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ON DIFFERENTIAL INVARIANTS OF HOLOMORPHIC PROJECTIVE CURVES

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1. Canonical forms

Let $(\varphi_1(u), \cdots, \varphi_n(u))$ be a system of holomorphic functions whose Wronskian does not vanish at origin, where holomorphic functions mean functions holomorphic around origin.

A variable transformation

$$(u, y) \mapsto (u(z), \lambda(z)y)$$

induces a map

$$\begin{aligned}(\varphi_1(u), \cdots, \varphi_n(u)) &\mapsto (\phi_1(z), \cdots, \phi_n(z)) \\ &= (\lambda(z)\varphi_1(u(z)), \cdots, \lambda(z)\varphi_n(u(z))) ,\end{aligned}$$

where $\dfrac{du}{dz}(0) \neq 0$ and $\lambda(0) \neq 0$.

We associate linear differential operators of rank n which is a projective invariant of a holomorphic curve[1]: $u \mapsto (\varphi_1(u), \cdots, \varphi_n(u))$ in $\boldsymbol{P}^{n-1}$.

$$\begin{aligned}L_n(p\,|\,u, y) &= \sum_{l=0}^{n}\binom{n}{l}p_l(u)\left(\frac{d}{du}\right)^{n-l}y \\ &= (-1)^{n-1}\begin{vmatrix}\varphi_1, & \cdots, & \varphi_n \\ \vdots & & \\ \left(\frac{d}{du}\right)^{n-1}\varphi_1, & \cdots, & \left(\frac{d}{du}\right)^{n-1}\varphi_n\end{vmatrix}^{-1}\begin{vmatrix}y, & \varphi_1, & \cdots, & \varphi_n \\ \vdots & & & \\ \left(\frac{d}{du}\right)^{n}y, & \left(\frac{d}{du}\right)^{n}\varphi_1, & \cdots, & \left(\frac{d}{du}\right)^{n}\varphi_n\end{vmatrix},\end{aligned}$$

$$L_n(q\,|\,z, y) = \sum_{l=0}^{n}\binom{n}{l}q_l\left(z\left(\frac{d}{dz}\right)\right)^{n-l}y$$

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1) We mean holomorphic maps by holomorphic curves.

$$= (-1)^{n-1} \begin{vmatrix} \phi_1, & \cdots, & \phi_n \\ \vdots & & \\ \left(\frac{d}{dz}\right)^{n-1}\phi_1, & \cdots, & \left(\frac{d}{dz}\right)^{n-1}\phi_n \end{vmatrix}^{-1} \begin{vmatrix} y, & \phi_1, & \cdots, & \phi_n \\ \vdots & & & \\ \left(\frac{d}{dz}\right)^{n} y, & \left(\frac{d}{dz}\right)^{n}\phi_1, & \cdots, & \left(\frac{d}{dz}\right)^{n}\phi_n \end{vmatrix} .$$

The variable transformation $(u, y) \mapsto (u(z), \lambda(z)y)$ induces a transformation:

$$\begin{aligned} L_n(p \mid u, y) &\mapsto L_n(q \mid z, y) \\ &= \lambda(z)^{-1}\left(\frac{du}{dz}\right)^{-n} \sum \binom{n}{l} p_l(u(z))\left(\frac{d}{du(z)}\right)^{n-l}(\lambda(z)y) . \end{aligned}$$

DEFINITION (Lagurre-Forsyth). *A linear differential operator $L_n(Q \mid z, y)$ is called to be canonical, if*

$$Q_1 \equiv Q_2 \equiv 0 .$$

We call z a canonical independent variable of a canonical form $L_n(Q \mid z, y)$.

THEOREM 1 (Forsyth)[2]. *For each $L_n(p \mid u, z)$ there exists a variable transformation*

$$(u, y) \mapsto (u(z), \lambda(z)y)$$

such that $L_n(p \mid u, z)$ is transformed to a canonical form. Moreover a variable transformation maps a canonical form to a canonical form, if and only if

$$u(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \lambda(z) = \frac{cy}{(\gamma z + \delta)^{n-1}} \qquad \left(\begin{pmatrix} \delta \beta \\ \gamma \alpha \end{pmatrix} \in SL(2, \boldsymbol{C}), c \in \boldsymbol{C} - \{0\}\right) .$$

Forsyth's theorem means that for each holomorphic projective curve:

$$u \mapsto (\varphi_1(u), \cdots, \varphi_n(u)) \in \boldsymbol{P}^{n-1}$$

we may associate a canonical variable and a canonical form which are unique up to Möbius transformations.

