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FORMAL MEROMORPHIC FUNCTIONS AND COHOMOLOGY ON AN ALGEBRAIC VARIETY

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Introduction

Let X be a projective Gorenstein variety, $Y \subset X$ a proper closed subscheme such that X is smooth at all points of Y, so that the formal completion of X along Y is regular. Writing $\mathscr{M}$ for the sheaf of total quotient rings of $\mathcal{O}_{\hat{X}}$, we have the ring of *formal meromorphic functions*

$$K(\hat{X}) = \Gamma(\hat{X}, \mathscr{M})$$

of X along Y, extending $K(X)$. Following [HM], we shall say Y is $G3$ in X if $K(\hat{X}) = K(X)$, $G2$ in X if $K(\hat{X})$ is a finite algebraic extension of $K(X)$, and $G1$ in X if

$$\Gamma(X, \mathcal{O}_X) = \Gamma(\hat{X}_1 \mathcal{O}_{\hat{X}}) .$$

Here $G3 \Rightarrow G2 \Rightarrow G1$, by [HM, p. 64.], and clearly $G1 \Rightarrow Y$ is connected.

The conditions Gi describe the infinitesimal structure of X around Y. On the other hand, we have the *cohomological dimension*

$$cd(X - Y) = \sup \left\{ i \,\middle|\, \begin{matrix} H^i(X - Y, F) \neq 0 \text{ for at least} \\ \text{one coherent sheaf } F \text{ on } X \end{matrix} \right\} .$$

By Lichtenbaum's Theorem [K], $Y \neq \varnothing \Leftrightarrow cd(X - Y) < \dim(X)$. The main contribution (3.2) of the present exposition is to show (among other things) that the conditions

(1) $cd(X - Y) < \dim(X) - 1$

and

(2) Y is $G3$ in X, and meets every divisor on X

are equivalent, improving the previous result [S_1, Th. 3, p. 20], which, in turn, generalized Hartshorne's "Second Vanishing Theorem" [CDAV, Th.

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7.5, p. 444] for the case $X = \boldsymbol{P}^n$.

Since a formal meromorphic function is given locally by Laurent series which, roughly speaking, converge only in directions tangent to Y, our main result can be viewed as a global convergence criterion—even in characteristic p—in terms of the geometry of $X - Y$.

The crucial new step (2.1) in the proof is to dominate the polar divisor of a formal meromorphic function with the completion of a suitable hypersurface section of X. With X instead of $\hat{X}$, this is clearly possible and, of course, well known. The delicate point is to characterize the linear systems on X which give immersions to $\boldsymbol{P}^n$ in such a way that the relevant properties are preserved under passage to $\hat{X}$, where the language of rational maps is not available.

On the way to our main results, we obtain (in § 2) a simplified proof of Hartshorne's criterion [CDAV, Th. 6.7 and Cor. 6.8, pp. 438–440]: if $Y \subset X$ is a connected local complete intersection with ample normal bundle, then $G2$ holds.

In earlier research, criteria for $G3$ were applied to compute $cd(X - Y)$. Our new results allow computations of $cd(X - Y)$, alone, to be used to establish $G3$. For a simple example, suppose $D_0, \cdots, D_r$ on X are divisors whose complements are affine, and let Y be a closed subscheme whose support is $D_0 \cap \cdots \cap D_r$. Then $X - Y$ is covered by a Čech r-simplex, so we have $cd(X - Y) \leq r$. If $r < \dim(X) - 1$, our main result gives Y $G3$ in X. (This was known previously for ample D_i intersecting such that Y has ample normal bundle- compare, for example, [ASAV, Cor. 2.3, p. 202].)

We conclude with a simple result 3.2 about smooth very ample divisors D on X: if $Y \subset D \subset X$, with Y $G1$ on X, then, under suitable assumptions, $cd(D - Y) < \dim D - 1 \Rightarrow cd(X - Y) < \dim(X) - 1$, giving a partial converse to [S_2, Th. B, p. 146].

