$$

A representation (U, H) of G determines a sequence of mutually singular σ-finite measures $\{\sigma_1, \sigma_2, \cdots, \sigma_\infty\}$ on $\hat{G} = \sum_{m=0}^\infty \ell_m$ (see the preceding paragraph of Lemma 2 of [6]) such that

$$U \simeq \int_{\hat G}^{\oplus} T_{m,\lambda} d\sigma_1 \oplus [2] \int_{\hat G}^{\oplus} T_{m,\lambda} d\sigma_2 \oplus \cdots \oplus [\aleph_0] \int_{\hat G}^{\oplus} T_{m,\lambda} d\sigma_\infty \tag{6}$$

(see the preceding paragraph of Lemma 3 of [6]). Denoting by $\sigma_{j,ac,p}$ the sum of the absolutely continuous and purely discontinuous parts in Lebesgue's decomposition of the measure σ_j, a representation $U_{ac,p}$ of G is understood to be one which is unitarily equivalent to

$$\int_{\hat{G}}^{\oplus} T_{m,\lambda} d\sigma_{1,ac,p} \oplus [2] \int_{\hat{G}}^{\oplus} T_{m,\lambda} d\sigma_{2,ac,p} \oplus \cdots \oplus [\aleph_0] \int_{\hat{G}}^{\oplus} T_{m,\lambda} d\sigma_{\infty,ac,p} .$$

Similarly, for a selfadjoint operator L, we denote by $L_{ac,p}$ the restriction of L to the subspace which is orthogonal to the singular continuous subspace (p. 517, [7]). We also define U_{ac} and L_{ac} similarly. Finally, for a σ-finite measure σ on R and a Borel set B, let $\int_B^{\oplus} \lambda d\sigma$ denote the selfadjoint multiplication operator: $f(\lambda) \rightarrow \lambda f(\lambda)$ in the Hilbert space $L^2(B, \sigma)$.

THEOREM 1. (i) *L_0^ρ is unitarily equivalent to $\int_{R_+}^{\oplus} \lambda d\lambda$, where $R_+ = (0, \infty)$. (ii) For a positive half-integer k, $L_{k,ac,p}^\rho$ is unitarily equivalent to $[k + 1/2] \int_R^{\oplus} \lambda d\lambda$. For a positive integer k, $L_{k,ac,p}^\rho$ is unitarily equivalent to $[k] \int_R^{\oplus} \lambda d\lambda \oplus [\aleph_0] \int_{\{0\}}^{\oplus} \lambda \delta(d\lambda)$, where δ denotes Dirac's measure.*

THEOREM 2. *Under the notation above it holds that*

$$U_{\rho,ac}^{+,e} \simeq \int_{\ell_0^+}^{\oplus} S_{0,\lambda} d\lambda \oplus \sum_{m=2,4,\cdots} \oplus \int_{\ell_m}^{\oplus} S_{m,\lambda} d\lambda ,$$
$$U_{\rho,ac,p}^{-,e} \simeq \sum_{m=1,3,\cdots} \oplus \int_{\ell_m}^{\oplus} S_{m,\lambda} d\lambda ,$$

where $S_{m,\lambda}$ denotes a representation of the so-called continuous series (§ 10, [10]) and $\ell_0^+ = \{(0, \lambda) : \lambda \geq 0\}$.

Remark. As will be shown in the appendix, corresponding measures σ_j of the representation $U_\rho^{\pm,e}$ are absolutely continuous, so it holds that

$$U_\rho^{+,e} \simeq U_{\rho,ac}^{+,e} , \qquad U_\rho^{-,e} \simeq U_{\rho,ac,p}^{-,e} .$$

Consequently it also holds that $L_k^\rho \simeq L_{k,ac,p}^\rho$.

§ 3. Derivation of differential operators (1) and (2)

We refer to [6] (especially, § 3) the notations in the following. As is verified easily, the operators ω_j corresponding to the representation $U_\rho^{\pm,e}$ acts on $H_0^{\pm,e} = \{f \in H^\pm : f(y, \psi) \in C_0^\infty(Y \times (0, 4\pi))\}$ as smooth differential operators, where Y denotes the image of the projection $p: \{p(\langle \tau, \theta, \varphi \rangle):$

$(\tau, \theta, \varphi) \in R \times (0, \pi) \times (0, 2\pi)\}$. After some tedious computation we obtain their explicit forms in terms of the coordinate $(\tau, \theta, \varphi, \psi)$.

$$
\begin{aligned}
&(y_0, y_1, y_2, y_3) = \tfrac{1}{2}e^{\tau}(1, -\sin\theta\sin\varphi, -\sin\theta\cos\varphi, -\cos\theta)\,, \\
&d\mu = e^{2\tau}\sin\theta d\tau d\theta d\varphi\,, \\
&\omega_1 = \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi - \frac{\cos\varphi}{\sin\theta}\partial_\psi\,, \\
&\omega_2 = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi + \frac{\sin\varphi}{\sin\theta}\partial_\psi\,, \\
&\omega_3 = \partial_\varphi\,, \\
&\omega_4 = -\sin\theta\cos\varphi\partial_\tau - \cos\theta\cos\varphi\partial_\theta + \frac{\sin\varphi}{\sin\theta}\partial_\varphi - \cot\theta\sin\varphi\partial_\psi \\
&\qquad + i\rho e^{-\tau}(\cos\theta\cos\varphi\cos\psi - \sin\varphi\sin\psi)\,, \\
&\omega_5 = \sin\theta\sin\varphi\partial_\tau + \cos\theta\sin\varphi\partial_\theta + \frac{\cos\varphi}{\sin\theta}\partial_\varphi - \cot\theta\cos\varphi\partial_\psi\,, \\
&\qquad + i\rho e^{-\tau}(-\cos\theta\sin\varphi\cos\psi - \cos\varphi\sin\psi)\,, \\
&\omega_6 = \cos\theta\partial_\tau - \sin\theta\partial_\theta + i\rho e^{-\tau}\sin\theta\cos\psi\,.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&H_+ = e^{-i\varphi}\left(i\partial_\theta + \cot\theta\partial_\varphi - \frac{1}{\sin\theta}\partial_\psi\right)\,, \\
&H_- = e^{i\varphi}\left(i\partial_\theta - \cot\theta\partial_\varphi + \frac{1}{\sin\theta}\partial_\psi\right)\,, \qquad H_3 = i\omega_3\,, \\
&\Delta_o = \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\varphi^2 - \frac{2\cos\theta}{\sin^2\theta}\partial_\varphi\partial_\psi + \frac{1}{\sin^2\theta}\partial_\psi^2 + \cot\theta\partial_\theta\,, \\
&F_+ = e^{-i\varphi}\Big\{-\sin\theta\partial_\tau - \cot\theta\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi - i\cot\theta\partial_\psi \\
&\qquad\qquad + i\rho e^{-\tau}(\cos\theta\cos\psi - i\sin\psi)\Big\}\,, \\
&F_- = e^{i\varphi}\Big\{\sin\theta\partial_\tau + \cot\theta\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi - i\cot\theta\partial_\psi \\
&\qquad\qquad + i\rho e^{-\tau}(-\cos\theta\cos\psi - i\sin\psi)\Big\}\,, \\
&F_3 = i\omega_6\,, \\
&\Delta = (F_+F_- + F_-F_+ + 2F_3^2)/2 + \Delta_o - 1\,, \\
&\Delta' = -2\partial_\tau\partial_\psi + 2i\rho e^{-\tau}\sin\psi\partial_\theta - \frac{2i\rho e^{-\tau}}{\sin\theta}\cos\psi\partial_\varphi \\
&\qquad + (-2 + 2i\rho e^{-\tau}\cot\theta\cos\psi)\partial_\psi\,.
\end{aligned}
$$