2. Differential invariants

Similarly as classical invariant theory, we define differential invariants of $L_n(P \mid u, y)$ first for generic coefficients $P_i(u)$ $(1 \leq i \leq n)$, and then specialization

2) See Theorem 6.1 and 6.2, Chap 6 [1], or Theorem 2.4, Chap 2 [2].

$$\left(\frac{d}{du}\right)^l P_i(u) \mapsto \left(\frac{d}{du}\right)^l p_i(u) \qquad (1 \leq i \leq n; 0 \leq l < \infty)$$

gives the definition for $L_n(p \mid u, y)$.

DEFINITION. A differential invariant of weight p of $L_n(P \mid u, y)$ (with generic coefficients) is a polynomial

$$\Phi\left(\cdots, \left(\frac{d}{dz}\right)^l P_j(u), \cdots\right)$$

such that

$$\Phi\left(\cdots, \left(\frac{d}{du}\right)^l P_j(u), \cdots\right)(du)^p$$

is invariant for every variable transformation

$$(u, y) \mapsto (u(z), \lambda(z)y) ,$$

where

$$\frac{du(0)}{dz} \neq 0 , \qquad \lambda(0) \neq 0 .$$

Forsyth gave the following fundamental system of differential invariants of a canonical form

$$L_n(Q \mid z, y) = \left(\frac{d}{dz}\right)^n y + \sum_{l=3}^{n} \binom{n}{l} Q_l(z) \left(\frac{d}{dz}\right)^{n-l} y;$$

$$(\theta_3(z), \cdots, \theta_n(z)) ,$$

$$\theta_p(z) = \frac{1}{2} \sum \frac{(-1)^s (p-2)!p!(2p-s-2)!}{(p-s-1)!(p-s)!(2p-3)!s!} \left(\frac{d}{dz}\right)^s Q_{p-s}(z) \qquad (3 \leq p \leq n) ,$$

where weight of $\theta_p\ (z)$ is p.

For a complex number $w\,(\neq 0, 1, 2, \cdots)$ we denote

$$\binom{w}{l} = \frac{w(w-1)\cdots(w-l+1)}{l1} .$$

Let $w_1, \cdots, w_n$ be complex numbers $(\neq 0, 1, 2, \cdots)$,

$$\xi_1 = (\xi_1^{(0)}, \xi_1^{(1)}, \xi_1^{(2)}, \cdots), \cdots, \xi_n = (\xi_n^{(0)}, \xi_n^{(1)}, \xi_n^{(2)}, \cdots)$$

be variable vectors of infinite length and denote

$$f_j(\xi_j\,|\,z)=\sum_{l=0}^{\infty}\binom{w_j}{l}\xi_j^{(l)}z^l \qquad (1\leq j\leq n)\,.$$

Germ of $SL(2, C)$ at $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ acts on $f_j(\xi_j\,|\,z)$ as follows

$$f_j\left(\begin{pmatrix}\delta\beta\\ \gamma\alpha\end{pmatrix}\xi_j\,\middle|\,z\right)=(\gamma z+\delta)^{w_j}f_j\left(\xi_j\,\middle|\,\frac{\alpha z+\beta}{\gamma z+\delta}\right) \qquad (1\leq j\leq n)\,,$$

where $(\gamma z+\delta)^{w_j}=\delta^{w_j}\sum_{l=0}^{\infty}\binom{w_j}{l}(\delta^{-1}\gamma z)^l$.

