Y. Komori
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COMPLETENESS OF TWO THEORIES ON ORDERED ABELIAN GROUPS AND EMBEDDING RELATIONS

YUICHI KOMORI

§ 0. Introduction

The first order language $\mathscr{L}$ that we consider has two nullary function symbols $0, 1$, a unary function symbol $-$, a binary function symbol $+$, a unary relation symbol $0 <$, and the binary relation symbol $=$ (equality). Let $\mathscr{L}'$ be the language obtained from $\mathscr{L}$, by adding, for each integer $n > 0$, the unary relation symbol $n|$ (read "n divides"). The terms $1 + \cdots + 1$ and $t + \cdots + t$ (1 and t repeated n times) will be written as n and nt, the term $t + (-s)$ as $t - s$, the atomic formula $0 < t - s$ as $t < s$, and the formulas $u < t \wedge t < s$ and $t = s \vee t < s$ as $u < t < s$ and $t \leqq s$, respectively. We now give some axiom systems for abelian groups with a semidiscrete total ordering.

(a) The axioms for abelian groups:

$(x + y) + z = x + (y + z)$

$x + y = y + x$

$x + 0 = x$

$x - x = 0.$

(b) The axioms for a total ordering compatible with group structures:

$0 < x \wedge 0 < y \rightarrow 0 < x + y$

$\neg (0 < x \wedge 0 < -x)$

$x = 0 \vee 0 < x \vee 0 < -x.$

(c) The axioms for a semi-discrete ordering:

$0 < 1$

$2x < 1 \vee 1 < 2x.$

(d) The axioms for infinitesimals:

$2x < 1 \rightarrow nx < 1$ for each $n > 2.$

The axioms for $n|$ are

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(e) $n|x \leftrightarrow \exists y \exists z(-1 < 2z < 1 \wedge x = ny + z)$ for each $n > 0$, and
(f) $n|x \vee n|x+1 \vee n|x+2 \vee \cdots \vee n|x+n-1$ for each $n > 1$.
(g) The axioms for divisable infinitesimals:
$-1 < 2x < 1 \rightarrow \exists y(x = ny)$ for each $n > 1$.
(h) The axiom for discrete orderings:
$\neg (0 < x < 1)$.
(i) The axiom for existence of infinitesimals:
$\exists x(0 < x < 1)$.

The language of the theory $\boldsymbol{S}$ is $\mathscr{L}$. The set of axioms of $\boldsymbol{S}$ is (a) ∪ (b) ∪ (c) ∪ (d) (We will write (a, b, c, d) for this set in future.). The language of the theory $\boldsymbol{D}$ is $\mathscr{L}$ as well. The set of axioms of $\boldsymbol{D}$ is $\boldsymbol{S}$ ∪ (h) (equivalent to (a, b, c, h)), which is equivalent to (a, b) ∪ $\{0 < x \leftrightarrow x = 1 \vee 1 < x\}$. We call a model of $\boldsymbol{S}$ (or $\boldsymbol{D}$) *an abelian group with a semi-discrete* (*or discrete*) *total ordering*. The languages of the theories $\boldsymbol{SS}$, $\boldsymbol{SC}$ and $\boldsymbol{DC}$ are all $\mathscr{L}'$. The sets of axioms of $\boldsymbol{SS}$, $\boldsymbol{SC}$ and $\boldsymbol{DC}$ are $\boldsymbol{S}$ ∪ (e, f), $\boldsymbol{SS}$ ∪ (g, i) and $\boldsymbol{SS}$ ∪ (g, h) (equivalent to $\boldsymbol{D}$ ∪ (e, f)), respectively.

Let Z and Q be the set of integers and the set of rationals, respectively. Consider the group $ZQ = Z \times Q$ ordered as follows: $0 < (x, y)$ if and only if either $0 < x$ or $x = 0$ and $0 < y$. ZQ is a model of $\boldsymbol{S}$ which contains Z as a submodel (identifying $(n, 0)$ with n). Of course, Z is a model of $\boldsymbol{D}$, but ZQ is not a model of $\boldsymbol{D}$. It is clear that each model of $\boldsymbol{S}$ is also a model of (e), or, more precisely, given a model of $\boldsymbol{S}$, there is a unique value of $n|$ so that (e) is satisfied. ZQ is a model of $\boldsymbol{SC}$ as well.

