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ON THE COMPOSITION SERIES OF PRINCIPAL SERIES REPRESENTATIONS OF A THREE-FOLD COVERING GROUP OF $SL(2, K)$[1]

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Introduction

In this paper, we study the composition series of certain principal series representations of the three-fold metaplectic covering group of $SL(2, K)$, where K is a non-archimedean local field. These representations are parametrized by unramified characters $\mu(x)=|x|^s$ of $K^\times$, and characters ω of the group of third roots of unity. We study only the genuine representations corresponding to nontrivial ω parameter, as the case where $\omega = 1$ gives nothing but representations of $SL(2, K)$. We show that, outside the line $\mathrm{Re}\, s = 0$ (where the representations may decompose simply), the genuine principal series are irreducible except when $s = \pm 1/3$. We find the composition series at $s = \pm 1/3$, and obtain a unique quotient, r_ω, which is spherical.

The motivation for this study is a paper of Gelbart and Sally (cf. [4]) where it is proved that an irreducible component of the Weil representation appears as a quotient of the genuine principal series representation corresponding to $s = 1/2$ of the two-fold covering group of $SL(2, K)$; this is the only spherical quotient of the representations corresponding to $s = \pm 1/2$, and all other genuine principal series representations parametrized by nonunitary unramified characters are irreducible.

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1. Metaplectic group

We fix once and for all a non-archimedean local field, K, of

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characteristic zero containing the cube roots of unity. We denote by $\mathcal{O}$ the ring of integers of K, τ a fixed generator of the prime ideal $\mathcal{P}$ of $\mathcal{O}$, $\mathcal{O}^\times$ its group of units, and q the order of $\mathcal{O}/\mathcal{P}$. We shall assume, for convenience, that q is odd.

The three-fold metaplectic group is defined by a two-cocycle on $G = SL(2, K)$ which involves the cubic power residue symbol of K. (This construction is given by Kubota for n-fold metaplectic groups in [7]). We shall, therefore, list some properties of the cubic power residue symbol, $(\ ,\ )_3$, which will be frequently used.

1.1. PROPOSITION.
i) $(\ ,\ )_3$ *is bilinear.*
ii) $(a, b)_3 = (b, a)_3^{-1}$
iii) $(\ ,\ )_3$ *is identically* 1 *on* $\mathcal{O}^\times \times \mathcal{O}^\times$
iv) *If a is a cube in K, (a, b) is identically* 1.

For proofs and more information cf. [7], [1].

Now, suppose $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $SL(2, K)$. We set $x(g)$ equal to c or d according as c is non-zero or not. The following theorem is proved in [7].

1.2. THEOREM. *The map $\alpha: SL(2, K) \times SL(2, K) \to Z_3$ defined by*:

$$\alpha(g_1, g_2) = (x(g_1), x(g_2))_3(-x(g_1)^{-1}x(g_2), x(g_1g_2))_3$$

is a cohomologically non-trivial two-cocycle on $SL(2, K)$.

We thus get a covering group, G', of G by Z_3 which is central as a group extension. This is the three-fold metaplectic group. The group law in G' is given by

$$(g_1, \tau_1)(g_2, \tau_2) = (g_1g_2, \alpha(g_1, g_2)\tau_1\tau_2) .$$

We denote by B the upper triangular subgroup of G; A is the diagonal subgroup, and N the subgroup $\left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\right\}$. We set $M = SL(2, \mathcal{O})$. If H is any subgroup of G, we shall denote its inverse image in G' by H'.

The cocycle α is trivial on $M \times M$ and $N \times N$. Therefore, M and N are isomorphic to subgroups of G' which we shall also denote by M and N. As a notational convenience, we shall write a for the element $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ of A when the meaning is clear from the context. We can easily see that

$$\alpha(a, b) = (a, b^{-1})_3 \,.$$

It is also clear that α is trivial on $A_0 \times A_0$, where A_0 is the subgroup of the diagonal group consisting of elements with entries whose order is divisible by 3—the order of a nonzero element x in K is the unique integer $v(x)$ for which $x\tau^{-v(x)}$ is a unit. We therefore have $A_0' = A_0 \times \boldsymbol{Z}_3$.

2. Principal series representations of G'

Any irreducible representation of A_0' is clearly of the form $L_{\omega,\mu}$ with

$$L_{\omega,\mu}(a, \zeta) = \omega(\zeta)\mu(a)$$

where μ is a quasi-character of the multiplicative group $K^\times$ of nonzero elements in K, and ω is a character of $\boldsymbol{Z}_3$.

2.1. PROPOSITION. *All finite dimensional irreducible representations of A' are obtained by inducing $L_{\omega,\mu}$ from A_0'.*

Proof. Let $L_0 = L_{\omega,\mu}$ be an arbitrary representation of A_0', and $h' = (h, \eta)$ any element of A'. Since we have

$$h'(b, \zeta)h'^{-1} = (b, (h, b^{-1})_3^2\zeta)$$

L_0 and $L_0^{h'}$, its conjugation by h' are identical on A' if and only if $\omega((h, b^{-1})_3^2) = 1$ for all b in A_0. Hence the set

$$H = \{h' \in A' : L_0^{h'} = L\}$$

is either A' or A_0' depending on whether ω is trivial or not. So, from the theory of representations for groups with normal subgroups of finite index (cf. [3], Lemma 5.2), we can see that all finite dimensional representations of A' are obtained by inducing from A_0'.

We put $\sigma_{\omega,\mu} = \mathrm{Ind}\,(A_0', A', L_{\omega,\mu})$. $\sigma_{\omega,\mu}$ acts by right translations on the space of $\boldsymbol{C}$-valued functions f on A' satisfying

$$f(x_0', y') = L_{\omega,\mu}(x_0')f(y')$$

whenever x_0' is in A_0'. We now compute the action of A' explicitly. Since any (x, ζ) in A' can be uniquely decomposed as

$$(x, \zeta) = (x_0, \zeta(x_0, \tau^{i(x)})_3)(\tau^{i(x)}, 1) \tag{2.1}$$

where x_0 is in A_0 and $0 \le i(x) \le 2$, $\{(\tau^i, 1)\ i = 0, 1, 2\}$ is a set of representatives for A'/A_0'. We have

$$(x, \zeta_x)(a, \zeta) = (a, \zeta(a, x^2)_3)(x, \zeta_x),$$

$$\sigma_{\omega,\mu}(a, \zeta)f(\tau^i, 1) = \begin{cases} \mu(a_0)\omega((a_0, \tau^{2i+i(a)})_3\zeta)f(\tau^{i+i(a)}, 1) \\ \mu(\tau^3 a_0)\omega((a_0, \tau^{2i+i(a)})_3\zeta)f(\tau^{i+i(a)-3}, 1) \end{cases} \tag{2.2}$$

according as $i + i(a) \leq 2$ or not.