DEFINITION. A formal power series $F(\xi; z)$ with coefficients in $C[\xi]$ is called a covariant of index u (a complex number), if

$$F(\xi; z)=\sum_{l=0}^{\infty}\binom{u}{l}c_l(\xi)z^l\,,$$

$$F\left(\begin{pmatrix}\delta\beta\\ \gamma\alpha\end{pmatrix}\xi; z\right)=(\gamma z+\delta)^u F\left(\xi; \frac{\alpha z+\beta}{\gamma z+\delta}\right)\,, \quad \left(\begin{pmatrix}\delta\beta\\ \gamma\alpha\end{pmatrix}\in \text{Germ of } SL(2, C)\right)\,.$$

DEFINITION. A polynomial $\varphi(\xi)$ is called a semi-invariant of index u, if

$$\sum_{j=1}^{n}\sum_{l=0}^{\infty}l\xi_j^{(l-1)}\frac{\partial}{\partial\xi_j^{(l)}}\varphi(\xi)=0$$

and

$$\sum_{j=1}^{n}\sum_{l=0}^{\infty}(w_j-2l)\xi_j^{(l)}\frac{\partial}{\partial\xi_j^{(l)}}\varphi(\xi)=u\varphi(\xi)\,.$$

THEOREM 2[3]. *The following three conditions are equivalent;*

i) $F(\xi; z)$ *is a covariant of index* u,

ii) $F(\xi; z)=\exp(z\Delta)\varphi(\xi)$ *with a semi-invariant of* $\varphi(\xi)$ *index* u, *where*

$$\Delta=\sum_{j=1}^{n}\sum_{l=0}^{\infty}(w_j-l)\xi_j^{(l+1)}\frac{\partial}{\partial\xi_j^{(l)}}\,,$$

iii) $F(\xi; z)=\varphi\left(\cdots, \dfrac{(d/dz)^l f_j(\xi_j\,|\,z)}{w_j(w_j-1)\cdots(w_j-l+1)}, \cdots\right)$ *with a semi-invariant* $\varphi(\cdots, \xi_j^{(l)}, \cdots)$ *of index* u.

THEOREM 3[4].

3) See Theorem 5.1 (Robert's Theorem), Chap. 5 [1], or Theorem 1.3, Chap 1 [2].

4) See Theorem 6.5, Chap. 6 [1], or Theorem 2.6, Chap 2 [2].

$$\{\textit{Differential invariants of } L_n(Q\,|\,z, y)\} \\ = \{\textit{covariants of } \theta_3(z), \cdots, \theta_n(z)\}\,,$$

where $\theta_p(z) = \sum_{l=0}^{\infty}\binom{-2p}{l}\eta_p^{(l)}z^l$ *and* $(w_3, \cdots, w_n) = (-6, -8, \cdots, -2n)$.

3. Defining differential equations of moduli

Two projective holomorphic curves in P^{n-1}

$$C_\varphi\colon u \to (\varphi_1(u), \cdots, \varphi_n(u))$$

and

$$C_\phi\colon z \to (\phi_1(z), \cdots, \phi_n(z))$$

are called to be equivalent, if a projective transformation in P^{n-1} and a variable transformation $(u, y) \to (u(z), \lambda(z)y)$ map C_φ to C_ϕ.

By virtue of Forsyth's theorem equivalence classes of projective holomorphic curves correspond bijectively to equivalence classes of canonical forms with respect to transformations

$$(z, y) \to \left(\frac{\alpha z + \beta}{\gamma z + \delta}, \frac{cy}{(\gamma z + \delta)^{n-1}}\right).$$

Here we shall give the moduli spaces of projective holomorphic curves in P^{n-1} and stable projective holomorphic curves by means of non-linear differential equations. The answer is not so difficult, that is a consequence of properties of Schwarzian derivatives.

Schwarzian

$$\{z, \tau\} = \frac{\dfrac{d^3z}{d\tau^3}}{\dfrac{d^2z}{d\tau^2}} - \frac{3}{2}\left[\frac{\dfrac{d^2z}{d\tau^2}}{\dfrac{dz}{d\tau}}\right]^2$$

is naturally generalized to pairs of differential forms as follows

$$\{(df)^m, dg\} = \{df, dg\} = \{f, g\}(dg)^2\,.$$

LEMMA 1. *If we put*

$$\chi_j(z) = \begin{vmatrix} \theta_j(z) & \dfrac{-1}{2j}\dfrac{d\theta_j(z)}{dz} \\ \dfrac{-1}{2j}\dfrac{d\theta_j(z)}{dz} & \dfrac{1}{2j(2j+1)}\dfrac{d^2\theta_j(z)}{dz^2} \end{vmatrix},$$

we have

$$(1) \qquad \chi_j(z)(dz)^{2j+2} = \frac{(\theta_j(z)(dz)^j)^2}{2(2j+1)}\{\theta_j(z)(dz)^j, dz\}.$$