For $k = 0, \frac{1}{2}, 1, \cdots$, denote by W_k the k-th heighest weight vectors, i.e. the solutions of equations

(7) $$H_3 f = kf\,, \quad \Delta_0 f = -k(k+1)f \qquad \text{for } f \in H^{\pm}$$

with respect to the representation $U_\rho^{\pm,e}$. Quite similarly to the § 4 of [6], we obtain the following table.

	k: $W_k \neq \{0\}$	W_k
$U_\rho^{+,e}$	$0, 1, \cdots$	$\left\{\sum_{\nu=-k}^{k} f_\nu(\tau) P_{k,-\nu}^k(\cos\theta) e^{-ik\varphi+i\nu\psi} : f_\nu \in L^2(R, e^{2\tau}d\tau)\right\}$
$U_\rho^{-,e}$	$1/2, 3/2, \cdots$	

Denote by W_k^o a subspace of W_k consisting of functions with $f_\nu \in C_o^\infty(R)$. Then making use of formulas on $P_{m,n}^k$ ([6], § 4), we calculate the restrictions $\Delta | W_0^o$, $\Delta' | W_k^o$ and $F_+ | W_k^o$.

(8) $$\Delta f_0 = [-\partial_\tau^2 - 2\partial_\tau + \rho^2 e^{-2\tau}] f_0\,,$$

(9) $$\begin{aligned} \Delta' f = \sum_{\nu=-k}^{k} [&-2i(\nu)\partial_\tau f_\nu - 2f_\nu - i\rho e^{-\tau}\sqrt{(k-\nu)(k+\nu+1)} f_{\nu+1} \\ &+ i\rho e^{-\tau}\sqrt{(k-\nu+1)(k+\nu)}] P_{k,-\nu}^k e^{-ik\varphi+i\nu\psi}\,, \end{aligned}$$

(10) $$\begin{aligned} F_+ f = \frac{1}{\sqrt{(2k+1)(2k+2)}} \sum_{\nu=-k-1}^{k+1} &[2i\sqrt{(k-\nu+1)(k+\nu+1)}(\partial_\tau f_\nu - kf_\nu) \\ &+ i\rho e^{-\tau}(\sqrt{(k-\nu)(k-\nu+1)} f_{\nu+1} + \sqrt{(k+\nu)(k+\nu+1)} f_{\nu-1})] \\ &\times P_{k+1,-\nu}^{k+1} e^{-i(k+1)\varphi+i\nu\psi}\,. \end{aligned}$$

Now it is clear that the natural isometry

$$J_k\colon W_k \to L^2(R)_{2k+1} = \sum_{\nu=-k}^{k} \oplus L^2(R)$$

defined by

$$J_k\Big(\sum f_\nu P_{k,-\nu}^k e^{-ik\varphi+i\nu\psi}\Big) = \left(\sqrt{\frac{2}{2k+1}} f_\nu e^\tau\right)$$

transforms $\Delta | W_0$ and $\Delta' | W_k$ to (1) and (2) respectively (notice that the differential operators (1) and (2), with domain $C_o^\infty(R)$ and $C_o^\infty(R)_{2k+1}$ respectively, are essentially selfadjoint).

§ 4. Proof of theorems

4.1. The spectral type of the differential operator (1). Since the operator L_0^ρ is unitarily equivalent to the differential operator

(1′) $$-\partial_\tau^2 + e^{2\tau}\,,$$

we will calculate the spectral matrix of the latter. Let φ_j $(j = 1, 2)$ be solutions of the equation

$$[-\partial_\tau^2 + e^{2\tau} - \lambda]\varphi(\tau) = 0 \qquad \text{with initial value } \begin{pmatrix} \varphi_1(0) & \varphi_2(0) \\ \varphi_1'(0) & \varphi_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $\operatorname{Im}\lambda$ is assumed to be positive. By change of variable $t = e^{-\tau}$, $\psi_j(t) = \varphi_j(\tau)$ satisfies

$$\left[\partial_t^2 + \frac{1}{t}\partial_t + \left(-1 + \frac{\lambda}{t^2}\right)\right]\psi_j = 0 . \tag{12}$$

Two independent solutions of (12) are $I_{\sqrt{-\lambda}}(t)$ and $K_{\sqrt{-\lambda}}(t)$ ([8], p. 161). Set $\nu = \sqrt{-\lambda}$. By our convention $\operatorname{Im}\nu$ is negative. Since $|I_\nu(t)/K_\nu(t)| \to 0$ (resp. ∞), as $t \to 0$ (resp. ∞), the functions $m_{-\infty}(\lambda)$ and $m_{+\infty}(\lambda)$ are equal to $I_\nu'(1)/I_\nu(1)$ and $K_\nu'(1)/K_\nu(1)$ respectively (for the definition of $m_\pm$, see § 5 in chap. 9 of [4]). By the aid of the integral representation ([8], pp. 186–187)