We extend $\sigma_{\omega,\mu}$ to B' which is the semi-direct product of A' and N, and then induce to G', and thus obtain the principal series representations of G'. We denote such a representation by $\rho_{\omega,\mu}$. It acts by right translations on the space $\phi_{\omega,\mu}$ of locally constant functions ϕ on $G' \times A'$ satisfying

(i) $\phi(g', a_0'a') = L_{\omega,\mu}(a_0')\phi(g', a')$ if $a_0' \in A_0'$
(ii) $\phi(b'g', a') = \delta(b')\phi(g', a'b')$ if $b' \in A'$

where $\delta(b')$ denotes the modulus of b if $b' = b(1, \zeta)$.

In the rest of this paper we shall restrict ourselves to the case of unramified characters of $K^\times$, so that $\mu(x) = |x|^s$ for a complex number s. Furthermore, if $\mu(x) = |x|^s$ and $\mu'(x) = |x|^{s'}$ where s and s' differ by an integer multiple of $2\pi i/3lnq$, then $L_{\omega,\mu}$ and $L_{\omega,\mu'}$ are equal on A_0'. We shall therefore restrict ourselves to the strip $-\pi/3lnq \leq \operatorname{Im} s \leq \pi/3lnq$. *Throughout this paper, we shall be referring to this strip when we say all complex numbers s.* Finally, we shall always assume ω to be nontrivial, and thereby consider only the genuine representations of G'.

An analogue of the Bruhat decomposition holds in G'; we have

$$G' = B' \cup B'(w, 1)N$$

where $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. We note that, we write g for the element $(g, 1)$ of G' when the meaning is clear from the context.

It follows from the above decomposition that all ϕ in $\phi_{\omega,\mu}$ are determined by their values on N and w. Hence, putting $f(x, a')$ equal to $\phi(w^{-1}n(x), a')$ with $n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ gives rise to a realization of $\rho_{\omega,\mu}$ on the space $F_{\omega,\mu}$ of locally constant functions on $K \times A'$ satisfying

(2.3)
(i) $f(x, a_0'a') = L_{\omega,\mu}(a_0')f(x, a')$ if $a_0' \in A_0'$
(ii) $|x|\sigma_{\omega,\mu}(x, 1)f(x, a')$ is constant for large $|x|$.

We fix a character χ of K once and for all. We assume, for convenience, that the conductor of χ is $\mathcal{O}$. For a function f in $F_{\omega,\mu}$ we define

$$\mathscr{F}f(x, a') = \sum_{n\in Z}\int_{v(y)=n} f(y, a')\chi(yx)dy$$

where dy is a fixed Haar measure on K normalized so that $\mathcal{O}$ has measure 1. This series converges uniformly on compact subsets of $K^\times$. (cf. [5], Lemma 9; essentially the same proof works here as we have $|f(x, a')| \sim |x|^{-1}|\mu(x)|^{-1}$ for large $|x|$.) $\mathscr{F}f$ will be called the Fourier transform of f, and sometimes be denoted by f^*. Moreover, for each fixed $a', f(x, a')$ is a square integrable function when $\mathrm{Re}\, s > -1/2$; in this case the Fourier transform of f in the L^2 sense coincides with $\mathscr{F}$.

From distribution theory, it can be seen that the kernel of the map $\mathscr{F}$ contains only functions which are constant on $K\times\{a'\}$ for each a' in A'. However, the only such function satisfying condition (ii) of (2.3) is zero. Hence, $\mathscr{F}$ maps $F_{\omega,\mu}$ injectively onto a space $\mathscr{H}_{\omega,\mu}$. We shall characterize this space only for certain μ and this will be done in § 5. We shall denote the realization of $\rho_{\omega,\mu}$ on $\mathscr{H}_{\omega,\mu}$ by $\rho^*_{\omega,\mu}$.

3. Intertwining operators

We shall fix some notation first: Let K_i denote the set of elements of K whose order is equal to i modulo 3, and ψ_i the characteristic function of K_i. $\mathscr{S}(K)$ (resp. $\mathscr{S}(K^\times)$) will denote the Schwartz-Bruhat space of K (resp. $K^\times$); i.e., the space of locally constant functions whose support is compact in K (resp. $K^\times$). We let $d^\times x$ be the Haar measure on K^* given by $\dfrac{dx}{|x|}$.

3.1. Lemma. *For any Φ in $\mathscr{S}(K)$, complex number s with $0<\mathrm{Re}\, s<1$, and $j=1,2$ we have*

$$\int_{K_i}\Phi(x)|x|^s\omega((x,\tau^j)_3)d^\times x = c_jq^{s-1/2}\int_{K_{2i-1}}\Phi^*(x)|x|^{1-s}\omega((x,\tau^{-j})_3)d^\times x$$

where Φ^* is the Fourier transform of Φ, and c_j are complex numbers of modulus 1 with $c_1c_2=1$.

Proof. We fix a unit D in K so that $(D,\tau)_3$ is a primitive cube root of 1. Then $(D,x)_3$ is a primitive root unless $v(x)\equiv 0 \bmod 3$. We can therefore write the characteristic function of K_i as

$$\psi_i(x) = 1/3 \sum_{l=0}^{2} (D, x\tau^{-i})_3^l .$$

Furthermore, since the character $(D, x)_3$ is unitary and unramified, it is of the form $|x|^d$ for some complex number d with $\mathrm{Re}\, d = 0$. We can now write the left hand side of the equality in the proposition as

$$(1/3) \sum_{l=0}^{2} q^{ldi} \int_K \Phi(x)\, |x|^{s+ld}\, \omega((x, \tau^j)_3) d^\times x .$$

Applying Tate's functional equation to each term and recalling that we have

$$\Gamma(|\cdot|^s \omega((\cdot, \tau^j)_3)) = c_j q^{s-1/2}$$

where Γ is the p-adic gamma function and c_j are complex numbers of modulus 1 such that $c_1 c_2 = 1$ (cf. [9], Theorem 1), the sum becomes

$$(1/3) c_j q^{s-1/2} \sum_{l=0}^{2} q^{ldi} \cdot q^{ld} \int_K \Phi^*(x)\, |x|^{1-s-ld}\, \omega((x, \tau^{-j})_3) d^\times x.$$

Observing that we have

$$(1/3) \sum_{l=0}^{2} q^{ldi} \cdot q^{ld} |x|^{-ld} = (1/3) \sum_{l=0}^{2} (D, x^{-1}\tau^{-i-1})_3 = \psi_{2i-1}(x)$$

we prove the proposition.

For the case $j = 0$ we have the following.