Proof. Putting $\theta_j(z)(dz)^j = (df)^j$, we have

$$\begin{aligned}\chi_j(z)(dz)^{2j+2} &= \left(\frac{(df/dz)^j}{2j(2j+1)}\left(\frac{d}{dz}\right)^2\left(\frac{df}{dz}\right)^j - \frac{1}{(2j)^2}\left(\frac{d}{dz}\left(\frac{df}{dz}\right)^j\right)^2\right)(dz)^{2j+2} \\ &= \frac{1}{2(2j+1)}\left(\frac{df}{dz}\right)^{2j}\{f, z\}dz^{2j+2}.\end{aligned}$$

LEMMA 2. *Let $\theta_j(u)(du)^j, \chi_j(u)(du)^{2j+2}$ $(3 \le j \le n)$ be holomorphic differential forms around origin. Then there exists a variable z such that*

$$\chi_j(u)(du)^{2j+2} = \frac{(\theta_j(u)(du)^j)^2}{2(2j+1)}\{\theta_j(u)(du)^j, dz\} \qquad (3 \le j \le n),$$

if and only if

$$(2) \qquad \begin{aligned} &2(2j+1)\chi_j(u)\theta_k(u)^2 - 2(2k+1)\chi_k(u)\theta_j(u)^2 \\ &= \frac{1}{j}\theta_j(u)\theta_k(u)^2\frac{d^2\theta_j(u)}{du^2} - \frac{1}{k}\theta_k(u)\theta_j(u)^2\frac{d^2\theta_k(u)}{du^2} \\ &\quad - \frac{2j+1}{2j^2}\theta_k(u)^2\left(\frac{d\theta_j(u)}{du}\right)^2 + \frac{2k+1}{2k^2}\theta_j(u)^2\left(\frac{d\theta_k(u)}{du}\right)^2 \\ &\qquad (3 \le j < k \le n).\end{aligned}$$

If $(\theta_3(u)(du)^3, \cdots, \theta_n(u)(du)^n) \neq (0, \cdots, 0)$, then the variable z is uniquely determined up to Möbius transformations.

Proof. If $\theta_j(u), \theta_k(u) \not\equiv 0$, then from properties of Schwarzian it follows.

$$\begin{aligned}&\{\theta_j(u)(du)^j, dz\} - \{\theta_k(u)(du)^k, dz\} \\ &= \{\theta_j(u)(du)^j, du\} + \{du, dz\} - \{\theta_k(u)(du)^k, du\} - \{du, dz\} \\ &= \{\theta_j(u)(du)^j, du\} - \{\theta_k(u)(du)^k, du\} \\ &= \left(\frac{1}{j}\frac{d^2\theta_j(u)}{d^2u}\theta_j(u)^{-1} - \frac{2j+1}{2j^2}\left(\frac{d\theta_j(u)}{du}\right)^2\theta_j(u)^{-2}\right)(du)^2 \\ &\quad - \left(\frac{1}{k}\frac{d^2\theta_k(u)}{d^2u}\theta_k(u)^{-1} - \frac{2k+1}{2k^2}\left(\frac{d\theta_k(u)}{du}\right)^2\theta_{jk}(u)^{-2}\right)(du)^2.\end{aligned}$$

For a fixed j the equation on z

$$\{\theta_j(u)(du)^j, du\} - 2(2j+1)\theta_j(u)^{-2}\chi_j(u)(du)^2 = \{dz, du\}$$

is solvable, and

$$\{\theta_j(u)(du)^j, dz\} = 2(2j+1)\theta_j(u)^{-2}\chi_j(u)(du)^2 .$$

Hence we can choose a variable z such that

$$\chi_j(u)(du)^{2j+2} = \frac{(\theta_j(u)(du)^j)^2}{2(2j+1)}\{\theta_j(u)(du)^j, dz\}$$

if and only if (2). Assume that $\theta_j(u)du \neq 0$. Then

$$\{\theta_j(u)(du)^j, dz\} - \{\theta_j(u)(du)^j, dz'\} = \{dz, dz'\} = 0$$

and thus z is uniquely determined by $\{\theta_j(u)(du)^j, dz\}$ up to Möbius transformations.

