

## ON THE INTEGER RING OF THE COMPOSITUM OF ALGEBRAIC NUMBER FIELDS

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### §1. Statement of the results

Let  $k$  be an algebraic number field of finite degree. For a finite extension  $L/k$  we denote by  $\mathfrak{D}_{L/k}$  the different of  $L/k$ , and by  $\mathfrak{O}_L$  the integer ring of  $L$ . Let  $K_1$  and  $K_2$  be finite extensions of  $k$ . It is known that we have  $\mathfrak{O}_{K_1K_2} = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  if  $K_1$  and  $K_2$  are linearly disjoint over  $k$  and  $\mathfrak{D}_{K_1K_2/k} = \mathfrak{D}_{K_1/k}\mathfrak{D}_{K_2/k}$  holds (see Shimura [2], 1.2).

In this paper we compute the conductor of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  with respect to  $\mathfrak{O}_{K_1K_2}$  and the module index of  $\mathfrak{O}_{K_1K_2}$  and  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  in terms of relevant differents and "Elements". We note that the conductor of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  with respect to  $\mathfrak{O}_{K_1K_2}$  is the largest ideal of  $\mathfrak{O}_{K_1K_2}$  which is contained in  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ . For a Dedekind domain  $R$  whose quotient field is  $L$  and  $R$ -lattices  $M, N$  of the same finite dimensional vector space over  $L$ , we denote by  $[M: N]_R$  the module index of  $M$  and  $N$ . We note that the index  $[M: N]$  is the absolute norm of  $[M: N]_R$  if  $L$  is a number field and  $R$  is its integer ring. For general properties of module indices we refer to Frölich [1]. For a finite extension  $L/K$  of algebraic number fields of finite degree and an embedding  $\sigma$  of  $L$  over  $K$ , we denote by  $e_\sigma$  the element with respect to  $\sigma$ . We recall that  $e_\sigma$  is the ideal generated by  $x - x^\sigma$ ,  $x \in \mathfrak{O}_L$ .

We state our results.

**THEOREM.** *Let  $k$  be an algebraic number field of finite degree, and  $K_1, K_2$  its finite extensions. Then we have*

- (1) *the conductor  $\mathfrak{f}$  of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  with respect to  $\mathfrak{O}_{K_1K_2}$  is  $\prod_{\sigma \neq 1} e_{\sigma|K_2} \mathfrak{D}_{K_1K_2/K_1}^{-1}$ , where  $\sigma$  runs through all the non-trivial embeddings of  $K_1K_2$  over  $K_1$ ,*
- (2)  *$[\mathfrak{O}_{K_1K_2} : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]_{\mathfrak{O}_L}^2 = N_{K_1K_2/L}(\mathfrak{f})$  holds, where  $L = k, K_1, K_2$ .*

**COROLLARY.** *Let notations be as in Theorem. Then we have  $\mathfrak{O}_{K_1K_2} = \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  if and only if  $\mathfrak{D}_{K_1K_2/K_1} = \prod_{\sigma \neq 1} e_{\sigma|K_2}$  holds.*

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We note that  $\prod_{\sigma \neq 1} e_{\sigma|K_2} = \mathfrak{D}_{K_2/k}$  holds if  $K_1$  and  $K_2$  are linearly disjoint over  $k$ .

We shall give another description of the conductor and some examples in § 3.

## § 2. Proofs

**2.1. Proof of Theorem (1).** Firstly we claim that there exists an element  $z \in \mathfrak{O}_{K_2}$  for any prime ideal  $\mathfrak{p}$  of  $K_2$  such that

(i)  $k(z) = K_2$ ,

(ii) the  $\mathfrak{P}$ -component of the conductor of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  is that of  $\mathfrak{O}_{K_1}[z]$  for all prime ideals  $\mathfrak{P}$  of  $K_1K_2$  above  $\mathfrak{p}$ .

We take  $z \in \mathfrak{O}_{K_2}$  which satisfies

(iii)  $\text{ord}_{\mathfrak{p}} f'_z(z) = \text{ord}_{\mathfrak{p}} \mathfrak{D}_{K_2/k}$ , and  $\text{deg } f_z = [K_2:k]$ .

