

## MODULI SPACES OF THE STABLE VECTOR BUNDLES OVER ABELIAN SURFACES

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Let  $X$  be a projective non-singular variety and  $H$  an ample line bundle on  $X$ . The moduli space of  $H$ -stable vector bundles exists by Maruyama [4]. If  $X$  is a curve defined over  $C$ , the structure of the moduli space (or its compactification)  $M(X, d, r)$  of stable vector bundles of degree  $d$  and rank  $r$  on  $X$  is studied in detail. It is known that the variety  $M(X, d, r)$  is irreducible. Let  $L$  be a line bundle of degree  $d$  and let  $M(X, L, r)$  denote the closed subvariety of  $M(X, d, r)$  consisting of all the stable bundles  $E$  with  $\det E = L$ . We know the global Torelli theorem holds for the mapping  $X \mapsto M(X, L, r)$  if the genus  $g$  of  $X \geq 2$  and  $(d, r) = 1$ . Namely, let  $X'$  be a non-singular projective curve of genus  $g$  and  $L'$  be a line bundle of degree  $d$  on  $X'$ . Then the variety  $M(X, L, r)$  is isomorphic to  $M(X', L', r)$  if and only if  $X$  is isomorphic to  $X'$ . In higher dimensional case, very little is known about the moduli space of  $H$ -stable vector bundles. The moduli spaces have been studied only on two types of surfaces. When  $X$  is a hyperelliptic surface, we determined the moduli spaces of  $H$ -stable vector bundles with trivial Chern classes in Umemura [14]. In this case the Moduli spaces are not connected when we fixed the numerical Chern classes. Barth [1] proved the moduli space of  $H$ -stable vector bundles with  $c_1 = 0$  of rank 2 over  $P_2$  is irreducible and rational. In this paper, we work over abelian surfaces  $A$  and we study the moduli spaces of some  $H$ -stable vector bundles. In the first example, a component of the moduli spaces is isomorphic to  $A \times A$  and in the second example, it is birationally isomorphic to the symmetric product  $S^n(A)$ . In both cases, the local Torelli theorem holds (see for precise statements, Theorem 5 and Theorem 21). We know nothing about the connectedness of the moduli spaces.

Let  $X$  be a non-singular algebraic surface defined over an algebraically closed field  $k$  and  $H$  an ample line bundle over  $X$ . We know that, if we

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fix numerical Chern classes  $c_1, c_2$  and a number  $r$ , the coarse moduli space  $M(c_1, c_2)$  of  $H$ -stable vector bundles of rank  $r$  exists. The moduli space  $M(c_1, c_2)$  is a scheme of finite type over  $k$  and the Zariski tangent space  $T_x$  at a closed point  $x \in M(c_1, c_2)$  is isomorphic to  $H^1(X, \text{End } E)$  where  $E$  is the  $H$ -stable vector bundle corresponding to the point  $x$  (see Maruyama [4]). The scheme consisting of all the irreducible components of  $M(c_1, c_2)$  passing through  $x$  is called the moduli space of  $E$  and denoted by  $X(E)$ . From now on, we assume all schemes are defined over  $C$ .

**LEMMA 1.** *Let  $Y$  be an irreducible non-singular algebraic variety and  $\mathcal{E}$  be a family of  $H$ -stable vector bundles over  $X$  parametrized by  $Y$ , i.e.  $\mathcal{E}$  is a locally free sheaf over  $Y \times X$  and for any closed point  $y \in Y$ , the vector bundle  $E_y$  over  $y \times X$  is  $H$ -stable. Let  $f: Y \rightarrow M(c_1, c_2)$  denote the map defined by  $\mathcal{E}$ . We assume that, for any closed point  $y \in Y$ ,  $\dim H^1(X, \text{End } E_y) = \dim Y$ . If  $f$  is injective, then  $f$  is an open immersion. In particular  $Y$  is birationally equivalent to an irreducible component of  $M(c_1, c_2)$ .*

*Proof.* In fact, for any closed point  $y \in Y$ ,  $\dim H^1(X, \text{End } E_y) = \dim Y \leq \dim X \leq \dim H^1(X, \text{End } E_y)$ . Hence  $\dim f(Y) = \dim X$  and  $f(X)$  is contained in the open subset of nonsingular points of  $M(c_1, c_2)$ . Therefore the analytic map  $f^{an}: X^{an} \rightarrow M(c_1, c_2)^{an}$  is an open immersion. It follows from Mumford [9],  $f$  is étale. Now the lemma follows.

**LEMMA 2.** *Let  $A$  be an abelian surface (abelian variety of dimension 2),  $H$  an ample line bundle over  $A$  and  $E$  an  $H$ -stable vector bundle over  $A$ . If  $E$  is absolutely simple, of type  $M_0$  and a model, then there is an injection  $\check{A} \times A \rightarrow X(E)$ .*

*Proof.* Let  $\mathcal{L}$  be the Poincaré line bundle over  $A \times \check{A}$ ,  $p: A \times (\check{A} \times A) \rightarrow A$  be the map defined by

$$p((a, b, c)) = a + c,$$

and  $p_{12}: A \times (A \times A) \rightarrow A \times A$  be the projection. Consider the vector bundle  $\mathcal{E} = p^*E \otimes p_{12}^*\mathcal{L}$  over  $A \times (\check{A} \times A)$ . This is a family of  $H$ -stable vector bundles over  $A$  parametrized by  $\check{A} \times A$ . The vector bundle over a point  $(L, a) \in \check{A} \times A$  is  $T_a^*E \otimes L$ . Let  $f: \check{A} \times A \rightarrow X(E)$  be the map defined by  $\mathcal{E}$ . If  $T_a^*E \otimes L \simeq T_a^*E \otimes L'$ ,  $T_a^*E \simeq T_a^*E \otimes L' \otimes L^{-1}$  hence  $T_{a-a}^*E \simeq E \otimes L' \otimes L^{-1}$ . In particular  $T_{a-a}^*P(E) \simeq P(E)$ . The hypothesis that  $E$  is a model implies  $H(P(E)) = 0$  where  $H(P(E)) = \{x \in A \mid P(E) \simeq$

$T_x^*P(E)$  (Umemura [15]). Hence  $a = a'$ . We get  $E \otimes L \simeq E \otimes L'$ . It follows from the absolute simplicity of  $E$  that  $L \simeq L'$ . This proves  $f$  is injective. See Umemura [15].