Notations and terminology

All schemes below will be of finite type over Spec (k), where k is an algebraically closed field, and all morphisms will be over Spec (k).

Following recent usage, a morphism $f: Y \to X$ will be called an *immersion* if it is an embedding at each point of Y; in other words, for every $y \in Y$ we have a surjection. $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. If an immersion is 1–1, we shall call it an *embedding*.

§1. Immersive linear systems

We begin by characterizing the linear systems which induce immersions (not necessarily embeddings—see above) into projective space. Our main result 1.2 is that an immersive linear system remains so under passage to a formal completion.

Let $\mathcal{O}$ be a local ring, $S \subset \mathcal{O}$ a multiplicative system. We shall say that S *generates the divisors of* $\mathcal{O}$ if every element of $\mathcal{O}$ has the form

$$\sum_{\text{finite}} \alpha_i s_i$$

for units $\alpha_i \in \mathcal{O}$ and elements $s_i \in S$.

EXAMPLE. Let X be a scheme, $p \in X$, $\mathcal{O} = \mathcal{O}_{X,p}$ the local ring at p. If S is the co-ordinate ring of any affine open subset containing p, then the image of S in $\mathcal{O}$ generates the divisors of $\mathcal{O}$.

The globalization we have in mind applies to formal schemes as well as to ordinary schemes. So let $\mathscr{X}$ be a formal scheme. A *linear system* V on $\mathscr{X}$ is a finite dimensional vector subspace of $\Gamma(\mathscr{X}, \mathscr{L})$, for some invertible sheaf $\mathscr{L}$ on $\mathscr{X}$. Identify $\mathscr{L}_p$ and $\mathcal{O}_{\mathscr{X},p}$, and let S_V be the multiplicative system generated by V in $\mathcal{O}_{\mathscr{X},p}$. We shall say V is *immersive at* p if S_V generates the divisors of $\mathcal{O}_{\mathscr{X},p}$. (Clearly this is independent of the particular isomorphism $\mathscr{L}_p \cong \mathcal{O}_{\mathscr{X},p}$.) We shall say V is *immersive* if V is immersive at all points $p \in \mathscr{X}$.

Our next result explains this terminology.

PROPOSITION 1.1. *Let X be a proper scheme, $\mathscr{L} \in \operatorname{Pic}(X)$. Suppose $V \subset \Gamma(X, \mathscr{L})$ is a linear system, with basis $s_0, \cdots, s_n$, spanning $\mathscr{L}$. For a closed point $p \in X$, the following are equivalent:*

(1) *V is immersive at p;*

(2) *the morphism $f: X \to \boldsymbol{P}^n$ given by the s_i is an immersion of p.*

Proof. (2) $\Rightarrow$ (1) is clear. If (1) holds, set $q = f(p)$. We need to show that the induced map $\mathcal{O}_{P^n,q} \to \mathcal{O}_{X,p}$ is surjective. On the one hand, let $\mathfrak{m}_q$ and $\mathfrak{m}_p$ be the maximal ideals at p and q. Choose coordinates $x_0, \cdots, x_n$ on $\boldsymbol{P}^n$ so that $q = (1, 0, \cdots, 0)$; hence $y_1 = x_1/x_0, \cdots, y_n = x_n/x_0$ generate $\mathfrak{m}_q$. Let $t_1, \cdots, t_r \in \mathcal{O}_{X,p}$ generate $\mathfrak{m}_p/\mathfrak{m}_q^2$. Since V is immersive, we have

$$t_i = \sum_j \alpha_{ij} s_{ij}$$

for units α_{ij}, with $s_{ij} \in S_V \subset \mathcal{O}_{X,p}$. Now each s_{ij} is a monomial in the generators $z_1 = s_1/s_0, \cdots, z_n = s_n/s_0$ of $V \subset \mathcal{O}_{X,p}$. Discarding terms in $\mathfrak{m}_p^2$, we can take the s_{ij} *linear* in the z_i. Since $z_i = f^*(y_i)$, we have an expression