Here  $f_z$  is the minimal polynomial of  $z$  over  $k$ . We show  $z$  satisfies (ii). We recall that the conductor of  $\mathfrak{O}_k[z]$  with respect to  $\mathfrak{O}_{K_2}$  is  $f'_z(z)\mathfrak{D}_{K_2/k}^{-1}$ , where  $f'_z$  is the derivative of  $f_z$ . We have

$$\begin{aligned} \mathfrak{O}_{K_1}[z] &= \mathfrak{O}_{K_1}\mathfrak{O}_k[z] \supset \mathfrak{O}_{K_1}f'_z(z)\mathfrak{D}_{K_2/k}^{-1} \\ &= \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}f'_z(z)\mathfrak{D}_{K_2/k}^{-1} \supset \mathfrak{f}'_z(z)\mathfrak{D}_{K_2/k}^{-1}. \end{aligned}$$

Therefore the conductor of  $\mathfrak{O}_{K_1}[z]$  contains  $\mathfrak{f}'_z(z)\mathfrak{D}_{K_2/k}^{-1}$ . Since  $\text{ord}_{\mathfrak{p}} f'_z(z)\mathfrak{D}_{K_2/k}^{-1} = 0$ , we get the claim.

By the claim  $\mathfrak{f}$  is the greatest common divisor of the conductors of  $\mathfrak{O}_{K_1}[z]$ , where  $z$  satisfies (i) and is contained in  $\mathfrak{O}_{K_2}$ . The conductor of  $\mathfrak{O}_{K_1}[z]$  with respect to  $\mathfrak{O}_{K_1K_2}$  is  $g'_z(z)\mathfrak{D}_{K_1K_2/K_1}^{-1}$ , where  $z$  is an element of  $\mathfrak{O}_{K_2}$  with (i) and  $g_z$  is the minimal polynomial of  $z$  over  $K_1$ . We show that

$$\prod_{\sigma \neq 1} e_{\sigma|K_2} = (g'_z(z) : z \in \mathfrak{O}_{K_2} \text{ with (i)}),$$

where  $\sigma$  runs through all the non-trivial embeddings of  $K_1K_2$  over  $K_1$  into a finite Galois extension  $L$  over  $k$  containing  $K_1K_2$ . Let  $\mathfrak{p}$  be a prime ideal of  $K_2$ . Let  $z$  be an element of  $\mathfrak{O}_{K_2}$  satisfying (iii). Then we have

$$\text{ord}_{\mathfrak{P}}(z - z^\sigma) \geq \text{ord}_{\mathfrak{P}} e_{\sigma|K_2}$$

for all the non-trivial embeddings  $\sigma$  of  $K_1K_2$  over  $K_1$  into  $L$  and all prime ideals  $\mathfrak{P}$  of  $L$  above  $\mathfrak{p}$ , and

$$\sum_{\sigma \neq 1} \text{ord}_{\mathfrak{P}}(z - z^\sigma) = \text{ord}_{\mathfrak{P}} \mathfrak{D}_{K_2/k} = \sum_{\sigma \neq 1} \text{ord}_{\mathfrak{P}} e_{\sigma}$$

for all prime ideals  $\mathfrak{P}$  of  $L$  above  $\mathfrak{p}$ . Here the sums are taken over all

the non-trivial embeddings  $\sigma$  of  $K_2$  over  $k$  into  $L$ . Therefore we have

$$\text{ord}_{\mathfrak{P}}(z - z^\sigma) = \text{ord}_{\mathfrak{P}} e_{\sigma|K_2}$$

for all non-trivial embeddings  $\sigma$  of  $K_1K_2$  over  $K_1$  into  $L$  and all prime ideals  $\mathfrak{P}$  of  $L$  above  $\mathfrak{p}$ . Using the decomposition  $g'_2(z) = \prod_{\sigma \neq 1} (z - z^\sigma)$ , we have

$$\text{ord}_{\mathfrak{P}} g'_2(z) = \text{ord}_{\mathfrak{P}} \prod_{\sigma \neq 1} e_{\sigma|K_2}$$

for all prime ideals  $\mathfrak{P}$  of  $L$  above  $\mathfrak{p}$ . Thus we proved the assertion. Hence we have (1).

