R. Carlsson
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N-ARY ALGEBRAS

RENATE CARLSSON

1. Introduction

N-ary algebras are modules with a n-fold multiplication which we assume to be associative if nothing else is stated. They are a canonical generalization of binary and ternary associative algebras. Ternary rings were first investigated by Lister [8]. The aim of this note is to show that the Wedderburn structure theory and the usual cohomology for binary associative algebras can be extended to n-ary algebras. For ternary algebras this has been done in [8] and [1]. Moreover analogous results are wellknown for Lie and alternative triple systems, and for ternary Jordan pairs.

N-ary algebras are a special case of algebras with n-ary multiplication operators which were studied especially by Kurosh [7]. Let A be a not necessarily associative n-ary algebra over a ring k. Assume that the k-module A is free over k of finite rank r. Then A corresponds to a system of r differential equations in r variables x_i, $i = 1, \cdots, r$, $\dot{x}_j = p_j(x)$, $x := (x_j)_{j=1}^r$, $p_j(x) \in k[x]$, p_j a homogeneous polynomial of degree n, shortly $\dot{x} = p(x)$, p a homogeneous vector polynomial. This has been established by Röhrl [11].

So one notes that the result analogous to the Wedderburn decomposition as shown in the following implies the complete reducibility of the corresponding system of differential equations.

The proofs in the subsequent make use of the theory for binary and ternary associative algebras.

2. Definitions and imbeddings

Let k denote a commutative and associative ring with a unit 1. The k-modules U considered are taken to be unitary. Let $U^{[n]}$ for $n \in N$ denote the n-fold cartesian product. Then a *n-multiple system* A for

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$n \in N\backslash\{1\}$ is a k-module together with a n-linear inner composition $\mu: A^{[n]} \to A$ with

$$(x_i)_{i=1}^n := (x_i)_{i=1,\cdots,n} \mapsto \langle x_i \rangle_{i=1}^n$$

where $x_i \in A$. The multiple system is called *associative* or in short a *n-ary algebra* if the following associativity conditions are valid. Let $y_i \in A$ with $y_1 := \langle x_i \rangle_{i=1}^n$, the x_i as afore. Then for any $r \in \{2, \cdots, n\}$

$$\langle y_i \rangle_{i=1}^n = \langle z_i \rangle_{i=1}^n \tag{1}$$

with $z_i := x_i$ when $1 \leq i < r$, $z_i := y_i$ for $r < i \leq n$, and $z_r := \langle v_i \rangle_{i=1}^n$ with $v_i := x_{r-1+i}$ if $1 \leq i \leq i_0 := n - r + 1$ and $v_{i_0+j} := y_{1+j}$ if $1 \leq j < r$. More informally, for any r

$$\mu(\mu(x_1, \cdots, x_n), y_2, \cdots, y_n) = \mu(x_1, \cdots, x_{r-1}, \mu(x_r, \cdots, y_r), y_{r+1}, \cdots, y_n) .$$

By (1) the multiplication in A is associative in the obvious sense. The length of any monomial is congruent to 1 modulo $n - 1$. If $n = 2$ then A is a binary associative algebra.

For an associative k-algebra B, $x_i \in B$, and $s \in N$, let $\prod_{i=1}^1 x_i := x_1$ further $\prod_{i=1}^{s+1} x_i := (\prod_{i=1}^s x_i) \cdot x_{s+1}$. Suppose that A is a r-ary algebra, and $x_i \in A$ for $i \in \{0, \cdots, r - 1\}$. Define the right multiplication ρ by

$$\rho(x_i)_{i=1}^{r-1} : x_0 \mapsto \langle x_i \rangle_{i=0}^{r-1} .$$

If $d \in N\backslash\{1\}$, $n := (r - 1) \cdot (d - 1) + 1$ then we consider A as a n-ary algebra $A_{r(d)}$ by

$$\langle x_i \rangle_{i=0}^{n-1} := \left(\prod_{j=0}^{d-2} \rho(x_{j(r-1)+i})_{i=1}^{r-1} \right)(x_0) .$$