$$t_i = \sum \alpha_{ij} f^*(y_i) ,$$

so the induced map $\mathfrak{m}_q \to \mathfrak{m}_p/\mathfrak{m}_p^2$ is surjective. (Therefore, dualizing, φ induces an injection of the tangent spaces at p and q.) On the other hand, φ is proper, so $\varphi_*\mathcal{O}_X$ is a coherent $\mathcal{O}_{P^n}$-module. Thus [AG, Lemma 7.4, p. 153] applies, giving (2). Our proof is complete.

Here is the main result on formal completions.

THEOREM 1.2. *Let X be a scheme, $Y \subset X$ a closed subscheme, $V \subset \Gamma(X, \mathcal{L})$ a linear system, with $\mathcal{L} \in \operatorname{Pic}(X)$. Suppose V is immersive at a point $p \in Y$. Then, passing to formal completions, the linear system $V \subset \Gamma(\hat{X}, \hat{\mathcal{L}})$ on $\hat{X}$ is also immersive at p.*

The question is local, so it follows immediately from the next result.

PROPOSITION 1.3. *Let $\mathcal{O}$ be a regular local k-algebra with residue field k and completion $\hat{\mathcal{O}}$, and consider a local subring A of $\hat{\mathcal{O}}$, with $\mathcal{O} \subset A$. If a multiplicative system $S \subset \mathcal{O}$ generates the divisors of $\mathcal{O}$, then S also generates the divisors of A.*

Proof. Let $t_1, \cdots, t_d$ be regular parameters for $\mathcal{O}$. We can take $\hat{\mathcal{O}} = k[[t_1, \cdots, t_d]]$. There are two cases.

Case 1: $A = \hat{\mathcal{O}}$. Pick $x \in A$. Since $1 \in S$, we may as well assume x is not a unit. Since each t_i is a linear combination of elements of S with unit coefficients, we may also assume S contains the t_i. If $d = 1$, $\hat{\mathcal{O}}$ is a discrete valuation ring, so up to a unit, $x = t_1^e \in S$, with $e \geq 0$. If $d > 1$, we may assume our result holds for dimensions $< d$. Hence we are done if x is a power series in $t_1, \cdots, t_{d-1}$. If not, by the Weierstrass Preparation Theorem we have, up to a unit, an equation

$$x = g_0 + g_1 t_d + \cdots + g_n t_d^n ,$$

with $g_i \in k[[t_1, \cdots, t_{d-1}]]$. By our induction assumption, each g_i can be written

$$g_i = \sum_j u_{ij} s_{ij} ,$$

with units $u_{ij} \in \hat{\mathcal{O}}$, $s_{ij} \in S$. Hence the expression for x simplifies to an equation

$$x = \sum \alpha_i s_i \ ,$$

with units $\alpha_i \in \hat{\mathcal{O}}$, $s_i \in S$. This completes Case 1.

Case 2: $A \neq \hat{\mathcal{O}}$. Again, consider $x \in A$. By Case 1 we have

$$x \in (s_1, \cdots, s_r)\hat{\mathcal{O}}$$

for suitable $s_1, \cdots, s_r \in S$. I claim we also have

$$x \in (s_1, \cdots, s_r)A \ .$$

Indeed, say $x = \sum c_i s_i$ with $c_i \in \hat{\mathcal{O}}$. Let $\mathfrak{m}$ be the maximal ideal of A. Then

$$c_i = \lim_{n \to \infty} (c_{i,n})$$

for suitable $c_{i,n} \in A$, with $c_i - c_{i,n} \in \mathfrak{m}^n$, $\forall n$. Thus, for each n, we find

$$x = \sum_i c_{i,n} s_i + \sum_i (c_i - c_{i,n}) s_i \ ,$$

hence

$$x \in (s_1, \cdots, s_r)A + \mathfrak{m}^n \ .$$

But Krull's Intersection Theorem gives the equality

$$(s_1, \cdots, s_r)A = \bigcap_{n=1}^{\infty} ((s_1, \cdots, s_r)A + \mathfrak{m}^n) \ ,$$

hence $x \in (s_1, \cdots, s_r)A$, as claimed.