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_0^\pi \operatorname{ch}(z\cos\theta)\sin^{2\nu}\theta d\theta \qquad \text{for } \operatorname{Re}\nu > -1/2 ,$$

$$K_\nu(z) = \int_0^\infty e^{-z\operatorname{ch}t}\operatorname{ch}\nu t dt \qquad \text{for } \operatorname{Re}\nu > -1/2,\ \operatorname{Re} z > 0 ,$$

we deduce that for $\lambda \geq 0$

$$\begin{aligned} d\rho_{11}(\lambda) &= \frac{\operatorname{sh}\pi\sqrt{\lambda}}{\pi^2}\left(\int_0^\infty e^{-\operatorname{ch}t}\cos\sqrt{\lambda}\,t dt\right)^2 d\lambda , \\ d\rho_{22}(\lambda) &= \frac{\operatorname{sh}\pi\sqrt{\lambda}}{\pi^2}\left(\int_0^\infty \operatorname{ch}t\, e^{-\operatorname{ch}t}\cos\sqrt{\lambda}\,t dt\right)^2 d\lambda , \\ d\rho_{21}(\lambda) &= d\rho_{12}(\lambda) \\ &= -\frac{\operatorname{sh}\pi\sqrt{\lambda}}{\pi^2}\left(\int_0^\infty e^{-\operatorname{ch}t}\cos\sqrt{\lambda}\,t dt\right)\left(\int_0^\infty \operatorname{ch}t e^{-\operatorname{ch}t}\cos\sqrt{\lambda}\,t dt\right) d\lambda . \end{aligned}$$

Since the rank of the matrix $(d\rho_{jk}/d\lambda)$ is equal to one almost everywhere and since the operator L_0^ρ is positive definite, (i) of Theorem 1 is now proved.

4.2. The spectral type of the differential operator. The next lemma shows that L_k^ρ contains $[k']\int_R^\oplus \lambda d\lambda$, where k' denotes the greatest integer such that $k' \leq k + 1/2$.

LEMMA 1. *For an* $f = (f_k, \cdots, f_{-k})^t$ *in* $C_o^\infty(R)_{2k+1} = \sum \oplus C_o^\infty(R)$ *with* f_ν

$= 0$ *for non-positive index* ν, $e^{itL_k^\rho}e^{-itL_k^0}f$ *converges strongly in* $L^2(R)_{2k+1}$ *as* $t \to \infty$.

Proof. As is well known ([7], Theorem 3.7 in chap. X) the convergence follows from the integrability of the norm $\|e^{-\tau}V_k e^{-itL_k}f\|$ on some interval (s, ∞), where V_k is the constant matrix such that $L_k^\rho = L_k^0 + 2i\rho e^{-\tau}V_k$. Assume that a finite interval $(-c, c)$ contains the support of f and denote the maximum of the matrix elements of V_k (resp. $\max_{\nu,\tau}|f_\nu(\tau)|$) by v (resp, a). Since $e^{-itL_k^0}f(\tau) = (f_\nu(\tau - 2\nu t))$, we have for any large t an inequality

$$\|e^{-\tau}V_k e^{-itL_k^0}f\| \leq \sqrt{2c}ave^{c-t} ,$$

which implies the integrability of the left side. Q.E.D.

For the time being k is assumed to be a positive half-integer. We recall the eigenfunction expansion for L_k^ρ. Suppose a matrix valued function $\Phi(\tau, \lambda)$ satisfies

$$[-2i(\nu)\partial_\tau + 2i\rho e^{-\tau}V_k - \lambda]\Phi(\tau, \lambda) = 0 \tag{13}$$

with initial value $\Phi(0, \lambda) = E_{2k+1}$ (the unit matrix). Then we have

PROPOSITION 1. *There exists a spectral matrix* $(\rho_{\nu\nu'})$ *with the following properties.*

(i) $(\rho_{\nu\nu'})$ *is an hermitian* $(2k+1) \times (2k+1)$*-matrix valued function on* R.

(ii) $(\rho_{\nu\nu'}(\lambda_2) - \rho_{\nu\nu'}(\lambda_1))$ *is non-negative definite for* $\lambda_2 > \lambda_1$.

(iii) *The total variation of* $\rho_{\nu\nu'}$ *is finite on any finite interval.*

(iv) *For an* $f \in L^2(R)_{2k+1}$, *put*

$$Ff(\lambda) = \lim_{N\to\infty}\int_{|\tau|<N} \Phi^*(\tau, \lambda)f(\tau)d\tau \qquad in\ L^2((\rho_{\nu\nu'})) .$$

Then F *is a unitary operator on* $L^2(R)_{2k+1}$, *and it transforms* L_k^ρ *to the self-adjoint multiplication operator* M $(Mg(\lambda) = \lambda g(\lambda))$ *in* $L^2((\rho_{\nu\nu'}))$. *The inverse* F^{-1} *of* F *is given by*

$$F^{-1}g(\tau) = \lim_{N\to\infty}\int_{|\lambda|<N} \sum_{\nu\nu'} \varphi_\nu(\tau, \lambda)g_{\nu'}(\lambda)d\rho_{\nu\nu'} \qquad in\ L^2(R)_{2k+1} , \tag{14}$$

where φ_ν *is the* ν*-th component column vector of the matrix* $\Phi = (\varphi_k, \cdots, \varphi_{-k})$.

We may allow to skip the proof of the proposition, because it follows the same development as the chapter 10 of [4].