Now we have the next theorem:

THEOREM 4. *Equivalence classes of holomorphic projective curves in P^{n-1} correspond bijectively to system of holomorphic differential forms*

$$(\theta_3(u)(du)^3, \cdots, \theta_n(u)(du)^n, \chi_3(u)(du)^8, \cdots, \chi_n(u)(du)^{2n+2})$$

such that

$$\begin{aligned} &2(2j+1)\chi_j(u)\theta_k(u)^2 - 2(2k+1)\chi_k(u)\theta_j(u)^2 \\ &\quad = \frac{1}{j}\theta_j(u)\theta_k(u)^2\frac{d^2\theta_3(u)}{du^2} - \frac{1}{k}\theta_k(u)\theta_j(u)^2\frac{d^2\theta_k(u)}{du^2} \\ &\quad - \frac{2j+1}{2j^2}\theta_k(u)^2\frac{d\theta_j(u)^2}{du} + \frac{2k+1}{2k^2}\theta_j(u)^2\left(\frac{d_k(u)}{du}\right)^2 \\ &\qquad (3 \leq j < k \leq n) . \end{aligned} \tag{3}$$

Let $L_n(Q|z, y)$ be a canonical form associating with a holomorphic projective curve C, and $(\theta_3(z), \cdots, \theta_n(z))$ be the system of fundamental differential invariants of $L_n(Q|z, y)$. Then the bijective correspondence is given by

$$\begin{aligned} &\theta_j(u)(du)^j = \theta_j(z)(dz)^j \\ &\chi_j(u)(du)^{2j+2} = \frac{(\theta_j(z)(dz)^j)^2}{2(2j+1)}\{\theta_j(z)(dz)^j, dz\} \qquad (3 \leq j \leq n) \end{aligned} \tag{4}$$

Proof. It follows from the following equivalence

$$L_n(Q|z, y) \leftrightarrow (\theta_3(z), \cdots, \theta_n(z)) \leftrightarrow (\theta_3(z)(dz)^3, \cdots, \theta_n(z)(dz)^n, z) .$$

THEOREM 5. *Denote*

$$\theta_{jh}(z)=\begin{vmatrix}\theta_j(z) & \theta_h(z)\\ \dfrac{-1}{2j}\dfrac{d}{dz}\theta_j(z) & \dfrac{-1}{2h}\dfrac{d}{dz}\theta_h(z)\end{vmatrix}\qquad(3\le j<h\le n)\,,$$

$$\chi_{jk}(z)=\begin{vmatrix}\chi_j(z) & \theta_k(z)\\ \dfrac{-1}{4j+4}\dfrac{d}{dz}\chi_j(z) & \dfrac{-1}{2k}\dfrac{d}{dz}\theta_k(z)\end{vmatrix}\qquad(3\le j,k\le n)\,.$$

Let $\tilde{C}$ be the portrait (equivalence class with respect to independent variable transformations) of the curve

$$z\mapsto(\cdots,\theta_j(z),\cdots,\theta_{jh}(z),\cdots,\chi_j(z),\cdots,\chi_{jk}(z),\cdots)$$

in weighted projective space with weight system

$$(\cdots,j,\cdots,j+k+1,\cdots,2j+2,\cdots,2j+k+3,\cdots)\,.$$

Then, if $(\cdots,\theta_{jh},\cdots,\chi_{jk},\cdots)\not\equiv(0,\cdots,0)$, the correspondence:

$$[\text{equivalence class of } C]\mapsto\tilde{C}$$

is bijective. If $(\cdots,\theta_{jh},\cdots,\chi_{jk}\cdots)\equiv(0,\cdots,0)$, then $\tilde{C}$ is a point and C is equivalent to a curve

$$(z^{\lambda_1},z^{\lambda_1}\log z,\cdots,z^{\lambda_1}(\log z)^{m_1},\cdots,z^{\lambda_r},z^{\lambda_r}\log z,\cdots,z^{\lambda_r}(\log z)^{m_r})$$

with $\lambda_1,\cdots,\lambda_r\in\boldsymbol{C}$ and $\sum m_i=n$.