Let  $F$  be an  $V$ -stable vector bundle with numerical Chern classes  $c_1, c_2$  and of rank  $r$  over an abelian surface  $A$ . Let  $y \in M(c_1, c_2)$  be the corresponding point. Let us put  $\Delta(F) = (r - 1)c_1^2 - 2rc_2$ .  $-\Delta(F)$  is equal to the second Chern class of  $\text{End } F$ .

LEMMA 3. *The dimension of the Zariski tangent space at  $y$  is equal to  $-\Delta(F) + 2$ .*

*Proof.* It follows from the Riemann-Roch theorem,  $\dim H^0(A, \text{End } F) - \dim H^1(A, \text{End } F) + \dim H^2(A, \text{End } F) = -c_2(\text{End } F) = \Delta(F)$ . By the Serre duality  $\dim H^0(A, \text{End } F) = \dim H^2(A, \text{End } F)$ . Since a stable bundle is simple,  $\dim H^0(A, \text{End } F) = 1$ . Hence  $\dim H^1(A, \text{End } F) = -\Delta(F) + 2$  which is equal to the dimension of the Zariski tangent space at  $y$ .

LEMMA 4. *Under the same hypothesis as in Lemma 2, moreover if  $\Delta(E) = -2$ , then  $X(E)$  is irreducible and  $\check{A} \times A$  is isomorphic to  $X(E)$ .*

*Proof.* Let  $x' \in X(E)$  and  $E'$  the corresponding  $H$ -stable bundle. By Lemma 3  $\dim H^1(A, \text{End } E') = -\Delta(E') + 2$ . Since  $E'$  has the same numerical Chern classes as  $E$ ,  $-\Delta(E') + 2 = -\Delta(E) + 2 = 4$ . Now it follows from Lemmas 1 and 2 that the map  $f$  constructed in the proof of Lemma 2 is an isomorphism.

EXAMPLE 1. Let  $C$  be a non-singular projective curve of genus 2,  $P$  a point of  $C$  and  $J$  the Jacobian variety of  $C$ . Let  $C^{(n)}$  be the  $n$ -th symmetric product of  $C$ . We assume  $n \geq 3$ . There is a projection  $\varphi: C^{(n)} \rightarrow J$  defined by

$$\varphi(Q_1 + \cdots + Q_n) = (Q_1 + \cdots + Q_n - nP),$$

where  $Q_i \in C$  for  $1 \leq i \leq n$ . We know that there exists a vector bundle  $E_{n-1}$  of rank  $n - 1$  over  $J$  such that  $C^{(n)}$  is  $J$ -isomorphic to  $P(E_{n-1})$ . We proved in Umemura [4] that  $E$  is  $\mathcal{O}(C)$ -stable, of type  $M_0$  and a model. The number  $\Delta(E)$  is also calculated  $-2$ . Hence we can apply Lemma 4. The moduli space  $X(E_{n-1})$  is isomorphic to  $\check{J} \times J$ . The determinant defines a map  $\check{J} \times J \rightarrow \check{J}$  ( $y \mapsto \det E_y$ ). This map is surjective and all the fibers which we denote  $X'(E_{n-1})$  are isomorphic to  $J$ . Hence we proved:

THEOREM 5. *The scheme  $X'(E_{n-1})$  is isomorphic to  $J$ . The local Torelli theorem holds for  $C \mapsto X'(E_{n-1})$ .*

EXAMPLE 2. Let  $A$  be an abelian surface and  $L$  be a principal polarization of  $A$ , i.e.,  $L$  is ample and  $\dim H^0(A, L) = 1$ . Let  $n \geq 3$  be an integer and  $L_i$  be a line bundle algebraically equivalent to  $L$  and  $\varphi_i$  a non-zero section of  $L_i$ ,  $1 \leq i \leq n$ .  $\varphi_i$  is uniquely determined up to the multiplication of a non-zero constant. Let  $E(L_1, L_2, \dots, L_n)$  be the coherent sheaf over  $A$  defined by the exact sequence:

$$(i) \quad \begin{array}{l} 0 \rightarrow \mathcal{O} \rightarrow L_1 \oplus L_2 \oplus \dots \oplus L_n \rightarrow E(L_1, L_2, \dots, L_n) \rightarrow 0, \\ 1 \mapsto (\varphi_1, \varphi_2, \dots, \varphi_n). \end{array}$$

The coherent sheaf  $E(L_1, L_2, \dots, L_n)$  does not depend on the choice of  $\varphi_i$ . For, let  $\varphi'_i$  be another non-zero section of  $L_i$ . There exists a non-zero constant  $c_i$  such that  $\varphi'_i = c_i \varphi_i$ . Hence the diagram

$$\begin{array}{ccc} 0 \longrightarrow \mathcal{O} & \xrightarrow{\Phi} & L_1 \oplus L_2 \oplus \dots \oplus L_n \\ & \text{Id} \parallel & \downarrow \Psi \\ 0 \longrightarrow \mathcal{O} & \xrightarrow{\Phi'} & L_1 \oplus L_2 \oplus \dots \oplus L_n \end{array}$$

is commutative where  $\Phi$  is the injection of the exact sequence (i),  $\Phi'$  is the injection obtained from  $\Phi$  by replacing  $\varphi_i$  by  $\varphi'_i$  and  $\Psi$  is the  $\mathcal{O}_A$ -linear map defined by the diagonal matrix

$$\begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}.$$

Let  $C$  be an irreducible non-singular projective curve of genus 2. Let  $P$  be a point of  $C$  and  $\varphi$  the map of  $C$  to  $J$  defined by  $\varphi(Q) = (Q - P)$ . We denote by  $C$  the image  $\varphi(C)$ . Let  $P_1, P_2, \dots, P_6$  be the points of  $C$  such that  $\mathcal{O}(2P_i)$  is isomorphic to the canonical bundle  $K$  of  $C$ . Let  $C_i$  be the image of  $C$  in  $J$  defined by  $\varphi_i(Q - P_i) = \mathcal{O}(Q - P_i) \in J$  for  $Q \in C$ .