Let $(\rho_{\nu\nu'})$ be the spectral matrix of L_k^ρ. Then there exist an hermitian matrix valued function h and a locally bounded measure σ such that

$$\int \sum_{\nu\nu'} g_\nu \bar{g}_{\nu'} d\rho_{\nu\nu'} = \int \sum_{\nu\nu'} g_\nu \bar{g}_{\nu'} h_{\nu\nu'} d\sigma \qquad \text{for any } g \in L^2((\rho_{\nu\nu'}))\,.$$

If the multiplicity of L_k^ρ is equal to m, there exist some finite Borel set B of R, a unitary matrix valued measurable function U and strictly positive measurable functions $h_1, \cdots, h_m$ such that

$$(h_{\nu\nu'}(\lambda)) = U(\lambda)[h_1(\lambda), \cdots, h_m(\lambda), 0, \cdots, 0]U^*(\lambda) \tag{15}$$

on B almost everywhere with respect to σ. We may assume that $K^{-1} < h_i < K$ for each j for some positive constant K.

Proposition 2. *For a positive half-integer k, the multiplicity m of L_k^ρ does not exceed $k + 1/2$.*

Our proof of the proposition is lengthy. Two lemmas will precede the proof. By change of variable $t = e^{-\tau}$, $\Psi(t, \lambda) = \Phi(\tau, \lambda)$ will satisfy

$$\left[\partial_t + \rho(\nu)^{-1}V_k - \frac{\lambda}{2it}(\nu)^{-1}\right]\Psi(t, \lambda) = 0\,. \tag{13'}$$

We note that $t = \infty$ is the irregular singular point of the equation above.

Lemma 2. *The matrix $(\nu)^{-1}V_k$ has Jordan's canonical form*

$$\begin{pmatrix} J(1, k+1/2) & 0 \\ 0 & J(-1, k+1/2) \end{pmatrix},$$

where $J(\alpha, j)$ denotes a Jordan block.

Proof. Define one-parameter groups ω_i $(i = 1, 3)$ of G by

$$\omega_1(t) = \begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix} \qquad \omega_3(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}.$$

For a one parameter subgroup $\omega(t)$ and a finite dimensional continuous representation T of G, denote $\dot{\omega}$ and $T(\dot{\omega})$ the derivatives at $t = 0$ of $\omega(t)$ and $T(\omega(t))$ respectively. We know that there exists a $(2k+1)$-dimensional analytic representation T_k of G such that the restriction $T_k | SU(2)$ is an irreducible unitary representation of $SU(2)$ satisfying $T_k(\dot{\omega}_3) = i(\nu)$ and $T_k(\dot{\omega}_1) = V_k$. Obviously the characteristic polynomial $P(x)$ of the matrix $(\nu)^{-1}V_k$ is proportional to $\det((\nu)x - V_k)$. Since there exists an element

g of G satisfying $-ix\dot{\omega}_3 - \dot{\omega}_1 = g^{-1}\sqrt{x^2-1}\dot{\omega}_3 g$ if and only if $x^2 \neq 1$, and since it holds that $T_k(-ix\dot{\omega}_3 - \dot{\omega}_1) = (\nu)x - V_k$ for any $x \in C$, only -1 and 1 are possible characteristic roots. The multiplicity of them are equal to $k+1/2$, because $P(x)$ is even. As one sees easily, eigen-spaces for eigenvalue -1 or 1 are of dimension one. Q.E.D.

LEMMA 3. *For any finite interval of real λ, there exists a positive number t_o (independent of λ) and a $(k+1/2)$-dimensional subspace S_λ of C_{2k+1} such that for any solution ψ of* (13′),

$$\int_{t_o}^{\infty} e^{-\alpha t}\frac{|\psi(t)|^2}{t^\beta}dt = \infty \qquad \text{for any } \alpha, \beta \ (0<\alpha<2\rho, 0<\beta)\,, \tag{16}$$

provide $\psi(t_o) \notin S_\lambda$.

Proof. Put $\eta(t) = e^{-2\rho t}\psi(t)$. Then η satisfies

$$\left[\partial_t + \rho(\nu)^{-1}V_k + 2\rho - \frac{\lambda}{2it}(\nu)^{-1}\right]\eta(t) = 0\,. \tag{17}$$

By Lemma 2, the Theorems 4.1 and 4.3 in the chapter 13 of [4], a non-zero solution η of (17) satisfies

$$\limsup_{t\to\infty} \frac{\log|\eta(t)|}{t} = -3\rho \quad \text{or} \quad -\rho\,.$$

Applying the Theorem 4.4 in the same chapter to the equation (17), we conclude that there exist a positive number t_o' and a $(k+1/2)$-dimensional subspace S_λ' of C_{2k+1} such that

$$\begin{aligned} \limsup_{t\to\infty} \frac{\log|\eta(t)|}{t} &= -3\rho \qquad \text{if } \eta(t_o') \in S_\lambda'\backslash\{0\} \\ &= -\rho \qquad \text{if } \eta(t_o') \notin S_\lambda'\,. \end{aligned} \tag{18}$$

Set $t_o = t_o'$ and $S_\lambda = S_\lambda'$. We claim that t_o and S_λ posess the desired property. In fact, if non-zero solution of (13′) with $\psi(t_o) \notin S_\lambda$ does not satisfy (16), then the derivative of the integrand in (16) is also integrable on (t_o, ∞) because ψ satisfies (13′). Thus the integrand converges to zero as $t \to \infty$, contradicting to (18). Q.E.D.

Proof of Proposition 2. Let B, U and h_j $(j=1,\cdots,m)$ be so chosen as stated in the paragraph just before Proposition 2. Set $\Psi(t,\lambda)U(\lambda) = (\tilde{\psi}_j(t,\lambda))$. The assumption $m > k+1/2$ shall lead to a contradiction. For the finite Borel set B, there exist a positive number t_o and $(k+1/2)$-dimensional subspace S_λ of C_{2k+1} for which Lemma 3 holds. Fix a vector valued measurable function $g_o = (g_1, \cdots, g_m)^t$ so that $|g_o(\lambda)| = 1$ and that $\sum \tilde{\psi}_j(t_o, \lambda) h_j g_j(\lambda) \notin S_\lambda$. Then for any $\tilde{g}$ in $L^2(B, \sigma)$, the function defined by

$$\left| \int_B \frac{\sum \tilde{\psi}_j(t) h_j g_j \tilde{g}}{\sqrt{t}} d\sigma \right|^2$$

is integrable on (t_o, ∞) as well as its derivative because of (13′) and (14). Thus the square of the integral converges to zero as $t \to \infty$. By the resonance theorem ([12], p. 69), we have

$$\sup_{t \geq t_o} \int_B |\sum \tilde{\psi}_j(t, \lambda) h_j g_j(\lambda)|^2 \, d\sigma / t < \infty \, .$$

This contradicts to Lemma 3. Q.E.D.