3.2. Lemma. *For any Φ in $\mathscr{S}(K)$, complex number s with $0 < \mathrm{Re}\, s < 1$, we have*

$$\begin{aligned}\int_{K_i} \Phi(x)\, |x|^s\, d^\times x = {} & \frac{1 - q^{-1}}{1 - q^{-3s}} \int_{K_{2i}} \Phi^*(x)\, |x|^{1-s}\, d^\times x \\ & + \frac{q^s(q^{-3s} - q^{-1})}{1 - q^{-3s}} \int_{K_{2i-1}} \Phi^*(x)\, |x|^{1-s}\, d^\times x \\ & + \frac{q^{-s}(1 - q^{-1})}{1 - q^{-3s}} \int_{K_{2i-2}} \Phi^*(x)\, |x|^{1-s}\, d^\times x .\end{aligned}$$

Proof. The left hand side is equal to

$$(1/3) \sum_{l=0}^{2} \mathrm{q}^{ldi} \int_K \Phi(x)\, |x|^{s+ld}\, d^\times x .$$

By Theorem 1 of [9], $\Gamma(|\cdot|^s) = (1 - q^{s-1})/(1 - q^{-s})$. Applying the functional equation of Tate, we see that the above expression is equal to

$$(1/3)\int_K \Phi^*(x)|x|^{1-s}\left[\frac{1-q^{s-1}}{1-q^{-s}} + (D, x^2\tau^{-i})_3\left(\frac{1-q^{s+d-1}}{1-q^{-d-s}}\right)\right.$$
$$\left. + (D, x\tau^{-2i})_3\left(\frac{1-q^{s+2d-1}}{1-q^{-2d-s}}\right)\right]d^\times x\,.$$

We compute the expression in brackets. We factor out $1-q^{-3s}$, the product of the three denominators; this leaves an expression with a q^0 term coefficient of $3\psi_{2i}(x)$, a q^{-s} term coefficient of $3\psi_{2i-2}(x)$, a q^{-2s} term coefficient of $3\psi_{2i-1}(x)$, a q^{s-1} term coefficient of $-3\psi_{2i-1}(x)$, a q^{-1} term coefficient of $-3\psi_{2i-2}(x)$. Therefore, the integral is

$$\frac{1}{1-q^{-3s}}\int_K \Phi^*(x)|x|^{1-s}[(1-q^{-1})\psi_{2i}(x) + q^s(q^{-3s}-q^{-1})\psi_{2i-1}(x)$$
$$+ q^{-s}(1-q^{-1})\psi_{2i-2}(x)]d^\times x\,.$$

This completes the proof.

For an element ϕ of $\phi_{\omega,\mu}$ we put

$$I\phi(g', a') = \int_K \phi\left(w\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} g', wa'\,w^{-1}\right)dx\,.$$

The integral converges for $\operatorname{Re} s > 0$ since

$$|\phi(wn(x)g', wa'w^{-1})| \approx |\mu(x)|^{-1}|x|^{-1}\,.$$

It is easy to see that I_ϕ is in $\phi_{\omega,\mu^{-1}}$ and that I commutes with right translations; I intertwines $\rho_{\omega,\mu}$ and $\rho_{\omega,\mu^{-1}}$. Furthermore, if $\phi\in\phi_{\omega,\mu}$ and $\phi'\in\phi_{\bar\omega,\mu^{-1}}$ then $\phi\cdot\phi'$ is invariant under left translations of the second variable by elements of A'_0. The function

$$g' \mapsto \int_{A'_0\backslash A'} \phi\cdot\phi'(g', a')da'$$

is in the space $L(G', B')$ of locally constant functions Φ satisfying

$$\Phi\left(\left(\begin{bmatrix}a & *\\ 0 & a^{-1}\end{bmatrix}, \zeta\right)g'\right) = |a|^2\,\Phi(g')\,.$$

If we denote the essentially unique positive linear form on $L(G', B')$ by

$$\Phi \mapsto \int_{B'\backslash G'} \Phi(g')dg'$$

then

$$\langle\phi, \phi'\rangle = \int_{B'\backslash G'}\int_{A'_0\backslash A'} \phi(g', a')\phi'(g', a')da'dg'$$

gives a non-degenerate bilinear form on $\phi_{\omega,\mu} \times \phi_{\bar{\omega},\mu^{-1}}$. Thus it follows that $\rho_{\bar{\omega},\mu^{-1}}$ is the contragradient representation of $\rho_{\omega,\mu}$. (cf. [5], p. 1.18). By well-known techniques, the above integral can be written as

$$\langle \phi, \phi' \rangle = \int_K \int_{A'_0 \backslash A'} \phi(w^{-1}n(x), a')\phi'(w^{-1}n(x), a')da'dx .$$

We shall now restrict ourselves to the case of real s with $0 < s < 1$. In this case the complex conjugate of $I\phi$ is in $\phi_{\bar{\omega},\mu^{-1}}$ if ϕ is in $\phi_{\omega,\mu}$. Thus, the following is an invariant bilinear form on $\phi_{\omega,\mu} \times \phi_{\omega,\mu}$.

$$\begin{aligned}
&\int_K \int_{A'_0 \backslash A'} \phi_1(w^{-1}n(x), a')\overline{I\phi_2(w^{-1}n(x), a')}da'dx \\
&\quad = \int_K \int_{A'_0 \backslash A'} f_1(x, a') \int_K \overline{\phi_2(wn(y)w^{-1}n(x), wa'w^{-1})}dyda'dx \\
&\quad = \int_K \int_{A'_0 \backslash A'} f_1(x, a') \int_K \overline{\phi_2}\left(\begin{bmatrix} y^{-1} & 1 \\ 0 & y \end{bmatrix}\overline{w^{-1}n(x+y^{-1}), wa'w^{-1}}\right)dyda'dx .
\end{aligned}$$

We note that the arguments of ϕ_2 in the last two expressions are only equal up to a central element of G'; the difference is absorbed by the integration over $A'_0\backslash A'$. We write the integral in the following form:

$$\int_K \int_{A'_0 \backslash A'} f_1(x, a') \int_K \overline{\sigma_{\omega,\mu}(y^{-1}, 1), f_2(x+y^{-1}, wa'w^{-1})}d^\times y \, da'dx .$$

By using the set of representatives $\{\tau^i : i = 0, 1, 2\}$ of $A'_0\backslash A'$, this invariant bilinear form becomes

$$\begin{aligned}
&\int f_1(x, 1) \int \overline{\sigma_{\omega,\mu}(y, 1)f_2(x+y, 1)}d^\times y \, dx \\
&\quad + \int f_1(x, \tau) \int \overline{\sigma_{\omega,\mu}(y, 1)f_2(x+y, \tau^{-1})}d^\times ydx \\
&\quad + \int f_1(x, \tau^2) \int \overline{\sigma_{\omega,\mu}(y, 1)f_2(x+y, \tau^{-2})}d^\times ydx.
\end{aligned}$$