LEMMA 5. *If  $x$  is a point of  $J$  such that  $x \notin \bigcup_{i=1}^6 C_i$ , then  $C \cap (C + x) \cap (C + 2x) = \emptyset$ .*

*Proof.* Suppose that  $C \cap C + x \cap C + 2x$  is not empty. Then there exist three points  $Q, Q', Q''$  of  $C$  such that  $\mathcal{O}(Q - P) = \mathcal{O}(Q' - P) + x$ ,  $\mathcal{O}(Q - P) = \mathcal{O}(Q'' - P) + 2x$ . Hence  $\mathcal{O}(Q - Q') = x$ ,  $\mathcal{O}(Q - Q'') = 2x$ . Therefore  $\mathcal{O}(2Q - 2Q') = \mathcal{O}(Q - Q'')$ . Finally we get  $\mathcal{O}(Q + Q'' - 2Q') = \mathcal{O}$ . We

study two cases separately.

*Case (i)  $K \neq \mathcal{O}(2Q')$ .* In this case, by the Riemann-Roch theorem  $\dim H^0(C, \mathcal{O}(2Q')) = 1$ . Hence it follows from  $\mathcal{O}(Q + Q'' - 2Q') = \mathcal{O}$  that  $Q + Q'' = 2Q'$  as divisors. Consequently  $Q = Q'$  and  $Q'' = Q'$ . This shows  $x = 0$ . Hence  $x \in C_i$  for every  $i$ . This is a contradiction.

*Case (ii)  $K = \mathcal{O}(2Q')$ .* Hence there exists an  $i$  such that  $Q' = P_i$ . Therefore  $x = \mathcal{O}(Q - Q') = \mathcal{O}(Q - P_i)$  is in  $C_i$ . This is impossible.

Let  $D$  be the effective divisor on  $A$  such that  $L \simeq \mathcal{O}(D)$ . Such divisors are limited: (a) There exist a non-singular curve  $C$  of genus 2 and a point  $P$  of  $C$  such that the abelian variety  $A$  is isomorphic to the Jacobian variety  $J$  of  $C$  and  $D$  coincides with  $C$ . (b) There exist two elliptic curves  $C_1, C_2$  such that the abelian variety  $A$  is isomorphic to  $C_1 \times C_2$  and  $D$  is  $C_1 \times 0 \cup 0 \times C_2$  (Weil [18]). Let us study first the case (a).

**COROLLARY 6.** *For any integer  $N_1$ , there exist an integer  $N \geq N_1$  and a point  $x \in J$  of order  $N$  such that  $C \cap (C + x) \cap (C + 2x) = \emptyset$ .*

*This is an easy consequence of Lemma 5.*

**LEMMA 7.** *Let  $x$  be a point of the Jacobian variety  $J(= A)$  of order  $N \geq 3$  and  $L \simeq \mathcal{O}(C)$ . If  $C \cap (C + x) \cap \cdots \cap (C + (N - 1)x) = \emptyset$ , then  $E(L, T_x^*L, \dots, T_{(N-1)x}^*L)$  is an  $L$ -stable locally free sheaf.*

*Proof.* The locally freeness of  $E(L, T_x^*L, \dots, T_{(N-1)x}^*L)$  is evident. The cyclic group  $(x)$  operates on the exact sequence;

$$(ii) \quad 0 \rightarrow \mathcal{O} \rightarrow L \oplus T_x^*L \oplus \cdots \oplus T_{(N-1)x}^*L \rightarrow E(L, T_x^*L, \dots, T_{(N-1)x}^*L) \rightarrow 0.$$

Hence there exists an exact sequence of vector bundles over  $A' = A/(x)$ ;

$$(iii) \quad 0 \rightarrow \mathcal{O} \rightarrow F \rightarrow E' \rightarrow 0$$

such that  $\pi^*$  (iii) is isomorphic to (ii) where  $\pi$  is the isogeny  $A \rightarrow A/(x)$ .  $F$  is nothing but the direct image  $\pi_*L$  (Morikawa [7]). Hence by Takemoto [11],  $F$  is  $\det F$ -stable. The line bundle  $\det F$  will be denoted by  $L'$ . Suppose that  $E'$  is not  $L'$ -stable. Then there exist a non-zero locally free sheaf  $G'$  of rank  $< N - 1$  and a morphism  $g'$  of  $\mathcal{O}_{A'}$ -modules  $E' \rightarrow G'$  such that  $g'$  is surjective on  $A'$  (a subvariety of codimension  $\geq 2$ ) and such that

$$(iv) \quad \frac{(c_1(E') \cdot L')}{r(E')} \geq \frac{(c_1(G') \cdot L')}{r(G')}.$$

Since  $F$  is  $L'$ -stable,

$$(v) \quad \frac{(c_1(F') \cdot L')}{r(F')} < \frac{(c_1(G') \cdot L')}{r(G')}.$$

Let now  $M'$  be a line bundle over  $A'$ . Then  $N(M' \cdot L') = \pi^*(M' \cdot L') = (\pi^*M' \cdot L^{\otimes N}) = N(\pi^*M' \cdot L)$ . Hence we proved  $(\pi^*M' \cdot L) = (M' \cdot L')$ . Applying this rule to the inequalities (iv) and (v), we get

$$\frac{(L^{\otimes N} \cdot L)}{N} < \frac{(c_1(G) \cdot L)}{r(G)} \leq \frac{(L^{\otimes N} \cdot L)}{N-1}$$

where  $G = \pi^*G'$ . Consequently

$$2 < \frac{(c_1(G) \cdot L)}{r(G)} \leq 2 + \frac{2}{N-1}.$$

Since  $r(G) < N-1$ , it follows that  $(c_1(G) \cdot L) = 2r(G) + 1$ . If we put  $\tilde{G} = G \otimes L^{-1}$ , then  $(c_1(\tilde{G}) \cdot L) = 1$ . Let  $\varphi$  be the isogeny  $y \mapsto Ny$  of  $A$  onto  $A$  itself. Then, since the morphism induced by  $g' \varphi^*(L \oplus T_x^*L \oplus \cdots \oplus T_{(N-1)x}^*L)$