From now on k denotes a positive integer. Denote by $L_k^{\rho\perp}$ the restriction of L_k^ρ to the orthogonal complement $L^2(R)_{2k+1}^\perp$ of the eigenspace for eigenvalue zero. We shall mention to the eigenfunction expansion for $L_k^{\rho\perp}$. Let $\Phi(\tau, \lambda) = (\varphi_k, \cdots, \varphi_1, \varphi_{-1}, \cdots, \varphi_{-k})(\tau, \lambda)$ be a $(2k+1) \times (2k)$-matrix valued function satisfying (13) with initial value $\check{\Phi}(0, \lambda) = (\check{\varphi}_k, \cdots, \check{\varphi}_{-k})\,(0, \lambda) = E_{2k}$, where $\check{\Phi}$ denotes the $(2k) \times (2k)$-matrix obtained by expelling the 0-th row of Φ.

PROPOSITION 3. *There exists a spectral matrix $(\rho_{\nu\nu'})$ for $L_k^{\rho\perp}$ with the following properties.*

(i) *$(\rho_{\nu\nu'})$ is an hermitian $(2k) \times (2k)$-matrix valued function on $R^* = R \backslash \{0\}$*

(ii) *$(\rho_{\nu\nu'}(\lambda_2) - \rho_{\nu\nu'}(\lambda_1))$ is a non-negative definite for $\lambda_2 > \lambda_1$, $\lambda_1 \lambda_2 > 0$*

(iii) *The total variation of $\rho_{\nu\nu'}$ is finite on any finite interval lying outside of a neighborhood of zero.*

(iv) *For an $f \in L^2(R)_{2k+1}^\perp$, put*

$$Ff(\lambda) = \lim_{N \to \infty} \int_{|\tau| < N} \Phi^*(\tau, \lambda) f(\tau) d\tau \quad \text{in } L^2((\rho_{\nu\nu'}))\, .$$

Then F is a unitary operator and transforms $L_k^{\rho\perp}$ to the selfadjoint multiplication operator M ($Mg(\lambda) = \lambda g(\lambda)$) in $L^2((\rho_{\nu\nu'}))$. The inverse F^{-1} of F is given by the formula:

$$F^{-1} g(\tau) = \lim_{N \to \infty} \int_{N^{-1} < |\lambda| < N} \sum \varphi_\nu(\tau, \lambda) g_{\nu'}(\lambda) d\rho_{\nu\nu'} \qquad \text{in } L^2(R)_{2k+1}\, .$$

The proof of the proposition is almost the same as that of Proposition 1. However, in connection with the proof of Parseval equality, we should note that the image $L_k^\rho(C_o^\infty(R)_{2k+1})$ of L_k^ρ is dense in $L^2(R)_{2k+1}^\perp$ and that the following inequalities hold.

$$\int_{\varepsilon<|\lambda|<1} \lambda^2 |g(\lambda)|^2 \, d\rho_\delta(\lambda) \leq \int_{\varepsilon<|\lambda|<1} |g(\lambda)|^2 \, d\rho_\delta(\lambda) \leq \int_R |L_k^\rho f|^2 \, d\tau ,$$
$$\int_{1<|\lambda|<\varepsilon} \lambda^2 |g(\lambda)|^2 \, d\rho_\delta(\lambda) \leq \varepsilon^{-2} \int_{1<|\lambda|<\varepsilon} \lambda^4 |g(\lambda)|^2 \, d\rho_\delta(\lambda) \leq \varepsilon^{-2} \int_R |L_k^{\rho 2} f|^2 \, d\tau .$$

PROPOSITION 4. *For a positive integer k, the multiplicity m of $L_k^{\rho\perp}$ does not exceed k.*

For the proof we again prepare some auxiliary lemmas. By change of variables $t = e^{-\tau}$ and $\psi(t) = \varphi(\tau)$, (13) takes the form

$$\left[(\nu)\partial_t + \rho V_k - \frac{\lambda}{2it}\right]\psi(t) = 0 . \tag{13''}$$

Since L_k^ρ and $J_{k+1}F_+J_k^{-1}$ act on $C_o^\infty(R)_{2k+1}$ as smooth differential operators, denote by $\hat{L}_k^\rho$ and $\hat{F}_{+,k}$ their trivial extensions to $C^\infty(R)_{2k+1}$.

LEMMA 4. *It holds that*

(i) $\hat{L}_{k+1}^\rho \hat{F}_{+,k} = \hat{F}_{+,k}\hat{L}_k^\rho$,

(ii) $(\hat{F}_{+,k}f)_{k+1} = i\rho\sqrt{(2k+1)/(2k+3)}e^{-\tau}f_k$ *for* $f = (f_k, \cdots, f_{-k})^t$,

(iii) *The kernel of* $\hat{F}_{+,k}$ *is* $\{0\}$.

Proof. The fact that Δ' and F_+ commute yields (i). The assertions (ii) and (iii) follow from (10). Q.E.D.

LEMMA 5. *For $\lambda \neq 0$, the k-th component ψ_k of a solution of (13'') satisfies a differential equation*

$$\left[\sum_{j=0}^{2k} a_{k,j}(t, \lambda)\partial_t^j\right]\psi_k(t) = 0 , \tag{19}$$

where the coefficients $a_{k,j}$ are of the form

$$a_{k,j}(t, \lambda) = \sum_{\ell\geq 0}^{\text{finite}} a_{k,j,\ell}(\lambda)t^{-\ell}$$

with

(i) $a_{k,2k} = 1$,

(ii) $a_{k,j,\ell}(\lambda)$ *is a polynomial of* λ *for* $\ell \geq 1$,

(iii) $\sum_{j=0}^{2k} a_{k,j,0}\, x^j = (x^2 - \rho^2)^k$.