**2.2. Proof of Theorem (2).** It suffices to prove the case  $L = K_1$ . For a  $\mathfrak{O}_{K_1}$ -lattice  $M$  of  $K_1K_2$  we denote by  $M^*$  the dual module of  $M$  with respect to  $\text{Tr}_{K_1K_2/K_1}$ . We have

$$[\mathfrak{O}_{K_1K_2} : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]_{\mathfrak{O}_{K_1}}^2 = [(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^* : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]_{\mathfrak{O}_{K_1}} [\mathfrak{O}_{K_1K_2}^* : \mathfrak{O}_{K_1K_2}]_{\mathfrak{O}_{K_1}}^{-1}$$

(see Fröhlich [1], Proposition 4, § 3). Since  $\mathfrak{O}_{K_1K_2}^* = \mathfrak{O}_{K_1K_2/K_1}^{-1}$ ,

$$\begin{aligned} [\mathfrak{O}_{K_1K_2}^* : \mathfrak{O}_{K_1K_2}]_{\mathfrak{O}_{K_1}} &= [\mathfrak{O}_{K_1K_2/K_1}^{-1} : \mathfrak{O}_{K_1K_2}]_{\mathfrak{O}_{K_1}} \\ &= N_{K_1K_2/K_1}(\mathfrak{O}_{K_1K_2/K_1}) \end{aligned}$$

holds.

From now on we compute the module index  $[(\mathfrak{O}_{K_1}\mathfrak{O}_{K_2})^* : \mathfrak{O}_{K_1}\mathfrak{O}_{K_2}]_{\mathfrak{O}_{K_1}}$ .

**LEMMA 1.** *Let  $S$  be a finite set of prime ideals of  $K_1K_2$ . Let  $\mathfrak{p}$  be a prime ideal of  $K_2$ . We denote by  $n$  a natural number. Then there exists an element  $\gamma$  of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  such that  $\gamma \equiv 1 \pmod{\mathfrak{P}^n}$  holds for all  $\mathfrak{P}$  of  $S$  above  $\mathfrak{p}$  and  $\gamma \equiv 0 \pmod{\mathfrak{P}^n}$  for all  $\mathfrak{P}$  of  $S$  not above  $\mathfrak{p}$ .*

*Proof.* We put

$$S_1 = \{\mathfrak{O}_{K_1}\mathfrak{O}_{K_2} \cap \mathfrak{P} : \mathfrak{P} \in S, \mathfrak{P} \mid \mathfrak{p}\},$$

and

$$S_2 = \{\mathfrak{O}_{K_1}\mathfrak{O}_{K_2} \cap \mathfrak{P} : \mathfrak{P} \in S, \mathfrak{P} \nmid \mathfrak{p}\}.$$

Elements of  $S_1 \cup S_2$  are maximal ideals of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$ .  $S_1 \cap S_2 = \emptyset$  holds. So by Chinese remainder theorem there exists an element  $\gamma$  of  $\mathfrak{O}_{K_1}\mathfrak{O}_{K_2}$  such that  $\gamma \equiv 1 \pmod{\mathfrak{M}^n}$  for all  $\mathfrak{M} \in S_1$  and  $\gamma \equiv 0 \pmod{\mathfrak{M}^n}$  for all  $\mathfrak{M} \in S_2$  hold, which proves Lemma 1.

For a  $\mathfrak{O}_k$ -module  $M$  and a prime ideal  $\mathfrak{p}$  of  $k$ , we denote by  $M_{\mathfrak{p}} S_{\mathfrak{p}}^{-1} M$ , where  $S_{\mathfrak{p}}$  is  $\mathfrak{O}_k - \mathfrak{p}$ .