$\otimes L^{-1} = \overbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}^N \rightarrow \varphi^*G$  is surjective on  $A$  (a subvariety of codimension  $\geq 2$ ),  $H^0(A, \varphi^*G \det \tilde{G}) \neq 0$ . On the other hand, since the spectral sequence degenerates, we get  $H^0(A, \varphi^* \det \tilde{G}) \simeq H^0(A, \varphi_* \varphi^* \det \tilde{G}) = \bigoplus_{\varphi^*\mathcal{L} = \mathcal{O}_A} H^0(A, \mathcal{L} \otimes \det \tilde{G})$ . Therefore, there exists a line bundle  $\mathcal{L}$  on  $A$  such that  $\varphi^*\mathcal{L} \simeq \mathcal{O}_A$ ,  $H^0(A, \mathcal{L} \otimes \det \tilde{G}) \neq 0$ . Let  $D$  be an effective divisor on  $A$  such that  $\varphi^*\mathcal{L} = \mathcal{O}_A$ ,  $\mathcal{O}(D) = \mathcal{L} \otimes \det \tilde{G}$ . Then  $(D \cdot L) = 1$ . This is impossible as we proved in Umemura [16].

**LEMMA 8.** *Using the same notation as in the preceding lemma, we assume moreover  $N \geq 4$  and  $C \cap (C+x) \cap (C+2x) = \emptyset$ . Then  $E(L, T_x^*L, \dots, T_{mx}^*L)$  is an  $L$ -stable locally free sheaf for  $3 \leq m \leq N$ .*

*Proof.* We put  $E_r = E(L, T_x^*L, \dots, T_{(r+1)x}^*L)$  for  $2 \leq r \leq N-1$ . The local freeness of  $E_r$  follows from the hypothesis  $C \cap (C+x) \cap (C+2x) = \emptyset$ . Now we prove the  $L$ -stability of  $E$  by the descending induction on  $r$ . Lemma 7 shows that  $E_{N-1}$  is  $L$ -stable. Let us assume  $E_r$  is  $L$ -stable for an  $r$ ,  $3 \leq r \leq N-1$  and show  $E_{r-1}$  is  $L$ -stable. The diagram

$$\begin{array}{ccc} 0 \longrightarrow \mathcal{O} & \xrightarrow{\phi_r} & L \oplus T_x^*L \oplus \cdots \oplus T_{rx}^*L \\ & \text{Id} \parallel & \downarrow \psi_r \\ 0 \longrightarrow \mathcal{O} & \xrightarrow{\phi_{r-1}} & L \oplus T_x^*L \oplus \cdots \oplus T_{(r-1)x}^*L \end{array}$$

is commutative where  $\Phi_r(1) = (\varphi, T_x^*\varphi, \dots, T_{rx}^*\varphi)$ ,  $0 \neq \varphi \in H^0(A, L)$ ,  $\Phi_{r-1}(1) = (\varphi, T_x^*\varphi, \dots, T_{(r-1)x}^*\varphi)$  and  $\Psi_r$  is the projection onto the first  $r$  factors. The projection  $\Psi_r$  induces a surjection:  $\psi_r: E_r \rightarrow E_{r-1}$  and the  $\text{Ker } \psi_r = T_{rx}^*L$ . Hence we get an exact sequence

$$0 \rightarrow L' \rightarrow E_r \rightarrow E_{r-1} \rightarrow 0,$$

where  $L' = T_{rx}^*L$ . Tensoring  $L'^{-1}$  with the exact sequence, we obtain a new exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow E'_r \rightarrow E'_{r-1} \rightarrow 0.$$

Our induction hypothesis is that  $E'_r$  is  $L$ -stable and we have to show  $E'_{r-1}$  is  $L$ -stable. Let  $G$  be a non-zero locally free sheaf of rank  $\leq r-1$  and  $E'_{r-1} \rightarrow G$  be a morphism which is surjective on  $X$  (a subvariety of co-dimension  $\geq 2$ ). By the stability of  $E'_r$ ,  $(c_1(E'_r) \cdot L)/r(E'_r) < (c_1(F) \cdot L)/r(G)$ . Since  $c_1(E'_r)$  is algebraically equivalent to  $L$ , we get  $2/r < (c_1(F) \cdot L)/r(G)$ . If  $(c_1(F) \cdot L) \geq 2$ , then

$$\frac{(c_1(E'_{r-1}) \cdot L)}{r(E'_{r-1})} = \frac{2}{r-1} < \frac{2}{r(G)} \leq \frac{(c_1(F) \cdot L)}{r(G)}.$$

Hence we may assume  $(c_1(F) \cdot L) = 1$ . Then there is a generically surjective homomorphism  $(L \oplus T_x^*L \oplus \dots \oplus T_{rx}^*L) \otimes T_{rx}^*L^{-1} \rightarrow G \otimes T_{rx}^*L^{-1}$ . The argument of the proof of Lemma 7 shows this is impossible.

Let us now examine the case (b).