Proof. Let $\psi = (\psi_\nu)$ be a solution of (13″). Through the recursion relation

$$\psi_\nu = \alpha_{\nu+1}\partial_t\psi_{\nu+1} + \frac{\lambda}{t}\beta_{\nu+1}\psi_\nu + \gamma_{\nu+1}\psi_{\nu+2}\,, \tag{20}$$

we can represent ψ_ν in terms of ψ_k and its derivatives $\psi_k^{(j)}$. Clearly (ψ_ν) is a solution of (13″) if and only if ψ_k satisfies

$$-k\partial_t\psi_{-k} - \rho\sqrt{2k}\psi_{-k+1} - \frac{\lambda}{2it}\psi_{-k} = 0\,. \tag{21}$$

It is easy to see that the left side of the equation (21) is of the form

$$\left[-k\alpha_k\alpha_{k-1}\cdots\alpha_1\frac{\lambda}{2it}\alpha_{-1}\cdots\alpha_{-k+1}\partial_t^{(2k)} + \sum_{j=0}^{2k-1} b_j(t,\lambda)\partial_t^j\right]\psi_k = 0\,, \tag{21′}$$

where the coefficients $b_j(t,\lambda) = \sum_{\ell\geq 0} b_{j,\ell}(\lambda)t^{-\ell}$ have the properties: 1) $b_{j,0}$ is independent of λ, 2) $b_{j,\ell}(\lambda)$ is a polynomial of λ such that $b_{j,\ell}(0) = 0$ for $\ell \geq 1$ and 3) $b_{j,1}(\lambda) = b'_{j,1}(0)\lambda$. We claim that $b_{j,0} = 0$. Indeed, for $\lambda = 0$ the equation (21′) is reduced to an equation with constant coefficient $b_{j,0}$. If $b_{j,0} \neq 0$ for some j, the dimension of the solutions of (13″) is finite, which is a contradiction because the image $\hat{F}_{+,k-1}\cdots\hat{F}_{+,0}(C^\infty(R)_1)$ is an infinite dimensional subspace of the solutions on account of (i) and (ii) of Lemma 4. Now (i) and (ii) are selfevident.

Our proof of (iii) is rather lengthy. Obviously $R\psi_k(t) = \psi_k(-t)$ satisfies

$$\left[\sum_{j=0}^{2k}(-1)^j a_{k,j}(-t,\lambda)\partial_t^j\right]R\psi_k(t) = 0\,. \tag{22}$$

On the other hand both $R\psi(t)$ and $(-1)^\nu\psi(t)$ satisfy (13″) provided ρ is replaced by $-\rho((-1)^\nu$ denotes a diagonal matrix whose (ν,ν) component is equal to $(-1)^\nu)$. Thus a solution of (19) is a solution of (22), from which it follows that $a_{k,j}(-t,\lambda)(-1)^j = a_{k,j}(t,\lambda)$, and in particular that $\sum a_{k,j,0}x^j$ is an even function of x. Put $P_k = \sum_{j=0}^{2k} a_{k,j,0}x^j$. We will show that P_k devides P_{k+1}. For this purpose we need

LEMMA 6. *For positive integers m and n $(m \geq n)$, consider differential equations with holomorphic coefficients at $t = \infty$.*

$$\text{(A)}\quad \left[\sum_{j=0}^{m} a_j(t)\partial_t^j\right]\psi(t) = 0 \qquad \text{(B)}\quad \left[\sum_{j=0}^{n} b_j(t)\partial_t^j\right]\psi(t) = 0\,.$$

Assume that $a_m = b_n = 1$ and that every solution of (B) *is a solution of*

(A). *Then the polynomial* $\sum_{j=0}^{n} b_{j,0}x^j$ *devides* $\sum_{j=0}^{m} a_{j,0}x^j$, *where* $a_{j,0} = a^j(\infty)$ *and* $b_{j,0} = b_j(\infty)$.

Assuming Lemma 6, we continue the proof of (iii). If ψ_k is a solution of (19), $t\psi_k$ satisfies a differential equation

$$\left[\sum_{j=0}^{2k} \tilde{a}_{k,j}(t, \lambda)\partial_t^j\right](t\psi_k) = 0 , \tag{21''}$$

where the coefficients are holomorphic at $t = \infty$ with $\tilde{a}_{k,j}(\infty, \lambda) = a_{k,j,0}$. We see, making use of Lemma 4, that $t\psi_k$ is a solution of (19) with the index $k + 1$. Now applying Lemma 6 to equations (19) with index $k + 1$ and (21''), it follows that P_k devides P_{k+1}. Since $P_1(x) = x^2 - \rho^2$ and since $P_k(x)$ is even, the assertion (iii) follows as soon as we verify that $a_{k,0,0}$ is equal to $(-1)^k\rho^{2k}$. From (20) and (21) we can deduce that $a_{k,0,0}$ is equal to

$$\sum_{\nu=0}^{k} (-\rho^2)^k \frac{(2\nu)!\,(2k - 2\nu)!}{4^k(\nu!)^2((k - \nu)!)^2} .$$

Put $c_\nu = 1\cdot 3\cdot \cdots \cdot(2\nu - 1)/(2\cdot 4\cdot \cdots \cdot 2\nu)$. We shall show that $\sum_{\nu=0}^{k} c_\nu c_{k-\nu} = 1$. This equality is a direct consequence of the relation $(1 - x)^{-1} = \{(1 - x)^{-1/2}\}^2$. Lemma 5 is now proved. Q.E.D.