LEMMA 2. *Let  $p$  be a prime ideal of  $k$ . Then there exist  $\alpha, \beta \in K_1K_2$  such that*

$$(1) \quad \text{ord}_{\mathfrak{P}} \alpha = \text{ord}_{\mathfrak{P}} \beta = \text{ord}_{\mathfrak{P}} \mathfrak{D}_{K_1K_2}(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*$$

*hold for all prime ideals  $\mathfrak{P}$  of  $K_1K_2$  above  $p$ ,*

$$(2) \quad \alpha(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_{\mathfrak{P}} \subset (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_{\mathfrak{P}}^* \subset \beta(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_{\mathfrak{P}}.$$

*Proof.* Firstly we prove the existence of  $\alpha$  satisfying the condition. Let  $\mathfrak{p}$  be a prime ideal of  $K_2$  above  $p$ . Let  $z$  be an element of  $\mathfrak{D}_{K_2}$  with (iii) in 2.1. Since the dual module of  $\mathfrak{D}_k[z]$  with respect to  $\text{Tr}_{K_2/k}$  is  $f'_z(z)^{-1}\mathfrak{D}_k[z]$ , we have

$$f'_z(z)^{-1}\mathfrak{D}_k[z] \supset \mathfrak{D}_{K_2/k}^{-1}.$$

We take  $d \in \mathfrak{D}_{K_2/k}$  which satisfies  $\text{ord}_{\mathfrak{q}} d = \text{ord}_{\mathfrak{q}} \mathfrak{D}_{K_2/k}$  for all prime ideals  $\mathfrak{q}$  of  $K_2$  above  $p$ . Then we have

$$df'_z(z)^{-1}\mathfrak{D}_k[z]_{\mathfrak{P}} \supset \mathfrak{D}_{K_2, \mathfrak{P}}.$$

So we have

$$df'_z(z)^{-1}\mathfrak{D}_{K_1}[z]_{\mathfrak{P}} \supset (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_{\mathfrak{P}}.$$

By taking dual, we get

$$g'_z(z)^{-1}d^{-1}f'_z(z)\mathfrak{D}_{K_1}[z]_{\mathfrak{P}} \subset (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_{\mathfrak{P}}^*.$$

We put  $\alpha_{\mathfrak{p}} = g'_z(z)^{-1}d^{-1}f'_z(z)$ . We take  $\gamma_{\mathfrak{p}}$  which satisfies the conditions in Lemma 1, where  $S$  is the set of all prime ideals of  $K_1K_2$  above  $p$  and  $n$  is sufficiently large. We put

$$\alpha = \sum_{\mathfrak{p}|p} \alpha_{\mathfrak{p}}\gamma_{\mathfrak{p}}.$$

$\alpha$  satisfies the condition of Lemma 2. In fact, for a prime ideal  $\mathfrak{P}$  of  $K_1K_2$  and a prime ideal  $\mathfrak{p}$  of  $K_2$  with  $\mathfrak{P}|\mathfrak{p}|p$ , we have

$$\text{ord}_{\mathfrak{P}} \alpha = \text{ord}_{\mathfrak{P}} \alpha_{\mathfrak{p}}\gamma_{\mathfrak{p}} = \text{ord}_{\mathfrak{P}} \alpha_{\mathfrak{p}} = \text{ord}_{\mathfrak{P}} \mathfrak{D}_{K_1K_2}(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*.$$

And clearly  $\alpha$  satisfies (2), Lemma 2.