It follows that there is a smallest integer $r = r(x)$ such that $x \in (s_1, \cdots, s_r)A$, for some $s_1, \cdots, s_r \in S$. If $r = 1$, then $x = \alpha s$, for some $\alpha \in A$. Since α is a unit of A, we are done. If $r > 1$, we can assume our result is true for all $w \in A$ with $r(w) < r$. Writting

$$x = y + z$$

with $y \in s_1 A$, $z \in (s_2, \cdots, s_r)A$, we have $r(y) = 1$, $r(z) < r$. Hence there are elements $s_i \in S$, and units α_i of A such that

$$y = \alpha_1 s_1$$

and

$$z = \alpha_2 s_2 + \cdots + \alpha_r s_r \ .$$

Adding, we obtain a similar expression for x. This completes the proof.

§ 2. Formal meromorphic functions

Let X be a projective variety, with very ample invertible sheaf $\mathcal{O}(1)$, and suppose $Y \subset X$ is a connected closed subscheme such that X is smooth at all points of Y. The formal completion $\hat{X}$ of X along Y is regular, hence integral. We shall denote by $\hat{F}$ the formal completion, along Y, of a quasicoherent sheaf F on X.

Consider $\mathcal{M}$, the sheaf associated to the presheaf of total rings of fractions of $\hat{\mathcal{O}} = \hat{\mathcal{O}}_X$. Then the ring of *formal meromorphic* (or *formal rational*) *functions*

$$K(\hat{X}) = \Gamma(\hat{X}, \mathcal{M})$$

is a field, extending $K(X)$. Our goal here will be to compare $K(\hat{X})$ with the sections of $\hat{\mathcal{O}}(\nu)$, for various $\nu \geq 0$.

For any $f \in K(\hat{X})$, we define the *pole sheaf* $\mathfrak{P} = \mathfrak{P}_f$ by the assignment

$$U \longrightarrow \begin{Bmatrix} t \in \Gamma(U, \mathcal{M}) \text{ such that} \\ t \cdot (f \mid U) \in (U, \hat{\mathcal{O}}) \end{Bmatrix}$$

for open sets $U \subset \hat{X}$. Then $\mathfrak{P}$ is invertible, and f is the quotient of two global sections of $\mathfrak{P}^{-1}$. (Indeed, both $(f\mathfrak{P})^{-1}$ and $\mathfrak{P}^{-1}$ contain the unit section 1 of $\mathcal{M}$, and $\mathfrak{P} \cong f\mathfrak{P}$, giving two sections of $\mathfrak{P}^{-1}$ whose quotient is f.)

Now fix a nonzero section $t \in \Gamma(\hat{X}, \hat{\mathcal{O}}(1))$. Multiplication by t defines k-linear inclusions

$$\Gamma(\hat{X}, \hat{\mathcal{O}}) \subset \Gamma(\hat{X}, \hat{\mathcal{O}}(1)) \subset \cdots$$

where the union

$$A_t = \bigcup_{\nu \geq 0} \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu)) \ ,$$

is a k-algebra under the multiplication induced by cup-product. The assignment $s \mapsto s/t^\nu \in K(\hat{X})$, for $s \in \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu))$, is compatible with the inclusions defining A_t, so we obtain a map

$$\alpha_t \colon A_t \to K(\hat{X})$$

which is clearly a homomorphism of k-algebras. Since $\hat{X}$ is integral, A_t is an integral domain, and α_t is injective.

Our main result here is the next one.