It is known that $\boldsymbol{DC}$ allows elimination of quantifiers and that it is complete (cf. Kreisel and Krivine [2] p. 54. There exists an error in the proof, but it is easy to correct the error.). In § 1, we shall show that $\boldsymbol{SC}$ allows elimination of quantifiers and that it is complete and decidable.

In § 2, we shall show that any model of $\boldsymbol{S}$ ($\boldsymbol{SS}$, $\boldsymbol{D}$) can be embedded in some model of $\boldsymbol{SS}$ ($\boldsymbol{SC}$, $\boldsymbol{DC}$, respectively). Embedding relations will be used to show some results on the first order semantics for $\mathscr{L}$ and $\mathscr{L}'$. One of them is that for any universal formula F of $\mathscr{L}'$ and any model A of $\boldsymbol{SS}$ ∪ (i), F is valid in ZQ if and only if F is valid in A. In the paper [1], we shall make use of it for giving a complete description of super-Łukasiewicz propositional logics.

Z is a model of $\boldsymbol{SS}$ ∪ (g), but not a model of $\boldsymbol{SC}$. Therefore, (i) is not a consequence of the set (a, b, c, d, e, f, g). Consider the group $ZZ = Z \times Z$ ordered as follows: $0 < (x, y)$ if and only if either $0 < x$ or $x = 0$

and $0 < y$. This group (identifying $(n, 0)$ with n) is a model of $\boldsymbol{SS} \cup$ (i), but not a model of $\boldsymbol{SC}$. Hence, (g) is not a consequence of the set (a, b, c, d, e, f, i).

§ 1. Elimination of quantifiers

To show that $\boldsymbol{SC}$ admits elimination of quantifiers, we consider a formula F of the form $\exists x(\alpha_1 \wedge \cdots \wedge \alpha_n)$ where each α_i is an atomic formula of $\mathscr{L}'$ or the negation of an atomic formula of $\mathscr{L}'$. Thus α_i is of one of the forms $t = s$, $t \neq s$, $0 < t$, $\neg (0 < t)$, $n | t$ or $\neg (n | t)$.

In this section, the derivations from $\boldsymbol{SS}$ are done without notice. As we shall prove by the way that $\boldsymbol{DC}$ admits elimination of quantifiers, we notice the use when we use (g) or (i). $t = s$, $t \neq s$, $\neg (0 < t)$ and $\neg (n | t)$ are equivalent to $t - s = 0$, $0 < t - s \vee 0 < s - t$, $t = 0 \vee 0 < -t$ and $n | t + 1 \vee \cdots \vee n | t + n - 1$, respectively. Hence we can suppose that each α_i is of one of the forms $t = 0$, $0 < t$ or $n | t$.

Each term t can be written in the form $px + s$ with $p \in Z$ and s a term which does not contain x. If $p = 0$, the atomic formula can be taken out of the scope of $\exists x$.

Thus the formula F can be written in the form

$$(*) \quad \begin{aligned} &\exists x(p_1 x < t_1 \wedge \cdots \wedge p_j x < t_j \wedge u_1 < q_1 x \wedge \cdots \wedge u_k < q_k x \wedge \\ &\qquad r_1 x = v_1 \wedge \cdots \wedge r_l x = v_l \wedge n_1 | s_1 x - w_1 \wedge \cdots \wedge n_m | s_m x - w_m) \end{aligned}$$

where the p, q, r, s are in N (the set of natural numbers which does not contain 0) and t, u, v, w are terms which do not contain x.