$A_{r(d)}$ is a d-ary algebra if $r = 2$. Define $A^1 := A$ and

$$A^{s(r-1)+1} := \left(\prod_{j=1}^{s} \rho(A_{ij})_{i=1}^{r-1} \right)(A) \qquad \text{where } A_{ij} := A \text{ and } s \in N .$$

For a n-ary algebra A the concepts of a (n-*ary*) *morphism*, *automorphism*, *ideal*, *subalgebra* etc. are defined obviously (cf. [2]). For a submodule B we denote by ${}^{(1)}B$ the k-module generated by the monomials $\langle x_i \rangle_{i=1}^n$ with $x_i \in A$ and $x_{i_1}, x_{i_2} \in B$ for two different subscripts. If B is an ideal in A then obviously ${}^{(1)}B$ is an ideal.

EXAMPLES. 1.) Let A be a ternary ring.

2). For $n \in N$, $n \geq 2$, Z_{n-1} the rational integers modulo $n - 1$, and

$i \in \boldsymbol{Z}_{n-1}$ let V_i denote a k-module. Set $F_i := \mathrm{Mor}_k(V_i, V_{i+1})$, the k-module of the module morphisms, and $F := \oplus F_i$ for $i \in \boldsymbol{Z}_{n-1}$. The structure of a n-ary algebra can be induced on F obviously as follows: If $f_i \in F_i$ and if $f_{j+1}f_j$ for $j \in \boldsymbol{Z}_{n-1}$ denotes the product of maps then with $s_t \in \boldsymbol{Z}_{n-1}$ let

$$\langle f_{s_t}\rangle_{t=1}^n := \begin{cases} \prod_{t=1}^n f_{s_t} & \text{if } s_{t-1} = s_t + 1 \text{ when } t > 1 . \\ 0 & \text{otherwise} , \end{cases}$$

$\prod$ defined obviously. If any V_i is free of finite rank $r_i > 0$ we denote this n-ary algebra by $k(r_i)_{i=1}^{n-1}$.

3.) If A is an associative algebra, φ an automorphism with $\varphi^{n-1} = \mathrm{Id}_A$, and $\xi \in k, \xi$ a $n-1$th root of unity then the corresponding rootspace is a n-ary subalgebra of $A_{\tau(n)}$.

Define a n-multiple module over A to be a k-module M together with n multilinear maps μ_j,

$$\mu_j : A^{[n-1]} \times M \to M$$

where $j \in \{1, \cdots, n\}$. The k-module direct sum $E_0 := A \oplus M$ can be considered as a n-multiple system with ${}^{(1)}M := \{0\}$, the n-linear multiplication in E_0 extending μ and the μ_j. M is called a (*associative*) *n-ary module* over A if E_0 is associative. E_0 is called the *semidirect sum* of A and M or the *split zero extension* of A by M. In the following we denote by A a n-ary algebra and by M a n-ary module over A if we state nothing different.

Similarly as in the ternary case we can imbed A into an associative algebra $S(A)$. For this let $E(A) := \oplus_{j=1}^{n-1} \otimes^j A$. We induce the structure of a binary algebra on the k-module $E(A)$ as follows. If $p, q \in \{1, \cdots, n-1\}$ let $x := \otimes_{i=1}^p x_i$, $y := \otimes_{j=1}^q y_j$ with $x_i, y_j \in A$. Then

$$x \circ y := \begin{cases} x \otimes y & \text{if } p + q \leq n - 1 \\ \bigotimes_{j=1}^{p+q-(n-1)} z_j & \text{if } p + q \geq n . \end{cases}$$

For the latter case let $u_i := x_i$ when $1 \leq i \leq p$, $u_i := y_{i-p}$ if $p < i \leq n$, further $z_1 := \langle u_i \rangle_{i=1}^n$, and $z_j := y_{j+(n-p)}$ otherwise. Then any product of length not exceeding n is associative.