By (2.2) this expression is equal to

$$\begin{aligned}
&\int f_1(x, 1) \int \psi_0(y)\,|y|^s\, \overline{f_2(x+y, 1)}d^\times ydx \\
&\quad + \int f_1(x, 1) \int \psi_1(y)\,|\tau^{-1}y|^s\, \overline{\omega((y, \tau)_3)f_2(x+y, \tau)}\, d^\times ydx \\
&\quad + \int f_1(x, 1) \int \psi_2(y)\,|\tau^{-2}y|^s\, \overline{\omega((y, \tau^2)_3)f_2(x+y, \tau^2)}d^\times y \, dx \\
&\quad + \int f_1(x, \tau) \int \psi_0(y)\,|\tau^{-3}y|^s\, \overline{\omega((y, \tau)_3)f_2(x+y, \tau^2)}d^\times y \, dx
\end{aligned}$$

$$+\int f_1(x,\tau)\int \psi_1(y)|\tau^{-1}y|^s\,\overline{\omega((y,\tau^2)_3)f_2(x+y,1)}\,d^\times y dx$$
$$+\int f_1(x,\tau)\int \psi_2(y)|\tau^{-2}y|^s\,\overline{f_2(x+y,\tau)}\,d^\times y dx$$
$$+\int f_1(x,\tau^2)\int \psi_0(y)|\tau^{-3}y|^s\,\overline{\omega((y,\tau^2)_3)f_2(x+y,\tau)}\,d^\times y dx$$
$$+\int f_1(x,\tau^2)\int \psi_1(y)|\tau^{-4}y|^s\,\overline{f_2(x+y,\tau^2)}\,d^\times y dx$$
$$+\int f_1(x,\tau^2)\int \psi_2(y)|\tau^{-2}y|^s\,\overline{\omega((y,\tau)_3)f_2(x+y,1)}\,d^\times y dx\,.$$

We now assume that f has compact support as a function of x for each a'. Then each term of the above sum can be thought of (by Fubini's theorem) as having the form of the expressions in Lemmas 3.1 and 3.2 where Φ is the convolution of f_1^v and f_2. (f_1^v is the translate by -1 of f_1). By these lemmas, we therefore write the invariant bilinear form as follows, if we write $P(r,t,v)$ for the sum of $(1-q^{-1})r$, $q^{-s}(1-q^{-1})t$ and $q^s(q^{-3s}-q^{-1})v$:

$$\int f_1^*(y,1)\overline{f_2^*(y,1)|y|^{1-s}(1/(1-q^{-3s}))P(\psi_0(y),\psi_1(y),\psi_2(y))}\,d^\times y$$
$$+\int f_1^*(y,1)\overline{f_2^*(y,\tau)|y|^{1-s}c_1q^{2s-1/2}\omega((y,\tau^2)_3)\psi_1(y)}\,d^\times y$$
$$+\int f_1^*(y,1)\overline{f_2^*(y,\tau^2)|y|^{1-s}c_2q^{3s-1/2}\omega((y,\tau)_3)\psi_0(y)}\,d^\times y$$
$$+\int f_1^*(y,\tau)\overline{f_2^*(y,\tau^2)|y|^{1-s}c_1q^{4s-1/2}\omega((y,\tau^2)_3)\psi_2(y)}\,d^\times y$$
$$(3.1)\qquad +\int f_1^*(y,\tau)\overline{f_2^*(y,1)|y|^{1-s}c_2q^{2s-1/2}\omega((y,\tau)_3)\psi_1(y)}\,d^\times y$$
$$+\int f_1^*(y,\tau)\overline{f_2^*(y,\tau)|y|^{1-s}q^{2s}(1-q^{-3s})^{-1}P(\psi_1(y),\psi_2(y),\psi_0(y))}\,d^\times y$$
$$+\int f_1^*(y,\tau^2)\overline{f_2^*(y,\tau)|y|^{1-s}c_2q^{4s-1/2}\omega((y,\tau)_3)\psi_2(y)}\,d^\times y.$$
$$+\int f_1^*(y,\tau^2)\overline{f_2^*(y,\tau^2)|y|^{1-s}q^{4s}(1-q^{-3s})^{-1}P(\psi_2(y),\psi_0(y),\psi_1(y)}\,d^\times y$$
$$+\int f_1^*(y,\tau^2)\overline{f_2^*(y,1)|y|^{1-s}c_1q^{3s-1/2}\omega((y,\tau^2)_3)\psi_0(y)}\,d^\times y\,.$$

By taking a suitable sequence of functions in $F_{\omega,\mu}$ which are compactly supported in their first variable for each a' we can easily see that the above is valid for any f_1 in $F_{\omega,\mu}$.

We can think of the expression (3.1) in the form

$$\int_K \int_{A'_0 \backslash A'} f_1^*(y, a') \overline{J f_2^*(y, a')}\, d^\times y da' \tag{3.2}$$

for some linear map J defined on $\mathscr{H}_{\omega,\mu}$. For any operator T let us denote by T_c the operator $f \mapsto \overline{T\bar{f}}$. Then (3.2) gives an invariant non-degenerate bilinear form on $\mathscr{H}_{\omega,\mu} \times$ Image of J_c. (J is not 0). Thus the image of J_c can be identified with a subspace of the contragradient of $\mathscr{H}_{\omega,\mu}$ i.e.,. $\mathscr{H}_{\bar{\omega},\mu^{-1}}$. If we denote by I^* the intertwining operator obtained by carrying I from the $\phi_{\omega,\mu}$ model to the $\mathscr{H}_{\omega,\mu}$ model, then it is clear that

$$\langle f^*, J_c g^* \rangle = \langle f^*, I_c^* g^* \rangle .$$

Thus $J = I^*$ for $0 < s < 1$.

We now write J in the matrix form by considering f^* to be a vector valued function on $K^\times$; we put $f^*(x)$ equal to

$$(f^*(x, 1), f^*(x, \tau), f^*(x, \tau^2))$$

in C^3—this vector determines $f^*(x, a')$ for all a'. We then have

$$J(x) = |x|^{1-s} \begin{bmatrix} (1 - q^{-3s})^{-1} P(\psi_0, \psi_1, \psi_2) & c_1 q^{2s-1/2} \omega^2 \psi_1 & c_2 q^{3s-1/2} \omega \psi_0 \\ c_2 q^{2s-1/2} \omega \psi_1 & (1 - q^{-3s})^{-1} q^{2s} P(\psi_1, \psi_2, \psi_0) & c_1 q^{4s-1/2} \omega^2 \psi_2 \\ c_1 q^{3s-1/2} \omega^2 \psi_0 & c_2 q^{4s-1/2} \omega \psi_2 & q^{4s}(1 - q^{-3s})^{-1} P(\psi_2, \psi_0, \psi_1) \end{bmatrix}$$

where we write ψ_i (resp. ω^j) instead of $\psi_i(x)$ (resp. $\omega((x, \tau^j)_3)$). We shall sometimes write $J_{\omega,s}$, to emphasize dependence on ω and s.