THEOREM 2.1. *Assumptions as above, α_t identifies $K(\hat{X})$ with the field of fractions of A_t.*

Proof. Using t, we have inclusions of sheaves $\hat{\mathcal{O}} \subset \hat{\mathcal{O}}(1) \subset \cdots$, where the union $\mathfrak{A}_t$ is a sheaf of algebras. As before, we have an algebra sheaf injection $\mathfrak{A}_t \to \mathcal{M}$, inducing α_t. Identifying $\hat{\mathcal{O}}(\nu), \mathfrak{A}_t$ and A_t with their images in $\mathcal{M}$ and $K(\hat{X})$, t identifies with $1 \in \Gamma(\hat{X}, \mathcal{M}) = K(\hat{X})$.

This understood, let $f \in K(\hat{X})$ be nonconstant. To prove 2.1 it will be enough to show that we have

$$\mathfrak{P}^{-1} \subset \hat{\mathcal{O}}(\nu)$$

for $\nu \gg 0$, where $\mathfrak{P}$ is the pole sheaf of f. Indeed, f will then be the quotient of two global sections of $\hat{\mathcal{O}}(\nu)$, hence of two elements of $\mathfrak{A}_t$.

Pick any point $p \in \hat{X}$. Then f is represented by a quotient

$$f_p = g_p / h_p$$

with $g_p, h_p \in \mathcal{O}_{\hat{X},p}$, relatively prime.

Suppose first that $t(p) \neq 0$. Here $\Gamma(X, \mathcal{O}(1))$ is an immersive linear system on X, so, by 1.1, A_t contains the co-ordinate ring S of the affine open subset of X where $t \neq 0$. Also, S generates the divisors of $\mathcal{O}$, by 1.2 or 1.3. Thus, for $\nu \gg 0$, there are units $u_1, \cdots, u_r \in \mathcal{O}_{\hat{X},p}$ and global sections of $\hat{\mathcal{O}}(\nu)$ with germs $s_1, \cdots, s_r$ at p, such that we have

$$h_p = u_1 s_1 + \cdots + u_r s_r \ .$$

Hence we have an inclusion

$$(*p) \qquad (\mathfrak{P}^{-1})_p \subset \hat{\mathcal{O}}(\nu)_p$$

for each p such that $t(p) \neq 0, \forall \nu \geq$ some ν_p depending on p.

Now suppose $t(p) = 0$. Since $\mathcal{O}(1)$ is very ample, we can choose $u \in \Gamma(X, \mathcal{O}(1)) \subset \Gamma(\hat{X}, \hat{\mathcal{O}}(1))$ with $u(p) \neq 0$. Reasoning as above with u instead of t, there are units $v_i \in \mathcal{O}_{\hat{X},p}$ and germs $s_1, \cdots, s_r$ of sections of some $\mathcal{O}(\nu)$ such that we have

$$h_p = v_1 \alpha_u(s_1) + \cdots + v_{r'} \alpha_u(s_{r'}) \ .$$

Now $\alpha_u(s_i) = (t/u)^\nu \alpha_t(s_i)$, so, identifying t with 1, we have

$$h_p = v_1\left(\frac{1}{u}\right)^\nu s_1 + \cdots + v_{r'}\left(\frac{1}{u}\right)^\nu s_{r'} .$$

Hence we find

$$fp = \frac{u^\nu g_p}{v_1 s_1 + \cdots + v_{r'} s_{r'}} ,$$

giving the inclusion $(*p)$ for every $p \in \hat{X}$. Therefore, since $\hat{X}$ is quasicompact, we have $\mathfrak{P}^{-1} \subset \hat{\mathcal{O}}(\nu)$ for all sufficiently large ν, and our proof is complete.

Remark. Theorem 2.1 extends Hartshorne's result [CDAV, Cor. 6.8, p. 439], which assumes Y is a local complete intersection in X, with ample normal bundle. The next corollary enables us to simplify the proof of a further result of Hartshorne.