Let K denote the submodule of $E(A)$ generated by

$$\begin{aligned} \{v \mid \exists j, 2 \leq j \leq n-1 \forall m_1, m_2 \in N, m_1 + m_2 = n - j : v \in A^j := \otimes^j A \\ \wedge v \circ A^{n-j} = A^{n-j} \circ v = A^{m_1} \circ v \circ A^{m_2} = \{0\}\} . \end{aligned} \tag{2}$$

It is shown by an obvious verification that K is an ideal in $E(A)$, and that $S(A) := E(A)/K$ is an associative algebra. Then $\iota: A \to S(A)$ with $a \mapsto a + K$ is a n-ary monomorphism of A into $S(A)_{\tau(n)}$. We call $S(A)$ the *standard imbedding* of A.

Let B denote an associative algebra with multiplication $(y, z) \mapsto y \circ z$ for $y, z \in B$, together with a map $\iota: A \to B$. Then (B, ι) is called an *imbedding* of A if ι is a monomorphism of A in $B_{\tau(n)}$, and $B = \sum_{j=1}^{n-1} \iota(B)^j$. Then $\prod_{i=1}^{n} \iota(x_i) = \iota(\langle x_i \rangle_{i=1}^{n})$ for $x_i \in A$, the product taken in B. If the sum is direct then the imbedding is *direct*. For a direct imbedding let $\varepsilon_j: B \to B_j := (\iota(A))^j$ be the canonical projection. Then $B_i \circ B_j = B_p$ with $p = i + j$ modulo $(n - 1)$, $p \in \{1, \cdots, n - 1\}$. If $\xi \in k$, ξ a primitive $n - 1$th root of unity, $w \in B$ and $w = \sum_{j=1}^{n-1} w_j$ with $w_j \in B_j$ then $\varphi: B \to B$ with $w \mapsto \sum_{j=1}^{n-1} \xi^j w_j$ is an automorphism of B. An imbedding (B, ι) of A is called *universal* if for any imbedding (C, κ) of A there exists a (unique) morphism $f: B \to C$ of the algebras with $f\iota = \kappa$.

PROPOSITION 1. *Let A denote an associative n-ary algebra over k. Then there exists a universal imbedding $(U(A), \iota)$ of A. $(U(A), \iota)$ is direct.*

Proof. Let $T^*(A) := \bigoplus_{n \in N} (\otimes^n A)$ denote the nonunitary tensor algebra of the k-module A. The ideal Q of $T^*(A)$ is generated by $\otimes_{i=1}^{n} x_i - \langle x_i \rangle_{i=1}^{n}$ for $x_i \in A$. Set $U(A) := T^*(A)/Q$, further $\iota: x \mapsto x + Q$. Then $(U(A), \iota)$ is an imbedding by the existence of the standard imbedding, and the elementary properties of the tensor product. From the latter the imbedding is direct. □

Note that $U(A) \cong \bigoplus_{j=1}^{n-1} (\otimes^j A)$. In the following we may suppose that ι is the inclusion map. Define the subset V of $U(A)$ by substituting $U(A)$ instead of $E(A)$ in (2). Let $J(A)$ denote the ideal spanned by V. Obviously $S(A) \cong U(A)/J(A)$.

3. Radical and semisimplicity

Let A be a n-ary algebra. We define the Jacobson radical $R(A)$ or R of A. If $n = 2$ let [9]

$$(3) \qquad R(A) := \{x \in A \mid \forall y \in A \exists z \in A : x + z = x \cdot y \cdot z = z \cdot y \cdot x\}.$$

If $n > 2$ set $R(A) := R(U(A)) \cap A$. $R(A)$ is an ideal of A. A is called *semisimple* if $R(A) = \{0\}$.

For $r \in N_0$ let $A_{(r)} := A^{r(n-1)+1}$. We call A *nilpotent if* $A_{(r)} = \{0\}$ for some r. For an ideal I set ${}^{(0)}I := I$ and ${}^{(r+1)}I := {}^{(1)}({}^{(r)}I)$ if $r \in N$. Then the ${}^{(r)}I$ define a descending chain of ideals in A. I is called *A-solvable* if ${}^{(r)}I = \{0\}$ for some $r \in N_0$. Obviously the sum of A-solvable ideals is A-solvable.