Proof of Proposition 4. Let a finite Borel set of R^*, a unitary $(2k) \times (2k)$-matrix valued function $U(\lambda)$ and strictly positive bounded functions h_j $(j = 1, 2, \cdots, m)$ be so chosen as before (see (15)). Put $\chi = (\psi_k^{(j)})_{j=0}^{2k-1}$ for a solution ψ_k of (19). Then χ satisfies

$$\partial_t\chi = \left[\sum_{\ell\geq 0}^{\text{finite}} A_\ell(\lambda)t^{-\ell}\right]\chi , \tag{23}$$

where A_ℓ is a smooth function of λ on R^* and

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -a_{k,0,0} & -a_{k,1,0} & \cdots & \cdots & \cdots & -a_{k,2k-1,0} \end{pmatrix}$$

Conversely, it follows, from (13'') and (20), that there exists a lower triangular matrix valued function $W(t, \lambda) = (w_{\nu j}(t, \lambda))$ such that the function $\psi = (\psi_\nu)$ given as follows is a solution of (13''):

$$
(24) \qquad
\begin{aligned}
\psi_\nu &= \sum_{j=0}^{k-\nu+1} w_{\nu j}\chi_j \qquad \text{for } |\nu| = 1, 2, \cdots, k\,, \\
\psi_0 &= \frac{2i\sqrt{k(k+1)}}{\lambda} t(-\psi_1 + \psi_{-1})\,.
\end{aligned}
$$

Since the matrix A_0 has characteristic roots $\pm\rho$ with multiplicity k, there exist, for the Borel set B, a positive number t_o and a k-dimensional subspace $\tilde{S}_\lambda$ of C_{2k} such that for any solution χ,

$$
(25) \qquad \int_{t_o}^{\infty} e^{-\alpha t} \frac{|\chi(t)|^2}{t^\beta} dt = \infty
$$

$$\text{for any } \alpha \text{ and } \beta \ (0 < \alpha < 2\rho, 0 < \beta)\,,$$

provided $\chi(t_o) \notin \tilde{S}_\lambda$. Let S_λ be the image $W(t_o, \lambda)\tilde{S}_\lambda$. Since there exist positive constants a and n such that

$$\| W^{-1}(t, \lambda) \| \leq at^n \qquad \text{on } (t_o, \infty) \times B\,,$$

any solution ψ of (13″) satisfies

$$
(26) \qquad \int_{t_o}^{\infty} e^{-\alpha t} \frac{|\check{\psi}(t)|^2}{t^\gamma} dt = \infty
$$

$$\text{for any } \alpha \text{ and } \gamma \ (0 < \alpha < 2\rho, 2n < \gamma)$$

provided $\check{\psi}(t_o) \notin S_\lambda$, where $\check{\psi}$ denotes the vector obtained by expelling the 0-th component of ψ. Set $\check{\Phi}(t, \lambda)U(\lambda) = (\tilde{\psi}_j(t, \lambda))$. Obviously the column vectors $\tilde{\psi}_j$ $(j = 1, \cdots, 2k)$ are independent solutions of the following equation

$$[\partial_t + B_{-1}t + B_0 + B_1 t^{-1}]\check{\psi} = 0\,,$$

where B_j are some smooth functions of λ on R^*. Assuming that the multiplicity m of $L_k^{\rho\perp}$ exceeds k, take some bounded measurable functions g_j $(j = 1, \cdots, m)$ on B such that $\sum_{j=1}^m \tilde{\psi}_j(t_o, \lambda)h_j g_j(\lambda) \notin S_\lambda$. On account of Proposition 3 (iv), the square of the integral

$$\int_B \sum_{j=0}^{m} \tilde{\psi}_j(t, \lambda) h_j g_j \tilde{g}(\lambda) d\sigma / t^\gamma$$

is integrable on (t_o, ∞) provided $\tilde{g} \in L^2(B, \sigma)$ and $\gamma > n$. Now the same reasoning as in the proof of Proposition 2 yields a contradiction to (26).

Q.E.D.

It remains to prove Lemma 6.

Proof of Lemma 6. Denote by S the solutions of the equation (B).

Put $Q_n(x)=\sum_{0\le j\le n} b_j(\infty)x^j$. We consider a class of differential operators $D_h=\{\sum_{0\le j\le h} p_j(t)\partial_t^j : p_j$ is holomorphic at $t=\infty\}$ for non-negative integer h. Observe that for any $D\in D_h$ there exists a unique differential operator in D_{n-1} (denote it by $r(D)$) such that D and $r(D)$ agree on S. To a differential operator $D=\sum_{0\le j\le h} p_j(t)\partial_t^j$ in D_h we assign a polynomial $\sum_{0\le j\le h} p_j(\infty)x^j$, which will be denoted by $f(D)$. Finally, for a polynomial $Q(x)$ $\pi(Q)$ stands for the remainder with respect to Q_n. It suffices to show that $f\circ r=\pi\circ f$. The equality clearly holds if $0\le h\le n-1$. So we proceed by induction. Assume that D has a form $D=\sum_{0\le j\le h+1} p_j(t)\partial_t^j$. Put $\tilde{D}=\sum_{0\le j\le h} p_j(t)\partial_t^j - p_{h+1}(t)\partial_t^{h+1-n}(\sum_{0\le j\le n-1} b_j(t)\partial_t^j)$. Then we have $f\circ r(D)=f\circ r(\tilde{D})=\pi\circ f(\tilde{D})$. On the other hand, it holds that

$$\begin{aligned}\pi\circ f(\tilde{D}) &= \pi\left(\left\{\sum_{0\le j\le h} p_j(\infty)x^j - p_{h+1}(\infty)\left(\sum_{0\le j\le n-1} b_j(\infty)x^j\right)x^{h+1-n}\right\}\right)\\ &= \pi\left(\left\{\sum_{0\le j\le h} p_j(\infty)x^j + p_{h+1}(\infty)x^{h+1}\right\}\right)\\ &= \pi\circ f(D)\,. \end{aligned}$$

Q.E.D.

Now we are ready to prove the theorems.

Proof of Theorem 1. We have proved (i) in 4.1. From Lemma 1, Propositions 2 and 4 it follows that $L^{\rho}_{k,ac}\simeq[k']\int_R^{\oplus}\lambda d\lambda$ (k' is the greatest integer such that $k'\le k+1/2$). Since the function $e^{-\tau}$ is rapidly decreasing as $\tau\to\infty$, neither L^{ρ}_k with a positive half-integer nor $L^{\rho\perp}_k$ with a positive integer k has an eigenvalue (§ 3, [5]). For a positive integer k, L^{ρ}_k maps an infinite dimensional space $\hat{F}_{+,k-1}\cdots\hat{F}_{+,0}(C_0^{\infty}(R)_1)$ to zero due to Lemma 4. This completes the proof of (ii) Q.E.D.