COROLLARY 2.2. *Suppose the function*

$$\psi(\nu) = \dim_k \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu)) \qquad (\nu \in \mathbf{Z})$$

is bounded above, for all $\nu \gg 0$, *by a polynomial* $P(\nu) \in \mathbf{Q}[\nu]$ *of degree* $n + 1$. *Then we have*

$$\text{tr. deg}_k\, K(\hat{X}) \leq n$$

and, if $\dim(X) = n$, *then* Y *is G2 in* X.

Proof. Apply [CDAV, Lemma 6.3, p. 435] to the graded k-algebra

$$B = \sum \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu)) .$$

Since $A_t = B/(1 - t)$, the corollary follows from 2.1.

COROLLARY 2.3 (Hartshorne, [CDAV, 6.7 and 6.8, pp. 438–9]). *Let* Y *be a connected local complete intersection in* X, *with ample normal bundle, and assume* $\dim(Y) > 0$. *Then* Y *is G2 in* X.

Proof. By [CDAV, Th. 6.2, p. 433], 2.2 applies.

COROLLARY 2.4. *With* $Y \subset X$ *as in* 2.1, *suppose the natural maps*

$$\Gamma(X, \mathcal{O}(\nu)) \to \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu))$$

are bijective for all $\nu \gg 0$. *Then* Y *is G3 in* X.

Proof. This is immediate.

§ 3. Cohomological dimension

Recall that the cohomological dimension of a scheme V ([CDAV], [ASAV]) is the integer

$$cd(V) = \sup \left\{ i \,\middle|\, \begin{matrix} H^i(V, F) \neq 0 \text{ for at least} \\ \text{one coherent sheaf } F \text{ on } V \end{matrix} \right\} .$$

Here we are interested in the case $V = X - Y$, where Y is a closed subset of a projective variety X. We always have $cd(X - Y) < \dim(X)$, if Y is nonempty. For lower cd, we can strengthen our previous criterion ([S, Th. 3, p. 20], [ASAV, Cor. 22, p. 202]) as follows, to relate the vanishing of cohomology on $X - Y$ to the function field $K(\hat{X})$ of the formal completion.

THEOREM 3.1. *Let X be a projective Gorenstein variety, $Y \subset X$ a closed subscheme such that X is smooth at all points of Y. Then the following are equivalent:*

(1) $cd(X - Y) < \dim(X) - 1$

(2) *the natural map*

$$\Gamma(X, F) \to \Gamma(\hat{X}, \hat{F})$$

is bijective for all locally free coherent sheaves F on X

(3) *for any very ample invertible sheaf $\mathcal{O}(1)$ on X, the natural map*

$$\Gamma(X, \mathcal{O}(\nu)) \to \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu))$$

is bijective, for all $\nu \gg 0$;

(4) *Y is G3 in X, and meets every divisor of X.*

Proof. Since X is Gorenstein, the dualizing sheaf ω_X^0 is invertible. Hence the duality arguments for [ASAV, Th. 3.4, p. 96, Assertion (6)] go through without change, giving (1) $\Leftrightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial. To show (3) $\Rightarrow$ (2), assume (3), and consider a locally free sheaf F of finite rank, with dual sheaf $F^\vee$. For $\nu \gg 0$ we have a surjection

$$\mathcal{O}(-\nu)^{\oplus d} \xrightarrow[\alpha]{} F^\vee .$$

Then $K = \ker(\alpha)$ is locally free, so the dual sequence

$$0 \to F \to \mathcal{O}(\nu)^{\oplus d} \to K^\vee \to 0$$

is exact. Taking global sections, we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \Gamma(\hat{X}, \hat{F}) & \longrightarrow & \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu)^{\oplus d}) & \longrightarrow & \Gamma(\hat{X}, \hat{K}^\vee) \\ & \uparrow a & & \uparrow b & & \uparrow c \\ 0 \longrightarrow & \Gamma(X, F) & \longrightarrow & \Gamma(X, \mathcal{O}(\nu)^{\oplus d}) & \underset{f}{\longrightarrow} & \Gamma(X, K^\vee) \end{array}$$

with injective verticals. The Snake Lemma gives an exact sequence

$$0 = \ker(c \,|\, \operatorname{im}(f)) \to \operatorname{cok}(q) \to \operatorname{cok}(b) ,$$

where $\operatorname{cok}(b) = 0$, by (3). Hence $\operatorname{cok}(a) = 0$, so we have (2).