LEMMA 9. *Let  $C_1, C_2$  be elliptic curves.  $A = C_1 \times C_2$  and  $L = \mathcal{O}(C_1 \times 0 + 0 \times C_2)$ . Let  $r$  be an integer  $\geq 2$  and  $x_i$  be a point of order  $r+1$  of  $C_i$ ,  $1 \leq i \leq 2$ . If we put  $x = (x_1, x_2)$ , then  $E(L, T_x^*L, \dots, T_{rx}^*L)$  is an  $L$ -stable locally free sheaf.*

*Proof.* Let us put  $E = E(L, T_x^*L, \dots, T_{rx}^*L)$ ,  $C_1 \times 0 \cup 0 \times C_2 = D$ . Since  $D \cap T_x^*D \cap T_{2x}^*D = \emptyset$ ,  $E$  is locally free. Let us show the restriction  $E|_{C_1 \times 0}$  (resp.  $E|_{0 \times C_2}$ ) on  $C_1 \times 0$  (resp.  $0 \times C_2$ ) of  $E$  is stable. We need

SUBLEMMA 10. *Let  $C$  be an elliptic curve and  $M$  a line bundle of degree 1 on  $C$ . Let  $s$  be an integer  $\geq 2$  and  $y$  be a point of  $C$  of order  $s+1$ . Let  $E$  be the coherent sheaf defined by the following exact sequence;*

$$(vi) \quad \begin{aligned} 0 \rightarrow \mathcal{O} \rightarrow L \oplus T_y^*L \oplus \dots \oplus T_{sy}^*L \rightarrow E \rightarrow 0 \\ 1 \mapsto (\psi, T_y^*, \dots, T_{sy}^*\psi) \end{aligned}$$

where  $0 \neq \psi \in H^0(C, L)$ . Then  $E$  is locally free and stable.

*Proof of the sublemma.* For the same reason as in the proof of the lemma,  $E$  is locally free. As in the proof of Lemma 7 the cyclic group  $(y)$  operates on the exact sequence and there exists a exact sequence of vector bundles

$$(vii) \quad 0 \rightarrow \mathcal{O} \rightarrow F \rightarrow E' \rightarrow 0$$

over  $C' = C/(y)$  such that  $\pi^*(vii)$  is isomorphic to (vi) where  $\pi$  is the isogeny  $C \rightarrow C/(y)$ . For the same reason as before,  $F$  is stable. Let  $E' \rightarrow G'$  be a non-trivial quotient vector bundle. Since  $F$  is stable,  $1/(s+1) = d(F)/r(F) < d(G')/r(G')$ . Hence  $d(G') \geq 1$ . Since  $r(E') > r(G')$ ,  $d(E')/r(E') = 1/r(E') < d(G')/r(G')$ . This shows  $E'$  is stable. Since the degree of  $\pi$  is  $s$ , and  $s$  is relatively prime to  $r(E) = s-1$ ,  $\pi^*E' = E$  is stable.

Let us come back to the proof of Lemma 9. Let  $G$  be a non-zero locally free sheaf of rank  $r$  on  $A$  and  $E(L, T_x^*L, \dots, T_{rx}^*L) \rightarrow G$  be a morphism which is surjective on  $A$  (a subvariety of codimension  $\geq 2$ ). Since the restrictions are stable,

$$\frac{(c_1(E) \cdot C_1 \times 0)}{r(E)} = \frac{d(E|C_1 \times 0)}{r(E)} < \frac{d(G|C_1 \times 0)}{r(G)} = \frac{(c_1(G) \cdot C_1 \times 0)}{r(G)}$$

and

$$\frac{(c_1(E) \cdot 0 \times C_2)}{r(E)} < \frac{(c_1(G) \cdot 0 \times C_2)}{r(G)}.$$

Therefore

$$\begin{aligned} \frac{(c_1(E) \cdot L)}{r(E)} &= \frac{(c_1(E) \cdot C_1 \times 0)}{r(E)} + \frac{(c_1(E) \cdot 0 \times C_2)}{r(E)} < \frac{(c_1(G) \cdot C_1 \times 0)}{r(G)} \\ &+ \frac{(c_1(G) \cdot 0 \times C_2)}{r(G)} = \frac{(c_1(G) \cdot L)}{r(G)}. \end{aligned}$$

LEMMA 11. *Let us assume that  $E(L_1, L_2, \dots, L_n) = E$  is locally free. Then the number  $\Delta(E) = (r(E) - 1)c_1(E)^2 - 2r(E)c_2(E)$  is equal to  $-2n$ .*

*Proof.* The first Chern class  $c_1(E)$  is numerically equivalent to  $L^{\otimes n}$  and the second Chern class  $c_2(E)$  is numerically equivalent to  $(n(n-1)/2)L^2$ . Hence,

$$\begin{aligned} \Delta(E) &= (n-2)n^2L^2 - 2(n-1)\frac{n(n-1)}{2}L^2 \\ &= 2(n-2)n^2 - 2(n-1)^2n = -2n. \end{aligned}$$

*Remark 12.* We do not know whether all  $E(L_1, L_2, \dots, L_n) = E$  is  $L$ -stable. We can prove the following assertion which will be used in the sequel.

LEMMA 13. *Suppose that if  $i = j$ ,  $L_i \neq L_j$ . If  $E(L_1, L_2, \dots, L_n) = E$  is locally free,  $E$  is simple ( $\dim H^0(A, \text{End } E) = 1$ ).*

SUBLEMMA 14. *For  $1 \leq i \leq n$ ,  $H^0(A, \check{E} \times L_i) = 0$ .*

*Proof.* We have an exact sequence;

$$(viii) \quad 0 \rightarrow \check{E} \rightarrow \check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n \rightarrow \mathcal{O} \rightarrow 0.$$

Tensoring with  $L_i$ , we get

$$(ix) \quad 0 \rightarrow \check{E} \otimes L_i \rightarrow (\check{L}_1 \oplus \dots \oplus \check{L}_n) \otimes L_i \rightarrow L_i \rightarrow 0.$$

The long exact sequence of cohomology group is;

$$0 \rightarrow H^0(\check{E} \otimes L_i) \rightarrow H^0((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) \rightarrow H^0(L_i) \rightarrow \dots.$$

From the hypothesis  $H^0((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) \simeq H^0(\mathcal{O})$  and the homomorphism  $H^0((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) \simeq H^0(\mathcal{O}) \rightarrow H^0(L_i)$  is not zero. Hence  $H^0(A, \check{E} \otimes L_i) = 0$ .