LEMMA 1. *An ideal I in a n-ary algebra is nilpotent if and only if I is A-solvable.*

Proof. If $r \in N_0$ define $d_r \in N_0$ by $n^r = d_r \cdot (n-1) + 1$. Since ${}^{(r)}I \supset I_{(d_r)}$, A-solvability implies nilpotency.

On the other hand the defining monomials z in ${}^{(r)}I$ have 2^r factors in I. Let $r = 2n$. We may assume that the product of any n consecutive factors in each z is in I. Now $2^{2n}/n \geq 2n - 1$. Thus

$$ {}^{(2n)}I \subset I_{(2)} \, . $$

Assume ${}^{(2sn)}I \subset I_{(2s)}$ for some $s \in N$. Then

$$ {}^{(2(s+1)n)}I \subset {}^{(2sn)}({}^{(2n)}I) \subset ({}^{(2n)}I)_{(2s)} \subset (I_{(2)})_{(2s)} \subset I_{(2(s+1))} \, . $$

Hence nilpotency implies A-solvability. □

LEMMA 2. *Any nilpotent ideal I in a n-ary algebra A generates a nilpotent ideal in $U(A)$.*

Proof. For an ideal I in A denote by $I[A]$ the ideal in $U(A)$ generated by I. If $r \in N_0$ hence

$$ (I[A])^{n^r} \subset ({}^{(r)}I)[A] \, . $$

Thus by Lemma 1 if I is nilpotent then $I[A]$ is nilpotent.

We call A *artinian* if $U(A)$ is an artinian ring. By Hopkins Theorem $R(U(A))$ is nilpotent. Hence

COROLLARY 1. *The radical R of an artinian n-ary algebra A is the maximal nilpotent ideal in A.*

If the ring k contains a primitive $n-1$th root of unity then k is called *$n-1$ split.* Let k and $\tilde{k}$ be fields with $k \subset \tilde{k}$: The base field extension $A_{\tilde{k}} := A \otimes_k \tilde{k}$ can be considered as a n-ary algebra in the obvious way. Call A *separable* if $A_{\tilde{k}}$ is semisimple for any $\tilde{k}$. By standard properties of scalar extensions one derives $S(A_{\tilde{k}}) \cong S(A)_{\tilde{k}}$ with canonical algebra isomorphism. If A is artinian then $(R(S(A)))_{\tilde{k}}$ corresponds to a submodule of $R(S(A_{\tilde{k}}))$.

THEOREM 1. *Let A be a n-ary algebra over a ring k.*

(1) *If A is semisimple and k $n-1$ split, or k a field with $\mathrm{char}(k) \nmid n-1$ and A artinian and separable over the field then $S(A)$ is semisimple. If in the latter case A is finite dimensional over k then $S(A) \cong U(A)$.*

(2) *Any semisimple direct imbedding of A is isomorphic to $S(A)$.*

Proof. (1) We may assume that k contains a primitive $n-1$th root of unity ξ, and that A is semisimple. Generate the automorphism φ of $U(A)$ by ξ. We consider the radical of $U(A)$, $\tilde{R} := R(U(A))$. R is φ-invariant by (3). Since $\varphi^{n-1} = \mathrm{Id}_{U(A)}$ then similarly as in the finite dimensional case there is a rootspace decomposition of $\tilde{R}$ for φ, $\tilde{R} = \oplus_{i=1}^{n-1} \tilde{R}_i$, ξ^i the root belonging to $\tilde{R}_i$. Assume $m_1, m_2 \in N$ and $m_1 + m_2 = n - i$. Since $A^{m_1} \circ \tilde{R}_i \circ A^{m_2} = A^{n-i} \circ \tilde{R}_i = \tilde{R}_i \circ A^{n-i} = \{0\}$ if $i \geq 2$, hence $\tilde{R} \subset J(A)$. Now $J(A)$ is nilpotent thus $J(A) = \tilde{R}$. Hence $S(A)$ is semisimple.