For any $k \in N$, $t = u$, $t < u$ and $n | t$ are equivalent to $kt = ku$, $kt < ku$ and $kn | kt$, respectively. Hence, taking the least common multiple (l.c.m.) of $p_1, \cdots, p_j, q_1, \cdots, q_k, r_1, \cdots, r_l, s_1, \cdots, s_m$ we can suppose that $p_1 = \cdots = p_j = q_1 = \cdots = q_k = r_1 = \cdots = r_l = s_1 = \cdots = s_m = p$ in the formula $(*)$. Then the formula $(*)$ is equivalent to

$$\begin{aligned} &\exists x(x < t_1 \wedge \cdots \wedge x < t_j \wedge u_1 < x \wedge \cdots \wedge u_k < x \wedge \\ &\qquad x = v_1 \wedge \cdots \wedge x = v_l \wedge n_1 | x - w_1 \wedge \cdots \wedge n_m | x - w_m \wedge \exists y(x = py)). \end{aligned}$$

It follows from $\boldsymbol{SS} \cup$ (g) that $\exists y(x = py)$ is equivalent to $p | x$. If $l \geqq 1$, then the above formula equivalent to

$$\begin{aligned} &v_1 < t_1 \wedge \cdots \wedge v_1 < t_j \wedge u_1 < v_1 \wedge \cdots \wedge u_k < v_1 \wedge \\ &\qquad v_1 = v_2 \wedge \cdots \wedge v_1 = v_l \wedge n_1 | v_1 - w_1 \wedge \cdots \wedge n_m | v_1 - w_m \wedge p | v_1 \end{aligned}$$

which is quantifier free. Hence we can assume that $l = 0$. If $j \geqq 2$, the formula F is equivalent to

$$(t_1 < t_2 \wedge \exists x(x < t_1 \wedge x < t_3 \wedge \cdots)) \vee (t_2 < t_1 \wedge \exists x(x < t_2 \wedge x < t_3 \wedge \cdots))$$

and we are reduced to the case of a formula with $j - 1$ atomic formulas of the form $x < t$. Hence we can assume that $j \leqq 1$. Similarly, we can assume that $k \leqq 1$.

Suppose that $j = k = 1$. Let n be l.c.m. of $n_1, n_2, \cdots, n_m, p$. Let C_q be the formula

$$\exists y \exists z(0 < ny + z + q < t_1 - u_1 \wedge -1 < 2z < 1) \wedge n_1 | q + u_1 - w_1 \wedge \\ \cdots \wedge n_m | q + u_1 - w_m \wedge p | q + u_1 \qquad (q = 0, 1, 2, \cdots, n-1) .$$

Then, F is equivalent to $C_0 \vee C_1 \vee \cdots \vee C_{n-1}$.

It suffices to show that $\exists y \exists z(0 < ny + z + q < t \wedge -1 < 2z < 1)$ is equivalent to some quantifier free formula. It follows from $\boldsymbol{SS} \cup$ (g) that it is equivalent to $\exists y(0 < ny + q < t)$. If $q = 0$, then it is equivalent to $0 < t$ in ***SC*** (equivalent to $n < t$ in ***DC***). If $0 < q \leqq n - 1$, then it is equivalent to $-1 < 2(t - q)$ in ***SC*** (equivalent to $q < t$ in ***DC***).

When $j = 0$ or $k = 0$, F is equivalent to $E_0 \vee E_1 \vee \cdots \vee E_{n-1}$ where E_q is the formula $n_1 | q + u_1 - w_1 \wedge \cdots \wedge n_m | q + u_1 - w_m \wedge p | q + u_1$.

This completes our proof that ***SC*** (and ***DC***) allows elimination of quantifiers.

Because any atomic formula without variables is equivalent to $0 = 0$ or $0 \neq 0$, any quantifier free formula without variables is equivalent to $0 = 0$ or $0 \neq 0$. Hence any closed formula is equivalent to $0 = 0$ or $0 \neq 0$. Therefore ***SC*** (and ***DC***) is complete.