SUBLEMMA 15.  *$H^l(A, E \otimes L_i) \simeq H^l(\mathcal{O})$ , for  $l = 1, 2$ .*

Let us write the long exact sequence of cohomology of (ix) again;

$$\begin{aligned} 0 &\rightarrow H^0((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) \rightarrow H^0(L_i) \rightarrow H^1(\check{E} \otimes L_i) \\ &\rightarrow H^1((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) \rightarrow H^1(L_i) \rightarrow H^2(\check{E} \otimes L_i) \\ &\rightarrow H^2((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) \rightarrow H^2(L_i). \end{aligned}$$

Now the assertion follows from the following;

- (1)  $H^j((\check{L}_1 \oplus \check{L}_2 \oplus \dots \oplus \check{L}_n) \otimes L_i) = H^j(\mathcal{O})$  for any  $j$ .
- (2)  $H^1(L_i) = 0$  for  $1 = 1, 2$ .

SUBLEMMA 16.  *$H^0(A, \check{E}) = 0$ ,  $H^1(A, \check{E}) \simeq H^0(\mathcal{O})$ ,  $\dim H^2(A, \check{E}) = n + 1$ .*

The first two assertions follow easily from the long exact sequence of cohomology groups of the exact sequence (viii). The last assertion follows from the Riemann-Roch theorem for  $\check{E}$ .

*Proof of Lemma 13.* Tensoring  $E$  with the exact sequence (viii), we get

$$(x) \quad 0 \rightarrow \check{E} \otimes E \rightarrow \check{L}_1 \otimes E \oplus \check{L}_2 \otimes E \oplus \dots \oplus \check{L}_n \otimes E \rightarrow E \rightarrow 0.$$

The last terms of the exact sequence are;

$$\begin{aligned} \cdots \rightarrow H^1(E) &\rightarrow H^2(\check{E} \otimes E) \\ &\rightarrow H^2(\check{L}_1 \otimes E \oplus \check{L}_2 \otimes E \oplus \cdots \oplus \check{L}_n \otimes E) \rightarrow H^2(E). \end{aligned}$$

By the Serre duality and Sublemmas 15 and 16,

$$\begin{aligned} \dim H^1(E) &= \dim H^1(\check{E}) = 1, \\ \dim H^2(L_1 \otimes \check{E} \oplus L_2 \otimes \check{E} \oplus \cdots \oplus L_n \otimes \check{E}) \\ &= \dim H^0(\check{L}_1 \otimes E \oplus \check{L}_2 \otimes E \oplus \cdots \oplus \check{L}_n \otimes E) = 0. \end{aligned}$$

Hence  $\dim H^2(\check{E} \otimes E) \leq 1$ . By the Serre duality  $\dim H^0(\check{E} \otimes E) \leq 1$ . But  $H^0(\check{E} \otimes E)$  contains  $k$  as homotheties. Hence  $\dim H^0(A, \check{E} \otimes E) = 1$ .

LEMMA 17. *If  $E(L_1, L_2, \dots, L_n) \simeq E(L'_1, L'_2, \dots, L'_n)$ , the set  $\{L_1, L_2, \dots, L_n\}$  coincide with the set  $\{L'_1, L'_2, \dots, L'_n\}$  counted with multiplicity.*

In fact let  $M$  be a line bundle algebraically equivalent to  $L$ . Tensoring  $M^{-1}$  with the exact sequence, we get

$$\begin{aligned} (xi) \quad 0 &\rightarrow M^{-1} \rightarrow (L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M^{-1} \\ &\rightarrow E(L_1, L_2, \dots, L_n) \otimes M^{-1} \rightarrow 0. \end{aligned}$$

Since  $M$  is ample,  $H^0(A, M^{-1}) = H^1(A, M^{-1}) = 0$ . Hence  $H^0((L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M^{-1}) \simeq H^0(E(L_1, L_2, \dots, L_n) \otimes M^{-1})$ . Since the dimension of  $H^0((L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M^{-1})$  is the number of times that  $M$  appears in the set  $L_1, L_2, \dots, L_n$ , the lemma follows.

LEMMA 18. *Let  $M, M'$  be line bundles algebraically equivalent to 0. If  $E(L_1, L_2, \dots, L_n) \otimes M \simeq E(L'_1, L'_2, \dots, L'_n) \otimes M'$ , then  $M \simeq M'$  and  $E(L_1, L_2, \dots, L_n) \simeq E(L'_1, L'_2, \dots, L'_n)$ .*

Tensoring  $M^{-1}$ , we may assume  $M' \simeq \mathcal{O}$ . Suppose that  $M$  is not isomorphic to  $\mathcal{O}$ . Then tensoring  $M$  with the exact sequence (i), we get

$$0 \rightarrow M \rightarrow (L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M \rightarrow E(L_1, L_2, \dots, L_n) \otimes M \rightarrow 0.$$

Since  $M$  is algebraically equivalent to 0,  $H^i(A, M) = 0$  for any  $i$  and

$$H^0((L_1 \oplus L_2 \oplus \cdots \oplus L_n) \otimes M) \simeq H^0(E(L_1, L_2, \dots, L_n) \otimes M).$$

Hence  $\dim H^0(E(L_1, L_2, \dots, L_n) \otimes M) = n$ . On the other hand, from the exact sequence (i),

$$\begin{aligned} 0 &\rightarrow H^0(\mathcal{O}) \rightarrow H^0(L'_1 \oplus L'_2 \oplus \cdots \oplus L'_n) \rightarrow H^0(E(L'_1, L'_2, \dots, L'_n)) \\ &\rightarrow H^1(\mathcal{O}) \rightarrow H^1(L'_1 \oplus L'_2 \oplus \cdots \oplus L'_n) = 0. \end{aligned}$$

Hence  $\dim H^0(E(L'_1, L'_2, \dots, L'_n)) = n + 1$  and  $E(L_1, L_2, \dots, L_n) \otimes M$  is not isomorphic to  $E(L'_1, L'_2, \dots, L'_n)$ .