Let A be finite dimensional over the field k and A separable. Then there exists a Wedderburn decomposition $U(A) = B \oplus J(A)$ of $U(A)$ with a semisimple subalgebra B where $\varphi(B) = B$ [11]. Since $(J(A))_1 = \{0\}$ one has $A \subset B$, thus $U(A) = B$. Hence $J(A) = \{0\}$.

(2) Let B denote an arbitrary direct and semisimple imbedding of A, and $f: U(A) \to B$ the corresponding morphism. If $z \in \ker(f)$ then $z_i := \varepsilon_i(z) \in \ker(f)$ for $i = 1, \cdots, n-1$. Now $z_1 = 0$. Hence with m_1 and m_2 as before then $A^{m_1} \circ z_i \circ A^{m_2} = A^{n-i} \circ z_i = z_i \circ A^{n-i} = 0$ if $i \geq 2$. Since $J(A)$ is nilpotent $\ker(f) \subset J(A) \subset \ker(f)$. Hence $S(A) \cong B$. $\square$

We call a n-ary algebra A with $A^n \neq \{0\}$ simply if $\{0\}$ and A are the only ideals in A. We then get as a corollary

THEOREM 2. *Let A be an artinian n-ary algebra over a ring k, k $n-1$ split, and A semisimple. Then*

$$A = \bigoplus_{i=1}^{r} S_i\ , \qquad r \in N_0\ ,$$

where each S_i is a simple ideal in A.

Proof. Let $A \neq \{0\}$. By Theorem 1 the standard imbedding $S(A)$ is semisimple. Thus $S(A)$ decomposes into a direct sum of φ-simple ideals C_j, $S(A) = \oplus_{j=1}^{r} C_j$, $r \in N$. Observing Theorem 1 then $A = \oplus_{j=1}^{r} (C_j)_1$ is a decomposition into a direct sum of simple ideals. $\square$

Let M be a n-ary module over A and $E_0 = A \oplus M$ the semidirect sum. If M_S denotes the ideal in $S(E_0)$ generated by M then M_S is a bimodule for $U(A)$ in the obvious way. M_S is called the *standard imbedding* of M. *Irreducible* n-ary modules over A and *complete reducibility* are defined obviously. We then have further

PROPOSITION 2. *Suppose that k is a $n-1$ split field, and M a n-ary module over a semisimple n-ary k-algebra A, the dimensions finite. For the center Z of $U(A)$ let $Z \otimes_k Z$ be semisimple. Then M is completely reducible over A.*

Proof. Let $M \neq \{0\}$ and $W: = U(A) \otimes_k U(A)^{\mathrm{op}}$. Then M_S can be considered as an associative left module over W where for $a, b \in U(A)$ and $m \in M_S$ one has $(a \otimes b)m = a \circ m \circ b$. Now M_S has a two sided φ-invariant Peirce decomposition for the unit $e \in U(A)$. Let

$$M_S = (M_S)_{11} \oplus (M_S)_{10} \oplus (M_S)_{01} \oplus (M_S)_{00}$$

be the decomposition over e. Now observe [5, p. 117] and [1], Lemma 2]. Any Peirce component is completely reducible over W and over $U(A)$ from the right and from the left. Thus $M_S = \oplus_{i=1}^r N(i)$, $r \in N$, any $N(i)$ a φ-invariant minimal subbimodule not zero for $U(A)$. Hence $M = \oplus_{i=1}^r (N(i))_1$ is a decomposition into irreducible n-ary submodules. $\square$

4. The cohomology

Let B and B' be direct imbeddings of the n-ary k-algebras A and A', V a submodule of B and $f: V^{[r]} \to B'$ a multilinear map. Let $I: = \{1, \cdots, n-1\}$. Similarly as in [1] f is *isovarying* if $x: = (x_i)_{i=1}^r \in V^{[r]}$ with $x_i \in B_{j(i)}$ for $j(i) \in I$ and $j \in I$ with $j: = \sum_{i=1}^r j(i)$ modulo $(n-1)$ implies $f(x) \in B'_j$. f is called *antivarying* if $f(x) \in \oplus B'_p$ for $p \in I \backslash \{j\}$. Any multilinear map f has a unique decomposition $f = f_1 + f_0$, f_1 isovarying and f_0 antivarying. Let $\varepsilon: f \mapsto f_1$. Hence $\varepsilon^2: = \varepsilon \cdot \varepsilon = \varepsilon$.