3.1. PROPOSITION. *The operator J is defined and is equal to I^* on the whole right half-plane* $\{s: \mathrm{Re}\,(s) > 0\}$.

Proof. For $i = 0, 1, 2$, we let F_i be a function from $\mathscr{S}(K^\times)$, and put $f^*(x, \tau^i) = F_i(x)$, and $f(x, \tau^i) = F_i^*(x)$. For each μ we can extend f to a function f_μ so that f_μ is in $F_{\omega,\mu}$. Then

$$\mathscr{F} f_\mu(x, \tau^i) = f^*(x, \tau^i)$$

for $i = 0, 1, 2$. Since $J = I^*$ on the interval $(0, 1)$ we have

$$J_{\omega,s} f^*(x, \tau^i) = I_{\omega,s}^* f^*(x, \tau^i)$$

for $i = 0, 1, 2$ when $0 < s < 1$. (Note that the values of f^* in question are independent of s.) Thus, from the principal of analytic continuation and the fact that every function in $F_{\omega,\mu}$ is the pointwise limit of a sequence of functions of the form f_μ, the proposition follows.

4. Composition series of $\rho^*_{\omega,\mu}$ for $\operatorname{Re} s > 0$

We start with an analogue of a theorem for p-adic reductive groups. A simple proof of this theorem for the semi-simple rank 1 case is in [2], pp. 3–4; this proof works verbatim in the case of G'. We therefore omit the proof.

4.1. THEOREM. *The length of $\rho^*_{\omega,\mu}$ is at most* 2.

Consequently, to determine the composition series of $\rho^*_{\omega,\mu}$, we only need the following theorem.

4.2. THEOREM. *The image of $J_{\omega,s}$ is irreducible for all s with* $\operatorname{Re} s > 0$.

This is a theorem of Langlands whose proof for the case of real reductive groups is contained in [8]. We include here a slight modification of Langlands' proof for the sake of completeness. We first need the following.

4.3. LEMMA. *Let x be in $K^\times$, ϕ in $\phi_{\omega,\mu}$ and ϕ' in $\phi_{\bar{\omega},\mu^{-1}}$ with s a real number. If we put*

$$F(x) = \langle \rho_{\omega,\mu}(x^3, 1)\phi, \phi' \rangle$$

then as $|x|$ approaches ∞, we have

$$F(x) \sim |x|^{3(s-1)} \int_{A_0'\backslash A'} I\phi(w, wa'w^{-1})\phi'(e, a')da'$$

where e is the identity element of G'.

Proof. We write $F(x)$ in the form

$$F(x) = \int_{A_0'\backslash A'} \int_{N_1} \rho_{\omega,\mu}(x^3, 1)\phi'(n_1, a')\phi'(n_1, a')dn_1 da'$$

where $N_1 = w^{-1}Nw$. By the "Iwasawa decomposition", $G' = B'M$, we can write n_1 as $n(t, \zeta)k$. We also put

$$(x^{-3}, 1)n_1(x^3, 1) = n_x(t_x, \zeta_x)k_x$$

so that

$$kx^3 = (t, \zeta)^{-1}n^{-1}x^3 n_x(t_x, \zeta_x)k_x \,.$$

Substituting in $F(x)$ first the expression for n_1, and then the one for kx^3, we get

$$\rho_{\omega,\mu}(x^3, 1)\phi(n_1, a') = |x|^3 |t_x| \sigma_{\omega,\mu}(x^3(t_x, \zeta_x))\phi(k_x, a')$$

and

$$\phi'(n_1, a') = |t| \sigma_{\bar{\omega},\mu^{-1}}(t, \zeta)\phi'(k, a') .$$

Now we change variables by putting $n' = x^{-3}n_1x^3$. Observing that $k = (t, \zeta)^{-1}n^{-1}x^3n'x^{-3}$, and that $x^3n'x^{-3}$ approaches e as $|x|$ approaches ∞, we find that

$$F(x) \sim |x|^{3(s-1)} \int_{A_0'\backslash A'} \int_{N_1} \phi(n_1, a')dn_1\phi'(e, a')da' .$$

We leave it to the reader to prove that one can interchange the integral and the limit as we just did. (cf. [8]). This completes the proof of the lemma.

Proof of the Theorem. Suppose V_1 is the kernel of I and V_2 is a proper invariant subspace of $\phi_{\omega,\mu}$ containing V_1. It clearly suffices to prove that any such V_2 is contained in V_1.

Pick a non-zero element ϕ_0' in $\phi_{\bar{\omega},\mu^{-1}}$ such that $\langle \phi, \phi_0' \rangle = 0$ for all ϕ in V_2. Fix an element ϕ_2 of V_2. We have

$$\langle \rho_{\omega,\mu}(g')\phi_2, \phi_0' \rangle = 0$$

for all g' in G'. Putting $g' = x^3$ for x in $K^\times$, and using Lemma 4.3, we get

$$\int_{A_0'\backslash A'} I\phi_2(w, wa'w^{-1})\phi_0'(e, a')da' = 0 .$$

As this equality holds for $\rho_{\omega,\mu}(g')\phi_2$ instead of ϕ_2 for any g', we must have $I\phi_2 = 0$, which proves the theorem.

As a consequence of this, we have the following theorem.

4.4. THEOREM. *The representations $\rho^*_{\omega,\mu}$ are irreducible for* $\operatorname{Re} s \neq 0$ *except when $s = \pm 1/3$. If r_ω denotes the representation of G' obtained by restricting $\rho^*_{\omega,-1/3}$ to the image of $J_{\omega,1/3}$, then*

$$0 \subsetneqq r_\omega \subsetneqq \rho^*_{\omega,-1/3}$$

is the composition series of $\rho^*_{\omega,-1/3}$.

Proof. It can be seen from (3.1) that for $\operatorname{Re} s > 0$ we have

$$\det J_{\omega,s} = \frac{(1-q^{3s-1})^2(q^{3s}-q^{-1})}{(1-q^{-3s})^3}|x|^{3(1-s)} .$$

The kernel of $J_{\omega,s}$ is therefore trivial for $\operatorname{Re} s > 0$ except at $s = 1/3$. The theorem now follows from Theorems 4.1, 4.2 and the equivalence of $\rho^*_{\omega,\mu}$ and $\rho^*_{\omega,\mu^{-1}}$.