THEOREM 1.1. *Both theories **SC** and **DC** allow elimination of quantifiers, and they are complete and decidable.*

§ 2. Embedding relations

THEOREM 2.1. *Any model of **SS** can be embedded in some model of **SC**.*

Proof. Let A be a model of ***SS***. When A satisfies (h), we consider $A \times Q$ ordered as $0 < (x, y)$ if and only if either $0 < x$ or $x = 0$ and $0 < y$. Then $A \times Q$ is a model of ***SC*** and the mapping $f: A \rightarrow A \times Q$ such that $f(x) = (x, 0)$ is an embedding of A in $A \times Q$. Suppose A does not satisfy (h), that is, satisfies (i). Let B be the set $\{(x, n) | n | x \text{ and } n > 0 \text{ and } x \in A\}$.

We define functions $-, +$ and a relation $0<$ on B as follows: $-(x, n) = (-x, n)$, $(x, m) + (y, n) = (nx + my, mn)$, $0 < (x, n)$ if and only if $0 < x$. The relation $\sim$ on B defined by $(x, m) \sim (y, n)$ if and only if $nx = my$ is a congruence relation. $B/\sim$ is a model of $\boldsymbol{SC}$ and the mapping $g: A \to B/\sim$ such that $g(x) = [(x, 1)]$ (equivalence class containing $(x, 1)$) is an embedding of A in $B/\sim$. Q.E.D.

By the Embedding Theorem (cf. [2] p. 40) and the fact that ZQ is a model of complete theory $\boldsymbol{SC}$, we have

COROLLARY 2.2. *For any universal formula F of $\mathscr{L}'$, F is valid in ZQ if and only if F is valid in every model of $\boldsymbol{SS}$.*

The following corollary is used for giving a complete description of super-Łukasiewicz propositional logics in a subsequent paper [1].

COROLLARY 2.3. *For any universal formula F of $\mathscr{L}'$ and any model A of $\boldsymbol{SS} \cup$ (i), F is valid in ZQ if and only if F is valid in A.*

Proof. By Corollary 2.2, F is valid in A if F is valid in ZQ. Conversely, suppose that F is valid in A. ZZ can be embedded in any model of $\boldsymbol{SS} \cup$ (i). Hence F is valid in ZZ. Any finitely generated submodel of ZQ is isomorphic to Z or ZZ. Hence it can be embedded in ZZ. Therefore, F is valid in ZQ. Q.E.D.

LEMMA 2.4. *For any model A of $\boldsymbol{S}$ and any elements $x_1, x_2, \cdots, x_q$ of A, if $m + n_1x_1 + \cdots + n_qx_q = 0$, then $m = n_1 = \cdots = n_q = 0$ or there exist elements $y_1, y_2, \cdots, y_{q-1}$ of A and integers k_{ij} $(1 \leqq i \leqq q, 0 \leqq j \leqq q-1)$ such that $x_i = k_{i0} + \sum_{j=1}^{q-1} k_{ij}y_j$ for any i $(1 \leqq i \leqq q)$.*

Proof. Choosing the signs of $x_1, x_2, \cdots, x_q$ suitably, we can assume that $n_1, n_2, \cdots, n_q$ are positive or zero. We prove this lemma by induction on $n_1 + n_2 + \cdots + n_q$.

Case 1. Suppose that at least two of $n_1, n_2, \cdots, n_q$ are positive. We can assume that $0 < n_1 \leqq n_2$. Then we have

$$m + n_1(x_1 + x_2) + (n_2 - n_1)x_2 + n_3x_3 + \cdots + n_qx_q = 0 .$$

By the hypothesis of induction, there exist elements $y_1, \cdots, y_{q-1}$ of A and integers p_{ij} such that

$$x_1 + x_2 = p_{10} + \sum_{j=1}^{q-1} p_{1j}y_j \quad \text{and} \quad x_i = p_{i0} + \sum_{j=1}^{q-1} p_{ij}y_j \qquad (i = 2, 3, \cdots, q)\,.$$

Then we have $x_1 = (p_{10} - p_{20}) + \sum_{k=1}^{q-1}(p_{1j} - p_{2j})y_j$. It completes our proof of Case 1 that we put $k_{1j} = p_{1j} - p_{2j}$ $(0 \leqq j \leqq q-1)$ and $k_{ij} = p_{ij}$ $(2 \leqq i \leqq q,\ 0 \leqq j \leqq q-1)$.