We consider the Hochschild cohomology of $U(A)$ for M_S. The modules of the cochain complex are $C^0: = M_S$, and the k-modules C^q of the q-linear maps from $U(A)^{[q]}$ in M_S if $q \in N$. If $r \in N_0$ the coboundary operator $\delta: C^r \to C^{r+1}$ is defined for $a_i \in U(A)$ by

$$\delta f(a_1) = a_1 \cdot f - f \cdot a_1 \qquad \text{if } f \in C^0$$

$$\delta f((a_i)_{i=1}^{q+1}) = a_1 \cdot f((a_i)_{i=2}^{q+1}) + \sum_{i=1}^{q} (-1)^i f((b_{ij})_{j=1}^{q}) + (-1)^{q+1} f((a_i)_{i=1}^{q}) \cdot a_{q+1} \qquad \text{if } f \in C^q$$

with $b_{ij} := a_j$ if $j < i$, $b_{ii} := a_i \circ a_{i+1}$, and $b_{ij} := a_{j+1}$ if $i < j$. The k-modules of r-cocycles and r-coboundaries are denoted Z^r and B^r where $B^0 := \{0\}$, and $H^r(U(A), M_S) := Z^r/B^r$ is the r-dimensional cohomology group. Extend ε to C^0 by $\varepsilon := \varepsilon_{n-1}$. Any ε-invariant submodule T of C^r decomposes, $T = T_1 \oplus T_0$ with $T_i := \{f \in C^r \mid \varepsilon f = i \cdot f\}$ for $i \in \{0, 1\}$. Let $C_i^r := (C^r)_i$. Thus $C^r = C_1^r \oplus C_0^r$, and $\delta(C_i^r) \subset C_i^{r+1}$.

Set $C^r(A, M) := C_1^r$ for $r \in N_0$. Then the $C^r(A, M)$ together with the restriction ∂ of δ onto the C_1^r define a cohomology complex of A for M. The *r-cocycles* and *r-coboundaries* are $Z^r(A, M) := Z_1^r$ respectively $B^r(A, M) := B_1^r$. Denote the *r-dimensional cohomology group* by $H^r(A, M)$. Hence

LEMMA 3. *If A is a n-ary algebra over a ring k then for $r \in N_0$*

$$H^r(A, M) = Z_1^r/B_1^r\,, \quad \textit{and}$$
$$H^r(U(A), M_S) \cong H^r(A, M) \oplus Z_0^r/B_0^r\,.$$

COROLLARY 2. *Let k be a field,* $\operatorname{char}(k) \nmid n-1$, *$A$ separable and finite dimensional over k. Then for all $r \in N$ and all n-ary A-modules M*

$$H^r(A, M) = \{0\}\,.$$

Proof. By Theorem 1, (1) $U(A)$ is separable. □

A linear map $D: A \to M$ is called a *derivation* of A in M if

$$D(\langle a_i \rangle_{i=1}^n) = \sum_{i=1}^n \langle b_{ij} \rangle_{j=1}^n$$

with $b_{ij} := a_j$ if $j \neq i$ and $b_{ii} := D(a_i)$.

If $D(A, M)$ denotes the k-module of the derivations of A in M then $D(A, M) \cong Z^1(A, M)$, the isomorphism given by the restriction of $f \in Z^1(A, M)$ onto A, $f|_A$. *Inner derivations* are the restrictions of the elements in $B^1(A, M)$ onto A. Hence

COROLLARY 3. *If A and M are as in Corollary 2 then any derivation of A in M is inner.*

A short exact sequence of n-ary k-algebras M, E, A, the connecting maps e and p n-ary algebra morphisms,

$$\{0\} \longrightarrow M \xrightarrow{e} E \xrightarrow{p} A \longrightarrow \{0\}$$

is an *extension of A by M.* The extension is *singular* if ${}^{(1)}(e(M)) = \{0\}$ in E. The *equivalence* and *splitting* of extensions is defined in the obvious way.