We next show (3) $\Rightarrow$ (4). First of all, (3) $\Rightarrow K(\hat{X}) = K(X)$, by 2.4. To see that Y meets every divisor on X, suppose not. Then $X - Y$ contains a complete variety V of dimension $d = \dim(X) - 1$. Since V supports coherent sheaves G with $H^d(V, G) \neq 0$, we have $H^d(X - Y, G) \neq 0$, contradicting (1). But (1) $\Leftrightarrow$ (3), so (4) holds.

Finally, we need to show (4) $\Leftrightarrow$ (1). Since X is Gorenstein and smooth at all points of Y, the proof of the corresponding equivalence of [S_1, Th. 3, p. 20ff] generalizes immediately. Putting our implications together, we have established 3.1.

COROLLARY 3.2. *Let X be a projective Gorenstein variety, $D \subset X$ a very ample divisor, $Y \subset D$ a closed, connected subscheme such that X and D are smooth at all points of Y. Assume Y is G1 in X, and that $H^1(X, \mathcal{O}(\nu \cdot D)) = 0$ for all $\nu \geq 0$. Then*

$$cd(D - Y) < \dim(D) - 1 \\ \Rightarrow \\ cd(X - Y) < \dim(X) - 1 .$$

Proof. Writing $\mathcal{O}(\nu)$ for $\mathcal{O}(\nu D)$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu - 1)) & \xrightarrow{t} & \Gamma(\hat{X}, \hat{\mathcal{O}}(\nu)) & \xrightarrow{h_\nu} & \Gamma(\hat{D}, \hat{\mathcal{O}}_D(\nu)) \\ & \uparrow a_\nu & & \uparrow b_\nu & & \uparrow c_\nu \\ 0 \longrightarrow & \Gamma(X, \mathcal{O}(\nu - 1)) & \xrightarrow{t} & \Gamma(X, \mathcal{O}(\nu)) & \xrightarrow{g_\nu} & \Gamma(D, \mathcal{O}_D(\nu)) , \end{array}$$

where "$\wedge$" denotes formal completion along Y, $\mathcal{O}_D(\nu) = \mathcal{O}(\nu) \otimes \mathcal{O}_D$, and the arrows marked "t" are given by multiplication by a global equation t defining D. For all ν, c_ν is bijective, by 3.1 and our hypothesis on D. Also, as completion maps, a_ν and b_ν are injective. Hence, if b_ν is bijective for $\nu \geq 0$, our result will follow from 2.5.

I claim first g_ν is surjective, $\forall \nu \geq 0$. For $\nu = 0$, since X and D are connected, $H^0(X, \mathcal{O}) = H^0(D, \mathcal{O}_D) = k$; for $\nu > 0$ we have $H^1(X, \mathcal{O}_X(\nu - 1)) = 0$ by hypothesis, so g_ν is surjective, by the exact cohomology sequence extending the bottom row.

Hence, for all $\nu \geq 0$, the Snake Lemma gives an exact sequence

$$\operatorname{cok}(a_\nu) \xrightarrow{t} \operatorname{cok}(b_\nu) \longrightarrow \operatorname{cok}(c_\nu) = 0 .$$

By definition, $a_\nu = b_{\nu-1}$. Hence, by induction on ν, b_ν is surjective (hence bijective) if b_0 is. But Y is $G1$ in X, so this holds!

Remarks. (1) If Y is a local complete intersection in X, with ample normal bundle, or, more generally, if Y is $G2$ in X, then Y is $G1$ in X, and 3.2 applies.

(2) Iterating, 3.2 holds for sufficiently general complete intersections D containing Y.

(3) For an implication going the other way, compare [S_2, Th. B, p. 146].

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