Let now  $p_{i_{n+1}}: \overbrace{A \times \dots \times A}^{n+1} \rightarrow A \times A$  ( $1 \leq i \leq n$ ) be the projection onto the product of  $i$ -th and  $(n + 1)$ -th factors. Let  $m: A \times A \rightarrow A$  be the group law of  $A$ . Let  $\mathcal{L}_i$  be the inverse image  $(m \circ p_{i_{n+1}})^*L$  and  $\Theta_i = (m \circ p_{i_{n+1}})^*(\theta)$  where  $\theta$  is a fixed non-zero section of  $L$ . The coherent sheaf  $\mathcal{E}$  on  $\overbrace{A \times \dots \times A}^{n+1}$  is defined by the exact sequence;

$$(xii) \quad \begin{array}{c} 0 \rightarrow \mathcal{O} \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_n \rightarrow \mathcal{E} \rightarrow 0 \\ 1 \mapsto (\Theta_1, \Theta_2, \dots, \Theta_n) \end{array}$$

The coherent sheaf  $\mathcal{E}$  is considered as a family of coherent sheaves on the last  $A$  parametrized by the first  $\overbrace{A \times \dots \times A}^n$ . Let  $(x_1, x_2, \dots, x_n)$  be a point of  $\overbrace{A \times \dots \times A}^n$ . The restriction of the exact sequence (xii) to the fibre  $(x_1, \dots, x_n) \times A$  is

$$\begin{array}{c} 0 \rightarrow \mathcal{O} \rightarrow T_{x_1}^*L + T_{x_2}^*L + \dots + T_{x_n}^*L \rightarrow E(T_{x_1}^*L, T_{x_2}^*L, \dots, T_{x_n}^*L) \rightarrow 0 \\ 1 \mapsto (T_{x_1}^*\Theta_1, T_{x_2}^*\Theta_2, \dots, T_{x_n}^*\Theta_n) \end{array}$$

The symmetric group  $\mathfrak{S}_n$  operates on  $\overbrace{A \times \dots \times A}^n$  hence on  $(\overbrace{A \times \dots \times A}^n) \times A$ . There is an operation of  $\mathfrak{S}_n$  on  $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_n$  covering its operation on  $(\overbrace{A \times \dots \times A}^n) \times A$ . This operation is compatible with the injection  $0 \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_n$ . Hence  $\mathfrak{S}_n$  acts on  $\mathcal{E}$ . It follows from the descent theory that there exists a coherent sheaf  $\mathcal{E}'$  over  $S^n(A) \times A$  such that  $\pi^*\mathcal{E}' \simeq \mathcal{E}$  where  $S^n(A)$  is the  $n$ -th symmetric product of  $A$  and  $\pi: \overbrace{A \times \dots \times A}^n \rightarrow S^n(A)$  is the projection.

Let  $\mathcal{P}$  be the Poincaré line bundle over  $\check{A} \times A \simeq A \times A$  and  $p_{23}: S^n(A) \times A \times A \rightarrow A \times A$  the projection. We put  $p_{23}^*\mathcal{P} = \mathcal{P}'$  and  $\mathcal{E}' \otimes \mathcal{P}' = \mathcal{E}''$ . Then  $\mathcal{E}''$  is a family of coherent sheaves on  $A$  parametrized by  $S^n(A) \times A$ . Let  $(x, y), (x', y') \in S^n(A) \times A$ . If  $\mathcal{E}''|_{(x,y) \times A}$  is isomorphic to  $\mathcal{E}''|_{(x',y') \times A}$ , then  $(x, y) = (x', y')$  by Lemma 17 and Lemma 18.

**PROPOSITION 19.** *There exists a non-empty open subset  $U$  of  $S^n(A)$  such that (i)  $\mathcal{E}''|_{U \times A \times A} = \mathcal{E}^{(3)}$  is locally free, (ii) for any point  $(x, y) \in U \times A$ ,  $\mathcal{E}''|_{(x,y) \times A}$  is  $L$ -stable.*

*Proof.* Let  $X = \{(x, y, z) \in S^n(A) \times A \times A \mid \theta_i(x, z) = 0 \text{ for any } 1 \leq i \leq n\}$ . Then  $X$  is a closed subset of  $S^n(A) \times A \times A$ . The coherent sheaf  $\mathcal{E}''$  is locally free outside  $X$ . Let  $p_1: S^n(A) \times A \times A \rightarrow S^n(A)$  be the projection onto the first factor. Since  $p_1$  is proper,  $p_1(X)$  is a proper closed subset of  $S^n(A)$  by Corollary 6. Let  $U' = S^n(A) - p_1(X)$ . Then  $\mathcal{E}''|_{U' \times A \times A}$  is locally free. By Corollary 6, Lemma 8 and Lemma 9, there exists a point  $x \in U'$  such that  $\mathcal{E}^{(3)}|_{(x, y) \times A}$  is  $L$ -stable for any point  $y \times A$ . Since the stability is an open condition by Maruyama [4], there exists non-empty open subset  $U$  of  $U'$  satisfying the condition (i) and (ii) of the proposition.

**THEOREM 20.** *The algebraic variety  $S^n(A) \times A$  is birationally isomorphic to a component of the moduli space of  $L$ -stable vector bundles.*

*Proof.* Since  $U \times A$  parametrizes a family of  $L$ -stable bundles, we get a morphism from  $U \times A$  to a component of the moduli space. This map is injective by Lemma 17 and Lemma 18. The theorem now follows from Lemma 1, Lemma 3 and Lemma 11.

This component will be denoted by  $X_n(A; L)$ . Let  $g: X_n(A; L) \rightarrow \check{A} = A$  be the map such that  $g(x) = (\det E) \otimes L^{-1}$ . Since this map is birationally equivalent to  $S^n(A) \times A \rightarrow A$ ,  $((x_1, \dots, x_r), y) \mapsto \sum x_j + (n-1)y$  for  $x \in \check{A} = A$ , the fibre  $g^{-1}(z) = Y_n(A; L)$  is birational to  $Z^n(A) = \{((x_1, \dots, x_n), y) \in S^n(A) \times A \mid \sum x_i + (n-1)y = 0\}$ .