Let $M(E)$ denote the ideal in $U(E)$ generated by $e(M)$, and $P := M(E) \cap J(E)$. Set $U'(E) := U(E)/P$.

For any singular extension there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & M & \xrightarrow{e} & E & \xrightarrow{p} & A & \longrightarrow & \{0\} \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\{0\} & \longrightarrow & M_S & \xrightarrow{e'} & U'(E) & \xrightarrow{p'} & U(A) & \longrightarrow & \{0\}
\end{array}$$

$\hookrightarrow$ the inclusion map and e', p' algebra morphisms invariant for ε (cf. [3], pp. 158–159). A k-module V is *k-projective* if any k-module extension of V is split. The tensor product of two projective k-modules is k-projective. Thus if A is k-projective there exist two k-linear sections $q: A \to E$ and $q': U(A) \to U'(E)$ with $\varepsilon q' = q'$ and $q'|_A = q$.

For a n-ary A-module M a *factor set* of A in M is a n-linear map $f: A^{[n]} \to M$ so that a n-ary algebra $E_f := A \oplus M$ is defined by

$$[x_i]_{i=1}^n := \langle x_i \rangle_{i=1}^n + f((x_i)_{i=1}^n) ,$$

together with ${}^{(1)}M := \{0\}$, and the A-module structure of M. f is *trivial* if there exists $\sigma \in \mathrm{Mor}_k(A, M)$ with

$$f((x_i)_{i=1}^n) = \sigma(\langle x_i \rangle_{i=1}^n) - \sum_{i=1}^n \langle b_{ij} \rangle_{j=1}^n$$

where $b_{ij} := x_j$ if $i \neq j$ and $b_{ii} := \sigma(x_i)$. Let $F(A, M)$ denote the k-module of factor sets of A in M, and $F_t(A, M)$ the submodule of trivial factor sets. Using an argument similar to the one in [3] we get

THEOREM 3. *Let A be a n-ary algebra and M a n-ary module over A, further A k-projective. Then*

$$H^2(A, M) \cong F(A, M)/F_t(A, M) .$$

There exists a 1–1 correspondence of an element in $H^2(A, M)$ to a class of equivalent singular extensions of A by M. The split extensions correspond to zero.

The proof of the statement for A with radical R analogous to the *Wedderburn Principal Theorem* reduces in the case of finite dimension over a field k to ${}^{(1)}R = \{0\}$. Hence by Corollary 2

COROLLARY 4. *Let k be a field, A a finite dimensional n-ary algebra over k, R the radical of A, A/R separable and* $\mathrm{char}(k) \nmid n-1$. *Then*

$$A = S \oplus R ,$$

S a semisimple n-ary subalgebra of A.

This corollary can also be derived directly generalizing an argument of Loos suggested for some ternary case: Since A/R is separable we may suppose that k is $n-1$ split. Then there is a Wedderburn decomposition

$$U(A) = S \oplus R(U(A)) ,$$

S a semisimple subalgebra, and S φ-invariant by [12]. Thus $A = S_1 \oplus R$. Obviously S is a semisimple direct imbedding of S_1. By Theorem 1 S_1 is semisimple.

Let D be a derivation of A in A, D nilpotent of index r, $r \in N$, and $\mathrm{char}(k) \nmid \prod_{j=1}^{2(r-1)} j$. Then $\alpha := \exp(D)$ is an automorphism of A.