Proof of Theorem 2. For the representation $U^{+,e}_{\rho}$ it holds that $[\Delta'\mid W_k\ominus F_+W_{k-1}]_{ac}\simeq\int_R^{\oplus}\lambda d\lambda$ for a positive integer k. Indeed, the equivalence follows from the fact that $[\Delta'\mid W_k]_{ac}\simeq[k]\int_R^{\oplus}\lambda d\lambda$ and that $\Delta'\mid W_{k-1}\simeq\Delta'\mid\overline{F_+W_{k-1}}$ ([6], Lemma 4). Moreover $\Delta\mid W_0\simeq\int_R^{\oplus}\lambda d\lambda$ and $W_k=\{0\}$ for a half-integer k. Therefore the result of § 3 of [6] implies the first assertion of the theorem. Similarly the second assertion follows from the fact that $[\Delta'\mid W_k\ominus\overline{F_+W_{k-1}}]_{ac,p}\simeq\int_R^{\oplus}\lambda d\lambda$ for a half-integer k and that $W_k=\{0\}$ for a positive integer k. Q.E.D.

Appendix

The following proposition as well as an elegant proof of it is due to Professors T. Hirai and N. Tatsuuma.

PROPOSITION A. *It holds that*

(i) $\mathrm{Ind}_{G_0 \uparrow G}\, \pi_\rho^{\pm} \simeq \mathrm{Ind}_{G_0 \uparrow G}\, \pi_{\rho'}^{\pm}$,

(ii) $\mathrm{Ind}_{\{e\} \uparrow G}\, I \simeq [\aleph_0]\, \mathrm{Ind}_{G_0 \uparrow G}\, \pi_\rho^{+} \oplus [\aleph_0]\, \mathrm{Ind}_{G_0 \uparrow G}\, \pi_\rho^{-}$,

where I denotes the unit representation of $\{e\}$. In particular the associated measures of the representation $\mathrm{Ind}_{G_0 \uparrow G}\, \pi_\rho^{\pm}$ are absolutely continuous ones on $\hat{G}$.

Proof. We consider subgroups

$$G_{-1} = \left\{ \begin{pmatrix} \varepsilon & 0 \\ \zeta & \varepsilon \end{pmatrix} : \varepsilon = \pm 1, \zeta \in C \right\}$$

and

$$G_1 = \left\{ \begin{pmatrix} k & 0 \\ \zeta k^{-1} & k^{-1} \end{pmatrix} : k \in C^*, \zeta \in C \right\} .$$

Note that G_1 is a semi-direct product group between C and C^*. The representations $\eta_{\hat\zeta}^{\pm}$ ($\hat\zeta \in C$) of G_{-1} will be defined as follows.

$$\eta_{\hat\zeta}^{+}\left(\begin{pmatrix} \varepsilon & 0 \\ \zeta & \varepsilon \end{pmatrix}\right) e^{i\langle \zeta, \hat\zeta \rangle} , \qquad \eta_{\hat\zeta}^{-}\left(\begin{pmatrix} \varepsilon & 0 \\ \zeta & \varepsilon \end{pmatrix}\right) = \varepsilon e^{i\langle \zeta, \hat\zeta \rangle} ,$$

where $\langle \zeta, \hat\zeta \rangle = \zeta_1 \hat\zeta_1 + \zeta_2 \hat\zeta_2$ for $\zeta = \zeta_1 + i\zeta_2$ and $\hat\zeta = \hat\zeta_1 + i\hat\zeta_2$. From Mackey's theorem on irreducible unitary representations of semi-direct product group it follows that if $\hat\zeta \hat\zeta' \neq 0$,

$$\mathop{\mathrm{Ind}}_{G_{-1} \uparrow G_1} \eta_{\hat\zeta}^{\pm} \simeq \mathop{\mathrm{Ind}}_{G_{-1} \uparrow G_1} \eta_{\hat\zeta'}^{\pm}$$

(in fact, both representations are irreducible representations of G_1). From Mackey's theorem on the induced representations, we have

$$\mathop{\mathrm{Ind}}_{G_{-1} \uparrow G} \eta_{\hat\zeta}^{\pm} \simeq \mathop{\mathrm{Ind}}_{G_1 \uparrow G} \left(\mathop{\mathrm{Ind}}_{G_0 \uparrow G_1} \left(\mathop{\mathrm{Ind}}_{G_{-1} \uparrow G_0} \eta_{\hat\zeta}^{\pm} \right) \right) .$$

Since the fact that $\mathrm{Ind}_{G_{-1} \uparrow G_0}\, \eta_\rho^{\pm} \simeq \pi_\rho^{\pm}$ is known, (i) has been proved. As one sees easily

$$\mathop{\mathrm{Ind}}_{\{e\} \uparrow G_{-1}} I \simeq \int_C^{\oplus} \eta_{\hat\zeta}^{+} d\hat\zeta \oplus \int_C^{\oplus} \eta_{\hat\zeta}^{-} d\hat\zeta .$$

Thus

$$\operatorname*{Ind}_{\{e\}\uparrow G} I \simeq \operatorname*{Ind}_{G_{-1}\uparrow G}\left(\operatorname*{Ind}_{\{e\}\uparrow G_{-1}} I\right) \simeq \int_C^{\oplus} \operatorname*{Ind}_{G_{-1}\uparrow G} \eta_\xi^+ d\hat{\zeta} \oplus \int_C^{\oplus} \operatorname*{Ind}_{G_{-1}\uparrow G} \eta_\xi^- d\hat{\zeta}$$
$$\cong [\aleph_0] \operatorname*{Ind}_{G_{-1}\uparrow G} \eta_\rho^+ \oplus [\aleph_0] \operatorname*{Ind}_{G_{-1}\uparrow G} \eta_\rho^- .$$

Now, since $\operatorname{Ind}_{G_{-1}\uparrow G}\eta_\rho^\pm \simeq \operatorname{Ind}_{G_0\uparrow G}\pi_\rho^\pm$, the assertion (ii) follows. The remaining part of the proposition follows from the fact that Plancherel measure of G is absolutely continuous with support $\hat{G}\backslash\{(o,\lambda)\colon -1<\lambda<0\}$ ([10], 6 in § 14). Q.E.D.

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