Let us denote by π_ω the representation obtained by restricting $\rho^*_{\omega,1/3}$ to the kernel of $J_{\omega,1/3}$. We shall devote the rest of this section to proving that r_ω and π_ω are inequivalent representations, neither of which is equivalent to an irreducible $\rho^*_{\omega,\mu}$.

4.5. PROPOSITION. *The representations $\rho^*_{\omega,\mu}$ and r_ω are spherical; i.e., they contain a nontrivial subspace fixed by M. π_ω is not spherical.*

Proof. We shall consider the $\rho_{\omega,\mu}$ realization. By the Iwasawa decomposition, there exists an element ϕ_0 in $\phi_{\omega,\mu}$ fixed by M if and only if there is a function Φ_0 on A' with the properties

(i) $\Phi_0(a'_0a') = L_{\omega,\mu}(a'_0)\phi_0(a')$ for $a'_0 \in A'_0$

(ii) $\Phi_0(a'b') = \Phi_0(a')$ for $b' \in A' \cap M$.

If $a' = (a, \zeta)$, $b' = (u, 1)$ with a unit element u, then $a'b' = (u, (u, a^2)_3)(a, \zeta)$. Therefore, the second condition is met if and only if $\omega((u, a^2)_3) = 1$ for all units u, whenever $\Phi_0(a')$ is nonzero. Thus, it is necessary that we have $\Phi_0(\tau) = \Phi_0(\tau^2) = 0$. Any such Φ_0 that also satisfies (i) will give a function ϕ_0 in $\phi_{\omega,\mu}$ which is fixed by M by putting $\phi_0(a'k, b')$ equal to $\Phi_0(a'b')$.

As the subspace fixed by M is thus shown to be one-dimensional, to complete the proof of the proposition it suffices to prove that the function ϕ_0 in $\phi_{\omega,1/3}$ is not in the kernel of I. But

$$\begin{aligned} I\phi_0(1,1) &= \int_K \phi_0(wn(x), 1)dx \\ &= \phi_0(1,1)\int_{|x|\le 1} dx + \int_{|x|>1} \phi_0(wn(x), 1)dx, \end{aligned}$$

and for $|x| > 1$ we have

$$wn(x) = \begin{bmatrix} x^{-1} & 0 \\ 0 & x \end{bmatrix} n(y)k$$

for some element y and element k of M. Hence the second integral is

$$\int_{|x|>1} \phi_0(1, x^{-1})d^\times x = \int_{|x|>1} \Phi_0(x^{-1})d^\times x .$$

However, since Φ_0 vanishes outside A'_0, this integral becomes

$$\phi_0(1, 1)\int_{|x|>1} |x^{-1}|^{1/3}\, \psi_0(x)d^\times x = \phi_0(1, 1)(1 - q^{-1})\sum_{n=1}^{\infty} q^{-n} = q^{-1}\phi_0(1, 1) .$$

So $I\phi_0$ takes on the value $\phi_0(1, 1)(1 + q^{-1})$, and therefore is not zero.

This proposition already proves that no irreducible $\rho^*_{\omega,\mu}$ or r_ω is equivalent to π_ω. We now want to show that r_ω is not equivalent to any irreducible $\rho^*_{\omega,\mu}$.

We consider the Iwahori subgroup

$$B_0 = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M\colon\ c \equiv 0 \bmod \mathscr{P}\right\}$$

and compute the subspace $V_{\omega,\mu}(B_0)$ of $\phi_{\omega,\mu}$ fixed under B_0. G' can clearly be written as the disjoint union of $B'B_0$ and $B'wB_0$. The elements of $V_{\omega,\mu}(B_0)$ vanishing on $B'wB_0$ are of the form

(4.1) $$\phi(b'b_0, a') = \delta(b')\phi(1, a'b')$$

where $\phi(1, a')$ is a function on A' satisfying

(4.2) $$\phi(1, a'_0a') = L_{\omega,\mu}(a'_0)\phi(1, a'_0a') \qquad \text{if } a'_0 \in A'_0$$

(4.1) and (4.2) give a well defined function if and only if

$$\delta(b')\phi(1, a'b') = \phi(1, a')$$

for all b' in $B' \cap B_0$. As in the proof of the last proposition, we see that $\phi(1, \tau) = \phi(1, \tau^2) = 0$. Therefore, the subspace of functions in $V_{\omega,\mu}(B_0)$ vanishing on $B'wB_0$ is one dimensional.

We proceed similarly to study the elements of $V_{\omega,\mu}(B_0)$ vanishing on $B'B_0$. They must be given by (4.2) and

(4.3) $$\phi(b'wb_0, a') = \delta(b')\phi(1, a'b') .$$

It is then necessary that

$$\delta(b')\phi(1, a'b') = \phi(1, a')$$

whenever b' is in $wB_0w^{-1} \cap B'$; i.e., for $b' = (u, \zeta)$ with a unit u. Hence, by (4.2) $\phi(1, \tau) = \phi(1, \tau^2) = 0$.

We have proved that $V_{\omega,\mu}(B_0)$ is a two-dimensional subspace with a basis consisting of the two functions ϕ_1, ϕ_2 given as follows: ϕ_1 vanishes on $B'wB_0$ and

$$\phi_1(b'b_0, a') = \begin{cases} \delta(b')L_{\omega,\mu}(a'b') \\ 0 \end{cases}$$

according as $a'b'$ is in A_0' or not; ϕ_2 vanishes on $B'B_0$ and

$$\phi_2(b'wb_0, a') = \begin{cases} \delta(b')L_{\omega,\mu}(a'b') \\ 0 \end{cases}$$

according as $a'b'$ is in A_0' or not.

We shall now consider the B_0 fixed elements of π_ω and r_ω. We shall, therefore, first compute $I_{\omega,1/3}\phi_1$ and $I_{\omega,1/3}\phi_2$. It suffices to compute their values at $(1, 1)$ and $(w, 1)$ by B_0 invariance.

$$I\phi_1(1, 1) = \int_{|x|\leq 1} \phi_1(wn(x), 1)dx + \int_{|x|>1} \phi_1(wn(x), 1)dx .$$

The first integrand is 0. In the second integral we write

$$wn(x) = \begin{bmatrix} x^{-1} & -1 \\ 0 & x \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -x^{-1} & 1 \end{bmatrix} .$$

Thus,

$$I\phi_1(1, 1) = \int_{|x|>1} |x|^{-1} \phi_1(1, x^{-1})dx$$

where the integrand is $|x|^{-4/3} \psi_0(x)$; we get q^{-1}.