**THEOREM 21.** *The map  $(A, L) \mapsto Y_n(A, L)$  separates locally the moduli space of the principally polarized abelian surfaces.*

*Proof.* Let  $A, B$  be abelian varieties of any dimension. We put

$$\begin{aligned} Z^n(A) &= \{((x_1, \dots, x_n), y) \in S^n(A) \times A \mid \sum x_i + (n-1)y = 0\}, \\ Z^n(B) &= \{((x_1, \dots, x_n), y) \in S^n(B) \times B \mid \sum x_i + (n-1)y = 0\}. \end{aligned}$$

It is sufficient to show that, if  $Z^n(A)$  is birational to  $Z^n(B)$ , then  $A$  is isomorphic to  $B$ . This will be proved in Proposition 24.

Let  $X$  be an algebraic variety and  $n > 0$  an integer. The symmetric group  $\mathfrak{S}_n$  acts on the product  $\overbrace{X \times \dots \times X}^n$ . The quotient variety  $\overbrace{X \times \dots \times X}^n / \mathfrak{S}_n$  is the  $n$ -th symmetric product of  $X$  and denoted by  $S^n(X)$ .

**LEMMA 22.** *Let  $A$  be an abelian variety. The closed subvariety  $\{(x_1, x_2, \dots, x_n) \in A \times \dots \times A \mid \sum_{i=1}^n x_i = 0\}$  of  $S^n(A)$  will be denoted by  $T^n(A)$ .*

Let  $\tilde{T}^n(A)$  be a non-singular model of  $T^n(A)$ . If  $n \geq 2$ ,  $H^0(\tilde{T}^n(A), \Omega^1) = 0$ .

Let  $V$  be the universal covering space of  $A$ . Hence there exists a lattice  $\Gamma \subset V$  such that  $V/\Gamma \simeq A$ .  $H^1(A, \mathcal{O}) \simeq T$  where  $T = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  (see Mumford [10]). By the Künneth formula,  $H^1(A \times \cdots \times A, \mathcal{O}) \simeq T \oplus \cdots \oplus T$ . The symmetric group acts on  $H^1(A \times \cdots \times A, \mathcal{O}) \simeq T \oplus \cdots \oplus T$  as permutation of factors. Let  $U^n(A)$  denote the closed subvariety  $\{(x_1, x_2, \dots, x_n) \in A \times \cdots \times A \mid \sum_{i=1}^n x_i = 0\}$  of  $\overbrace{A \times \cdots \times A}^n$  and  $i$  the inclusion  $U^n(A) \hookrightarrow \overbrace{A \times \cdots \times A}^n$ .  $U^n(A)$  is isomorphic to  $\overbrace{A \times \cdots \times A}^{n-1}$ . Notice that the symmetric group  $\mathfrak{S}_n$  operates on  $U^n(A)$  and  $U^n(A)/\mathfrak{S}_n = T^n(A)$ . It is easy to check that the restriction of the map  $i^*: H^1(A \times \cdots \times A, \mathcal{O}) = \overbrace{T \oplus \cdots \oplus T}^n \rightarrow H^1(U^n(A), \mathcal{O}) = \overbrace{T \oplus \cdots \oplus T}^{n-1}$  to the subspace  $U = \{(t_1, t_2, \dots, t_n) \in T + \cdots + T \mid \sum_{i=1}^n t_i = 0\}$  is an isomorphism. Hence  $H^1(U^n(A), \mathcal{O})^{\mathfrak{S}_n} = U^{\mathfrak{S}_n} = 0$ . On the other hand  $H^1(U^n(A), \mathcal{O})^{\mathfrak{S}_n} = H^1(\tilde{T}^n(A), \mathcal{O})$  by Proposition 9. 24, Ueno [12] hence the lemma is proved.

LEMMA 23. Any rational map  $f$  from  $\tilde{T}^n(A)$  to  $B$  is trivial, i.e.,  $f(\tilde{T}^n)$  is a point.

*Proof.* Blowing up the given  $\tilde{T}^n(A)$  if necessary, we may assume  $f$  regular. Suppose that the dimension of  $f(T^n(A))$  is positive. Then there exist a holomorphic 1-form  $\omega$  on  $B$  such that  $f^*\omega \neq 0$  which conducts Lemma 22.

PROPOSITION 24. Let  $A, B$  be abelian varieties. If  $Z^n(A)$  is birationally isomorphic to  $Z^n(B)$ , then  $A$  is isomorphic to  $B$ .

*Proof.* If  $n = 1$ , then  $Z^1(A) = A$ ,  $Z^1(B) = B$  and the assertion is well known (see. Weil [17]). We assume  $n \geq 2$ . Let  $f: Z^n(A) \rightarrow Z^n(B)$  be a birational map. Let  $\pi_A$  (resp.  $\pi_B$ ) be the map from  $Z^n(A)$  to  $A$  (resp. from  $Z^n(B)$  to  $B$ ) defined by  $\pi_A((x_1, \dots, x_n), y) = \sum_{i=1}^n x_i$  (resp.  $\pi_B((x_1, \dots, x_n), y) = \sum_{i=1}^n x_i$ ). Then by Lemma 23,  $f$  induces a birational map  $\bar{f}: A \rightarrow B$  such that  $\pi_B \circ f = \bar{f} \circ \pi_A$ . Hence  $A$  is biregularly isomorphic to  $B$ .

In our examples, the moduli spaces have irregularity 4. The abelian variety  $A \times \hat{A}$  operates on the moduli spaces;  $E \mapsto T_x^*E \otimes L$ ,  $x \in A$ ,  $L \in \hat{A}$ . In most cases this operation is effective modulo finite group and hence by a theorem of Matsumura-Nishi [6], the irregularity of the moduli space  $\geq 4$ .

QUESTION 25. When is the irregularity of the moduli space 4?

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