For a subset B of a binary associative k-algebra C the *nilindex* t denotes $t \in N$, t minimal with $x^t = 0$ for all $x \in B$ if it exists. Define $\lambda_x, \rho_x \in \mathrm{End}(C)$ by $\lambda_x : y \mapsto xy$, $\rho_x : y \mapsto yx$, and $\delta(x) := \lambda_x - \rho_x$. One derives the statement analogous to the Theorem of Malcev-Harish-Chandra, using an argument similar to the one in Harish-Chandra's proof for Lie algebras (cf. [5]):

COROLLARY 5. *Let k be a field, the n-ary algebra A finite dimensional over k, R the radical of A, and A/R separable. Assume that W and T are semisimple subalgebras and $A = T \oplus R$. When R_s denotes the ideal in $S(A)$ generated by R, and t the nilindex of $\delta(R_s)_{n-1}$ for A suppose* $\mathrm{char}(k) \nmid (n-1) \prod_{j=1}^{2(t-1)} j$. *Then there is an automorphism α of A with $\alpha(W) \subset T$.*

5. Classification

We determine the simple n-ary algebras which are finite dimensional over an algebraically closed field k with $\mathrm{char}(k) \nmid n-1$. For $n = 2, 3$ these are wellknown. We generalize the arguments in [8] for $n = 3$.

Lemma 4. *Let k be a field, A a n-ary algebra over k.*

(1) *If $A = k(r_i)_{i=1}^{n-1}$, $r_i \in N$, then A is simple.*

(2) *Let $d, r \in N\backslash\{1\}$, and A a simple r-ary algebra. Then $A_{\tau(d)}$ is simple.*

Proof. (1) follows in the obvious manner using a standard basis.

For (2) assume that A is a simple r-ary algebra and I an ideal in $A_{\tau(d)}$. Let $z = \langle y_i \rangle_{i=1}^{r}$ with $y_i \in A$, and $y_{i_0} \in I$ for some subscript i_0. Choose y_i with $i \neq i_0$. Since $y_i \in A = A^{(r-1)(d-2)+1}$ trivially $z \in I$. Hence $I = \{0\}$, or $I = A$. □

The lemma proves one direction of

THEOREM 4. *Let k be an algebraically closed field,* $\operatorname{char}(k) \nmid n-1$, *and A a n-ary algebra over k. Then A is simple if and only if*

$$A \cong (k(r_i)_{i=1}^{r-1})_{\tau(d)}$$

with $r_i \in N$, and $r, d \in N\backslash\{1\}$ where $n = (r-1)\cdot(d-1)+1$.

Proof. If A is a simple n-ary algebra then by Theorem 1 $U(A)$ is simple or φ-simple. The latter is evident from the proof of that theorem. Assume that $U(A)$ is simple. Then $U(A)$ is isomorphic to a full matrix algebra $k(r)$ of $r \times r$ matrices in k, $r \in N$. Since any automorphism of $U(A)$ is inner then

$$\varphi\colon x \mapsto yxy^{-1}$$

for some $y \in U(A)$. Observing $\varphi^{n-1} = \mathrm{Id}_{U(A)}$, Schur's Lemma, and the Jordan normal form we may take y as a diagonal matrix. Further the diagonal of y consists of $n-1$ matrices $\xi^i E_{r_i}$ for $i = 1, \cdots, n-1$ and $r_i \in N$ where E_{r_i} denotes the unit matrix with r_i rows, as is shown easily. Hence $A \cong k(r_i)_{i=1}^{n-1}$.

Let now $U(A)$ be φ-simple but not simple. Obviously the restriction of the canonical projection π of $U(A)$ onto a simple ideal B in $U(A)$ maps A monomorphically into $B_{\tau(n)}$. Let $U(A) = \oplus_{i=0}^{q-1} \varphi^i(B)$ with $q > 1$. Then $q | n-1$. Set $r\colon = (n-1)/q + 1$. Further let

$$B_q\colon = \{x \in B | \varphi^q(x) = \xi^q x\} .$$

Obviously $B_q \neq \{0\}$. Moreover

$$U(A)_1 = \left\{x | x\colon = \sum_{i=0}^{q-1} \xi^{-i}\varphi^i(y), y \in B_q\right\} .$$

Now B_q is a subalgebra of $B_{\tau(r)}$, and B is a simple direct imbedding of the r-ary algebra B_q. For suitable $r_i \in N$ therefore $A \cong (k(r_i)_{i=1}^{r-1})_{\tau(q+1)}$. $\square$

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Mathematisches Seminar
Universität Hamburg
Bundesstraße 55
2 Hamburg 13