Also,

$$I\phi_1(w, 1) = \int_{|x|<1} \phi_1(wn(x)w, 1)dx + \int_{|x|\geq 1} \phi_1(wn(x)w, 1)dx .$$

In the first integral we have $\phi_1\left(\begin{bmatrix} -1 & 0 \\ x & -1 \end{bmatrix}, 1\right)$ which is 1. The second integrand is 0 since

$$wn(x)w = \begin{bmatrix} -x^{-1} & 1 \\ 0 & -x \end{bmatrix} wn(-x^{-1}) .$$

Therefore $I\phi_1(w, 1) = q^{-1}$.

In exactly the same manner we compute $I\phi_2$ and get $I\phi_2(1, 1) = I\phi_2(w, 1) = 1$. We thus see that π_ω contains a one-dimensional subspace fixed under B_0; it is generated by $\phi_2 - q\phi_1$. Therefore the B_0-fixed subspace of r_ω is also one-dimensional. This, along with Proposition 4.5 proves the following theorem.

4.6. THEOREM. *No two representations in the collection consisting of irreducible* $\rho^*_{\omega,\mu}$, r_ω *and* π_ω *are equivalent.*

5. The representation r_ω

In this section we shall study the irreducible representation r_ω more closely, and obtain a more explicit description.

We start by computing $\mathscr{H}_{\omega,\mu}$ for $\mu(x) = |x|^{1/3}$. We recall that this space consists of Fourier transforms of functions in $F_{\omega,\mu}$. $F_{\omega,\mu}$ is the direct sum of $\mathscr{S}_{\omega,\mu}$, which is the subspace of functions vanishing for large $|x|$, and the subspace generated by the function $g(x, a')$ given by

$$g(x, a') = \begin{cases} |x|^{-1}\sigma_{\omega,\mu}(x^{-1}, 1)G(a') \\ 0 \end{cases}$$

according as $|x| \geq 1$ or not, where G is a function on A' satisfying

$$G(a_0'a') = L_{\omega,\mu}(a_0')G(a') . \tag{5.1}$$

Thus $\mathscr{H}_{\omega,\mu}$ is the direct sum of $\mathscr{S}_{\omega,\mu}$ and the space generated by g^*. We shall now compute g^*; it suffices to compute its values when a' is 1, τ and τ^2. We have

$$g^*(y, 1) = \sum_{n=0}^{\infty} \int_{v(x)=n} G(x)\chi(x^{-1}y)d^\times x .$$

We break the sum into three parts, $\Sigma^0, \Sigma^1, \Sigma^2$ where Σ^i indicates that summation is to be carried out over those nonnegative integers which are equal to i modulo 3. We observe that by (5.1), $G(x)$ is nothing but $\mu(x)G(1)$ when x is in K_0. When x is in K_1 we write $(x, 1)$ in the form $(x\tau^{-1}, (x, \tau)_3)(\tau, 1)$ so that $G(x) = \mu(x\tau^{-1})\omega((x, \tau)_3)G(\tau)$; when x is in K_2, we find similarly that

$$G(x) = \mu(x\tau^{-2})\omega((x, \tau^2)_3)G(\tau^2) .$$

We thus have

$$\begin{aligned} g^*(y, 1) &= \Sigma^0 G(1) \int_{v(x)=n} \mu(x)\chi(x^{-1}y)d^\times x \\ &= \Sigma^1 G(\tau) \int_{v(x)=n} \mu(x\tau^{-1})\omega((x, \tau)_3)\chi(x^{-1}y)d^\times x \end{aligned}$$

$$= \Sigma^2 G(\tau^2) \int_{v(x)=n} \mu(x\tau^{-2})\omega((x, \tau^2)_3)\chi(x^{-1}y)d^\times x .$$

We have for $i = 1, 2$

$$\int_{v(x)=n} \mu(x)\omega((x, \tau^i)_3)\chi(x^{-1}y)d^\times x = \begin{cases} \mu(y)\omega((y, \tau^i)_3)q^{-s-1/2}c_{-i} \\ 0 \end{cases} \tag{5.2}$$

according as $v(y) = n - 1$ or not, where the c_i are the constants that arise as in Lemma 3.1 from the gamma function. (We put $c_i = c_{i+3m}$ for all integers m.)

We now compute Σ^0. We have

$$\int_{v(x)=n} \mu(x)\chi(x^{-1}y)d^\times x = q^{-ns} \int_{\mathcal{O}^\times} \chi(\tau^{-n}yu)du = q(h(\tau^{-n}y) - q^{-1}h(\tau^{-n+1}y))$$

in which $h(y)$ is 1 or 0 according as $v(y) \geq 0$ or not. Therefore,

$$\Sigma^0 \int_{v(x)=n} \mu(x)\chi(x^{-1}y)d^\times x = F_s^0(y) - q^{-1}F_s^0(\tau y)$$

where $F_s^0(y) = \Sigma^0 q^{-ns}h(\tau^{-n}y)$. Changing variables by putting $n = 3m$ in this summation, we easily find that

$$F_s^0(y) = \frac{1 - q^{-3s[v(y)/3]-3s}}{1 - q^{-3s}}$$

where [ ] is the Gauss symbol. We thus get

$$\Sigma^0 = \begin{cases} \dfrac{1}{1 - q^{-3s}}(1 - q^{-1} - q^{-3s[v(y)/3]-3s}(1 - q^{-1-3s})) \\ \dfrac{1}{1 - q^{-3s}}(1 - q^{-1} - q^{-3s[v(y)/3]-3s}(1 - q^{-1})) \end{cases}$$

according as $v(y) \equiv 2$ or $v(y) \not\equiv 2 \bmod 3$. Taking $s = 1/3$, putting the above together with Σ^1, Σ^2 and using (5.2) we find that

$$g^*(y, 1) = G(1) + |y|^{1/3} \begin{cases} G(\tau)c_2 q^{-1/2}\omega((y, \tau)_3) - G(1)q^{-1} \\ G(\tau^2)c_1 q^{-1/6}\omega((y, \tau^2)_3) - G(1)q^{-2/3} \\ -G(1)q^{-1/3}(1 + q^{-1}) \end{cases}$$

according as $v(y) \equiv 0$, $v(y) \equiv 1$ or $v(y) \equiv 2 \bmod 3$, if $|y|$ is sufficiently small—$g^*(y, 1)$ is 0 for large $|y|$. The computations of $g^*(y, \tau)$ and $g^*(y, \tau^2)$ are quite similar; we omit them and collect the results in the following proposition.