Case 2. We can assume that $n_1 \neq 0$ and $n_2 = n_3 = \cdots = n_q = 0$. Since it follows from $\boldsymbol{S}$ that $\forall x(mx \neq n)$ for every mutually prime integers m, n, n_1 is a factor of m. Hence, we have $x_1 = -m/n_1$. Put $y_i = x_{i+1}$ $(i = 1, 2, \cdots, q-1)$. Q.E.D.

THEOREM 2.5. *Any model of* $\boldsymbol{S}$ *can be embedded in some model of* $\boldsymbol{SS}$.

Proof. It suffices to show that any model of $\boldsymbol{S}$ generated by a finite set can be embedded in some model of $\boldsymbol{SS}$ (cf. Theorem 13 in [1] p. 41). Let A be a model of $\boldsymbol{S}$ which have n generators $a_1, a_2, \cdots, a_n$ but can not be generated by $n-1$ generators. We define a relation $0<$ on $Z \times Q^n$ as follows: $0 < (m, q_1, \cdots, q_n)$ if and only if $0 < pm + (pq_1)a_1 + \cdots + (pq_n)a_n$ where p is l.c.m. of denominators of $q_1, \cdots q_n$,

In order to prove that $Z \times Q^n$ is a model of $\boldsymbol{SS}$, it suffices to show that $m = k_1 = \cdots = k_n = 0$ if $m + k_1a_1 + \cdots + k_na_n = 0$. By Lemma 2.4, $a_1, a_2, \cdots, a_n$ can be generated by $n-1$ generators if $m + k_1a_1 + \cdots + k_na_n = 0$, and $m \neq 0$ or $k_i \neq 0$ for some i. Hence, $Z \times Q^n$ is a model of $\boldsymbol{SS}$.

Let f be a function from A into $Z \times Q^n$ such that $f(m + k_1a_1 + \cdots + k_na_n) = (m, k_1, \cdots, k_n)$. Then f is an embedding of A in $Z \times Q^n$. Q.E.D.

By Theorem 2.1 and Theorem 2.5, we have

THEOREM 2.6. *Any model of* $\boldsymbol{S}$ *can be embedded in some model of* $\boldsymbol{SC}$.

The following corollary can be proved similarly to Corollary 2.3.

COROLLARY 2.7. *For any universal formula* F *of* $\mathscr{L}$ *and any model* A *of* $\boldsymbol{S} \cup$ (i), F *is valid in* ZQ *if and only if* F *is valid in* A.

We can prove the following theorem quite similarly to Theorem 2.5.

THEOREM 2.8. *Any model of* $\boldsymbol{D}$ *can be embedded in some model of* $\boldsymbol{DC}$.

Since any model of $\boldsymbol{D}$ has a submodel Z which is a model of $\boldsymbol{DC}$, we have the following corollary.

COROLLARY 2.9. *For any universal formula F of $\mathscr{L}$ and any model A of $\boldsymbol{D}$, F is valid in Z if and only if F is valid in A.*

Any model of (a, b) $\cup \{0 < 1\}$, that is, any totally ordered abelian group with 1 contains Z as a submodel. Hence, for any model A of (a, b) $\cup \{0 < 1\}$, the set of open formulas valid in A is included in the set of open formulas valid in Z which equals to the set of open theorems of $\boldsymbol{D}$. The theory $\boldsymbol{D}$ is an open theory, that is, axiomatizable by only open formulas. Therefore, we have

THEOREM 2.10. *The theory $\boldsymbol{D}$ is the greatest element of the class of open consistent theories in $\mathscr{L}$ containing* (a, b) $\cup \{0 < 1\}$.

REFERENCES

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Shizuoka University