Proof. Let $(\theta'_3(z'),\cdots,\theta'_n(z'))$ be the system of fundamental differential invariants of a canonical form $L_n(Q'|z',y)$ corresponding to the same curve $\tilde{C}$ in weighted projective space. Then there exists $\mu(z)\not\equiv0$ such that

$$\begin{aligned}\theta'_j(z')&=\mu(z)^j\theta_j(z)\,,\\ \theta'_{jk}(z')&=\mu(z)^{j+k+1}\theta_{jk}(z)\,,\\ \chi'_j(z')&=\mu(z)^{2j+2}\chi_j\,(z)\,,\\ \chi'_{jk}(z')&=\mu(z)^{2j+k+3}\chi_{jk}(z)\,.\end{aligned}$$

On the other hand

$$\mu(z)^{j+k+1}\theta_{jk}(z)=\theta'_{jk}(z')=\begin{vmatrix}\theta'_j(z') & \theta'_k(z')\\ \dfrac{-1}{2j}\dfrac{d}{dz'}\theta'_j(z') & \dfrac{-1}{2k}\dfrac{d}{dz'}\theta'_k(z')\end{vmatrix}$$

$$= \mu(z)^{j+k} \frac{dz}{dz'} \theta_{jk}(z) ,$$

$$\mu(z)^{2j+k+3}\chi_{jk}(z) = \chi'_{jk}(z') = \begin{vmatrix} \chi'_j(z) & \theta'_k(z') \\ \dfrac{-1}{4j+4} \dfrac{d}{dz'}\chi'_j(z') & \dfrac{-1}{2k} \dfrac{d}{dz'}\theta'_k(z') \end{vmatrix}$$

$$= \mu(z)^{2j+k+2} \frac{dz}{dz'} \chi_{jk}(z) .$$

If $(\cdots \theta_{jk}, \cdots, \chi_{jk} \cdots) \not\equiv (0, \cdots, 0)$ then we have

$$\mu(z)\frac{dz'}{dz} \equiv 1$$

and

$$\theta'_j(z')(dz')^j = \theta_j(z)(\mu(z)dz')^j = \theta_j(z)(dz)^j ,$$
$$\chi'_j(z')(dz')^{2j+2} = \chi_j(z)(\mu(z)dz')^{2j+2} = \chi_j(z)(dz)^{2j+2} .$$

By virtue of (4) this shows

$$\{\theta_j(z)(dz)^j, dz\} = \{\theta_j(z)(dz)^j, dz'\} \qquad (3 \le j \le n) ,$$

and thus z' is a Möbius transformation of z. Namely $L_n(Q'|z', y)$ is equivalent to $L_n(Q|z, y)$. Assume that $(\cdots, \theta_{jk}, \cdots, \chi_{j,k}, \cdots) = (0, \cdots 0)$. Then by virtue of Lemma 2 there exist a constant c and a function $u(z)$ such that

$$\theta_j(z) = c_j\left(\frac{du}{dz}\right)^j ,$$

$$\chi_j(z) = \frac{c_j^2(du/dz)^{2j}}{2(2j+1)}\{u, z\} = \frac{-c_j^2(du/dz)^{2j}}{2(2j+1)}\{z, u\} ,$$

$$\chi_{jk}(z) = \begin{vmatrix} \chi_j & \theta_k \\ \dfrac{-1}{4j+4} \dfrac{d\chi_j}{dz} & \dfrac{-1}{2k} \dfrac{d\theta_k}{dz} \end{vmatrix}$$

$$= \frac{-c_j^2(du/dz)^{2j+2}}{4(2j+1)(2j+2)} \frac{d}{dz}\{z, u\} = 0 .$$

This shows $\{z, u\} = -\dfrac{\alpha}{2}$ with α in $\boldsymbol{C}$, and thus

$$(\cdots \theta_j(z), \cdots, \theta_{jk}(z), \cdots, \chi_j(z), \cdots, \chi_{jk}(z), \cdots)$$
$$= \left(\cdots, c_j, \cdots, 0 \cdots, \frac{-c_j^2\alpha}{4(2j+1)}, \cdots 0 \cdots 0\right) .$$