5.1. PROPOSITION. *$\mathscr{H}_{\omega,1/3}$ consists of functions f on $K^\times \times A'$ with*

$$f(x, a_0'a') = L_{\omega,1/3}(a_0')f(x, a')$$

which for any fixed a' are locally constant functions on $K^\times$ vanishing outside some compact subset of K and which behave in a neighborhood of 0 as $\eta(x, a') + \nu(x, a')$ for some functions η and ν where $\eta(x, a')$ is constant for a fixed a', and

$$\nu(x, 1) = |x|^{1/3}\begin{cases} -Aq^{-1} + Bc_2q^{-1/2}\omega((x, \tau)_3) \\ -Aq^{-2/3} + Cc_1q^{-1/6}\omega((x, \tau^2)_3) \\ -Aq^{-1/3}(1 + q^{-1}) \end{cases}$$

$$\nu(x, \tau) = |x|^{1/3}\begin{cases} -C(1 + q^{-1}) \\ Ac_2q^{-7/6}\omega((x, \tau)_3) - Cq^{-2/3} \\ Bc_1q^{-5/6}\omega((x, \tau^2)_3) - Cq^{-1/3} \end{cases}$$

$$\nu(x, \tau^2) = |x|^{1/3}\begin{cases} Ac_1q^{-3/2}\omega((x, \tau^2)_3) - Bq^{-1} \\ -Bq^{-2/3}(1 + q^{-1}) \\ Ccq^{-5/6}\omega((x, \tau)_3) - Bq^{-4/3} \end{cases}$$

according as $v(x) \equiv 0$, $v(x) \equiv 1$ or $v(x) \equiv 2 \bmod 3$, for some constants A, B, and C.

We now consider $J_{\omega,1/3}$ as given by (3.2). The following lemma is easily proved.

5.2. LEMMA. *The kernel of $J_{\omega,1/3}$ consists of functions f in $\mathscr{H}_{\omega,1/3}$ which satisfy the following:*

$$\begin{aligned} f(x, 1) &= -c_2q^{1/2}\omega((x, \tau)_3)f(x, \tau^2) && \text{if } v(x) \equiv 0 \bmod 3 \\ f(x, 1) &= -c_1q^{1/2}\omega((x, \tau^2)_3)f(x, \tau) && \text{if } v(x) \equiv 1 \bmod 3 \\ f(x, \tau) &= -c_1q^{1/2}\omega((x, \tau^2)_3)f(x, \tau^2) && \text{if } v(x) \equiv 2 \bmod 3\ . \end{aligned}$$

Consequently, the functions which behave as $\nu(x, a')$ around 0 are in the kernel. Thus to characterize the image it suffices to consider the subspace $\mathscr{S}_{\omega,1/3}$ of $\mathscr{H}_{\omega,1/3}$. We obtain the following easily.

5.3. LEMMA. *The image of $J_{\omega,1/3}$ consists of locally constant functions on $K^\times \times A'$ which satisfy*

(i) $f(x, a_0'a') = L_{\omega,-1/3}(a_0')f(x, a')$,

(ii) *one of the following according as $v(x) \equiv 0$, $v(x) \equiv 1$ or $v(x) \equiv 2 \bmod 3$.*

$$\begin{aligned} f(x,1) &= c_2 q^{-1/2}\omega((x,\tau)_3) f(x,\tau^2), & f(x,\tau) &= 0 \\ f(x,\tau) &= c_2 q^{1/2}\omega((x,\tau)_3) f(x,1), & f(x,\tau^2) &= 0 \\ f(x,\tau^2) &= c_2 q^{1/2}\omega((x,\tau)_3) f(x,\tau), & f(x,1) &= 0 \end{aligned}$$

and which behave as $\psi(x, a')$ *around* 0, *where*

$$\psi(x,1) = |x|^{-1/3}\begin{cases} A + Bc_2\omega((x,\tau)_3) \\ Aq^{-1/3} + Cc_1 q^{-1/2}\omega((x,\tau^2)_3) \\ 0 \end{cases}$$

$$\psi(x,\tau) = |x|^{-1/3}\begin{cases} 0 \\ C + Ac_2 q^{1/6}\omega((x,\tau)_3) \\ Cq^{1/3} + Bc_1\omega((x,\tau^2)_3) \end{cases}$$

$$\psi(x,\tau^2) = |x|^{-1/3}\begin{cases} Bq^{1/2} + Ac_1 q^{1/2}\omega((x,\tau^2)_3) \\ 0 \\ Bq^{5/6} + Cc_2 q^{1/6}\omega((x,\tau)_3) \end{cases}$$

according as $v(x) \equiv 0$, $v(x) \equiv 1$ *or* $v(x) \equiv 2 \bmod 3$, *for some constants* A, B, C.

Given any function f on $K^\times$, we define a function ιf on $K^\times \times A'$ by putting

$$\iota f(x,1) = \begin{cases} f(x) \\ c_1 q^{-1/2}\omega((x,\tau^2)_3) f(x) \\ 0 \end{cases}$$

$$\iota f(x,\tau) = \begin{cases} 0 \\ f(x) \\ c_1 q^{1/2}\omega((x,\tau^2)_3) f(x) \end{cases}$$

$$\iota f(x,\tau^2) = \begin{cases} c_1 q^{1/2}\omega((x,\tau^2)_3) f(x) \\ 0 \\ f(x) \end{cases}$$

according as $v(x) \equiv 0$, $v(x) \equiv 1$ or $v(x) \equiv 2 \bmod 3$, and requiring that

$$\iota f(x, a_0' a') = L_{\omega,-1/3}(a_0')\iota f(x, a').$$

5.4. THEOREM. *The representation* r_ω *has a realization on the space of locally constant functions on* $K^\times$, *which have compact support in* K, *and which behave around* 0 *as*

$$\psi(x) = |x|^{-1/3}\begin{cases} A + Bc_2\omega((x,\tau)_3) \\ Ac_2q^{1/6}\omega((x,\tau)_3) + C \\ Bq^{5/6} + Cc_2q^{1/6}\omega((x,\tau)_3) \end{cases}$$

according as $v(x) \equiv 0$, $v(x) \equiv 1$ *or* $v(x) \equiv 2 \bmod 3$. *The action of* G' *is given by*

$$r_\omega(g')f = (\iota^{-1}\rho_{\omega,-1/3}(g')\iota)f\ .$$

Moreover, r_ω *is a pre-unitary representation with the inner product*

$$(f_1, f_2) = -\int_K \int_{A'_0\backslash A'} \iota f_1(y, a')\overline{J_{\omega,-1/3}\iota f_2(y, a')}da'\, d^\times y\ .$$

Proof. It only remains to prove that (,) is positive definite. $J_{\omega,-1/3}$ does not vanish on the image of $J_{\omega,1/3}$—in fact $J_{\omega,-1/3}\circ J_{\omega,1/3}$ is a scalar. Furthermore, for each y, $-J_{\omega,-1/3}(y)$ is a Hermitian matrix with positive diagonal elements whose principal minors have nonnegative determinants. Thus at each y, $-J_{\omega,-1/3}(y)$ can be written as B^*B for some matrix B (which does not vanish on the image of $J_{\omega,1/3}$). This completes the proof.

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