Hence $\tilde{C}$ is a point. We may assume $z = e^{\alpha u}$ within Möbius transformation. This means

$$\frac{du}{\alpha z} = \frac{1}{\alpha z}, \qquad \theta_j(z) = c_j(\alpha z)^{-j}$$
$$z^{-n}L_n(Q\,|\,z, y) = \sum \binom{n}{l}\gamma_l\left(\frac{1}{z}\frac{d}{dz}\right)^{n-l} y$$

with $\gamma_3, \cdots, \gamma_n$ in $\boldsymbol{C}$. The fundamental solution of this type of linear differential operator is given by

$$(z^{\lambda_1}, z^{\lambda_1}\log z, \cdots, z^{\lambda_1}(\log z)^{m_1}, \cdots, z^{\lambda_r}, z^{\lambda_r}\log z, \cdots, z^{\lambda_r}(\log z)^{m_r}) .$$

4. Coordinate-free formulation

We shall reformulate the above results in terms of bundles. Let u be an independent variable, and let

$$d^l u \qquad (l = 1, 2, 3, \cdots)$$

be independent variables over ring $\boldsymbol{C}\{u\}$ of convergent power series in u. A derivation d is defined in polynomial algebra

$$\boldsymbol{C}\{u\}[\cdots, d^l u, \cdots]$$

as follows

$$df = \frac{df}{du}du \qquad (f \in \boldsymbol{C}\{u\}) ,$$
$$d(d^l u) = d^{l+1}u \qquad (l = 1, 2, 3, \cdots) .$$

Wronskian of $(\varphi_1(u), \cdots, \varphi_n(u))$ is defined by

$$W_{(\varphi_1, \cdots, \varphi_n)} = \begin{vmatrix} \varphi_1, & \cdots, & \varphi_n \\ d\varphi_1, & \cdots, & d\varphi_n \\ \vdots & & \vdots \\ d^{n-1}\varphi_1, & \cdots, & d^{n-1}\varphi_n \end{vmatrix} . \tag{3}$$

The relation between this Wronskian and the usual one is given by

$$(4)\qquad W_{(\varphi_1,\cdots,\varphi_n)} = \begin{vmatrix} \varphi_1, & \cdots, & \varphi_n \\ \frac{d}{du}\varphi_1, & \cdots, & \frac{d}{du}\varphi_n \\ \vdots & & \vdots \\ \left(\frac{d}{du}\right)^{n-1}\varphi_1, & \cdots, & \left(\frac{d}{du}\right)^{n-1}\varphi_n \end{vmatrix} (du)^{n(n-1)/2} .$$

We denote

$$\tilde{L}_n(\tilde{p}\,|\,y) = d^n y + \sum_{l=1}^{n} \binom{n}{l} \tilde{p}_l(\cdots, d^h\varphi_g, \cdots) d^{n-l} y$$

$$= (-1)^n W^{-1}_{(\varphi_1,\cdots,\varphi_n)} \begin{vmatrix} y, & \varphi_1, & \cdots, & \varphi_n \\ dy, & d\varphi_1, & \cdots, & d\varphi_n \\ \vdots & \vdots & & \vdots \\ d^n y, & d^n\varphi_1, & \cdots, & d^n\varphi_n \end{vmatrix} ,$$

then

$$L_n(\tilde{p}\,|\,y) = L_n(p\,|\,u, y)(du)^n ,$$

$\tilde{L}_n(\tilde{p}\,|\,y)$ $(L_n(p\,|\,u, y))$ is called to be semi-canonical if $\tilde{p}_1 \equiv 0$. $\tilde{L}_n(\tilde{p}\,|\,u, y)$ is semi-canonical if and only if

$$W_{(\varphi_1,\cdots,\varphi_n)} \equiv \gamma(du)^n$$

with $\gamma \neq 0$ in C.

Changing the dependent variable

$$y \mapsto \lambda(u)y$$

with a suitable $\lambda(u)$ $(\lambda(0) \neq 0)$, we may transform any $\tilde{L}_n(\tilde{p}\,|\,u, y)$ to a unique semi-canonical form.

A differential invariant of weight p of $\tilde{L}_n(\tilde{p}\,|\,u, y)$ is defined by

$$\Phi\left(\cdots, \left(\frac{d}{du}\right)^l p_j, \cdots\right)(du)^m$$

with a differential invariant of weight m

$$\Phi\left(\cdots, \left(\frac{d}{du}\right)^l p_j, \cdots\right) .$$

For a semi-canonical form $\tilde{L}_n(\tilde{p}\,|\,y)$ the set of differential invariants of weight m coincides with the set of differential polynomial with coefficients in C

$$\Phi(\cdots, d^l\tilde{p}_j, \cdots)$$

such that

$$\Phi(\cdots, d^l\tilde{p}_j, \cdots) = \Phi\Big(\cdots, \Big(\frac{d}{du}\Big)^l p_j, \cdots\Big)(du)^m .$$

For each differential invariant Φ of weight m of $L_n(p\,|\,u, y)$ we denote

$$\tilde{\Phi} = \Phi(\cdots, d^l\tilde{p}_j, \cdots) = \Phi\Big(\cdots, \Big(\frac{d}{du}\Big)^l p_j, \cdots\Big)(du)^m ,$$

then we get the system of fundamental differential invariants

$$(\tilde{\theta}_3, \cdots, \tilde{\theta}_n)$$

of $\tilde{L}_n(\tilde{p}\,|\,y)$. Moreover the system of differential invariants

$$(\tilde{\theta}_3, \cdots, \tilde{\theta}_n, \cdots, \tilde{\theta}_{jk}, \cdots, \tilde{\chi}_j, \cdots, \tilde{\chi}_{jk}, \cdots)$$

corresponding to

$$(\theta_3, \cdots, \theta_n, \cdots, \theta_{jk}, \cdots, \chi_j, \cdots, \chi_{jk}, \cdots) .$$

5. Several variable case

Let $u_1, \cdots, u_r$ be independent variables and let

$$d^l u_j \qquad (l = 1, 2, 3, \cdots; j = 1, 2, \cdots, r)$$

be independent variables over ring of convergent power series $C\{u_1, \cdots, u_r\}$. We define a derivation d on commutative polynomial algebra

$$C\{u_1, \cdots, u_r\}[\cdots, d^l u_j, \cdots]$$

over $C\{u_1, \cdots, u_r\}$ as follows

$$df = \sum_{j=1}^{r} \frac{\partial f}{\partial u_j} du_j \qquad (f \in C\{u_1, \cdots, u_r\}) ,$$

$$d(d^l u_j) = d^{l+1} u_j \qquad (l = 1, 2, 3, \cdots; j = 1, 2, \cdots, r) .$$

Remark. For any holomorphic curve

$$t \mapsto (u_1(t), \cdots, u_r(t))$$

we can associate a differential algebra homomorphism

$$C\{u_1, \cdots, u_r\}[\cdots, d^l u_j, \cdots] \to C\{t\}[\cdots, d^l t, \cdots]$$

such that

$$d^l\varphi(u_1, \cdots, u_r) \mapsto d^l\varphi(u_1(t), \cdots, u_r(t)) .$$

For a system $(\varphi_1(u_1, \cdots, u_r), \cdots, \varphi_n(u_1, \cdots, u_r))$ Wronskian of $(\varphi_1, \cdots, \varphi_n)$ is defined by

$$W_{(\varphi_1,\cdots,\varphi_n)} = \begin{vmatrix} \varphi_1, & \cdots, & \varphi_n \\ d\varphi_1, & \cdots, & d\varphi_n \\ \vdots & & \vdots \\ d^{n-1}\varphi_1, & \cdots, & d^{n-1}\varphi_n \end{vmatrix} . \tag{5}$$

We denote

$$\begin{aligned} \tilde{L}_n(\tilde{p}\,|\,y) &= (-1)^n W^{-1}_{(\varphi_1,\cdots,\varphi_n)} \begin{vmatrix} y, & \varphi_1, & \cdots, & \varphi_n \\ dy, & d\varphi_1, & \cdots, & d\varphi_n \\ \vdots & \vdots & & \vdots \\ d^n y, & d^n\varphi_1, & \cdots, & d^n\varphi_n \end{vmatrix} \\ &= d^n y + \sum_{l=1}^{n} \binom{n}{l} \tilde{p}_l(\cdots, d^l\varphi, \cdots) d^{n-l} y . \end{aligned} \tag{6}$$

We may define differential invariants of weight p of $\tilde{L}_n(p\,|\,y)$ by the same differential polynomials as differential invariants of one variable case.

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