Secondly we prove the existence of  $\beta$ . For a prime ideal  $\mathfrak{p}$  of  $K_2$  above  $p$  we take an element  $z$  of  $\mathfrak{D}_{K_2}$  with (iii) in 2.1. We put  $\beta_{\mathfrak{p}} = g'_z(z)$ . By taking dual of  $\mathfrak{D}_{K_1}[z] \subset \mathfrak{D}_{K_1}\mathfrak{D}_{K_2}$ , we have

$$\beta_{\mathfrak{p}}(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* \subset \mathfrak{D}_{K_1}[z].$$

Therefore we have

$$(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* \left( \sum_{\mathfrak{p}|p} \beta_{\mathfrak{p}} \mathfrak{D}_{K_1}\mathfrak{D}_{K_2} \right) \subset \mathfrak{D}_{K_1}\mathfrak{D}_{K_2}.$$

For a prime ideal  $\mathfrak{p}$  of  $K_2$  we take  $\gamma_{\mathfrak{p}}$  satisfying the condition in Lemma 1, where  $S$  is the set of all prime ideals of  $K_1K_2$  above  $p$  and  $n$  is sufficiently large. We put

$$\beta = 1 / \sum_{\mathfrak{p}|p} \beta_{\mathfrak{p}} \gamma_{\mathfrak{p}}.$$

$\beta$  satisfies the conditions of Lemma 2. In fact, for a prime ideal  $\mathfrak{P}$  of  $K_1K_2$  and a prime ideal  $\mathfrak{p}$  of  $K_2$  with  $\mathfrak{P}|\mathfrak{p}|p$ , we have

$$\text{ord}_{\mathfrak{P}} \beta = -\text{ord}_{\mathfrak{P}} \beta_{\mathfrak{p}} \gamma_{\mathfrak{p}} = -\text{ord}_{\mathfrak{P}} \beta_{\mathfrak{p}} = \text{ord}_{\mathfrak{P}} \mathfrak{D}_{K_1K_2} (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^*.$$

And clearly  $\beta$  satisfies (2), Lemma 2. Thus we proved Lemma 2.

Let  $p$  be a prime ideal of  $k$  and  $\alpha, \beta$  elements of  $K_1K_2$  satisfying (1), (2) in Lemma 2. Since

$$\begin{aligned} & [\mathfrak{D}_{K_1K_2} (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* : \alpha (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{D}_{k,p}} \\ &= [\mathfrak{D}_{K_1K_2} (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* : \beta (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{D}_{k,p}} \\ &= [\mathfrak{D}_{K_1K_2,p} : (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{D}_{k,p}}, \end{aligned}$$

we have

$$\alpha (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p = (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p^* = \beta (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p.$$

So we have

$$\begin{aligned} & ([(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})^* : \mathfrak{D}_{K_1}\mathfrak{D}_{K_2}]_{\mathfrak{D}_{K_1}})_p \\ &= [(\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p^* : (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{D}_{K_1,p}} \\ &= [\alpha (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p : (\mathfrak{D}_{K_1}\mathfrak{D}_{K_2})_p]_{\mathfrak{D}_{K_1,p}} \\ &= N_{K_1K_2/K_1} (\alpha^{-1}) \mathfrak{D}_{K_1,p} \\ &= N_{K_1K_2/K_1} \left( \prod_{\sigma \neq 1} e_{\sigma|K_2} \right) \mathfrak{D}_{K_1,p}. \end{aligned}$$

Hence we get

$$([\mathfrak{D}_{K_1}\mathfrak{D}_{K_2}]_{\mathfrak{D}_{K_1}})^* : \mathfrak{D}_{K_1}\mathfrak{D}_{K_2}]_{\mathfrak{D}_{K_1}} = N_{K_1K_2/K_1} \left( \prod_{\sigma \neq 1} e_{\sigma|K_2} \right).$$

### §3. Examples

**3.1.** Let  $k$  be a number field and  $K$  its finite Galois extension with

Galois group  $G$ . Let  $K_1, K_2$  be intermediate fields of  $K/k$ . We denote by  $H_1, H_2$  the subgroups of  $G$  corresponding to  $K_1, K_2$  respectively. We define  $\sum_{K_1, K_2}$ , a subset of  $G$ , by  $H_1 H_2 - H_1 - H_2$ , where  $H_1 H_2$  is  $\{h_1 h_2 : h_1 \in H_1, h_2 \in H_2\}$ . Then the conductor  $f_{K_1, K_2}$  of  $\mathfrak{D}_{K_1} \mathfrak{D}_{K_2}$  with respect to  $\mathfrak{D}_{K_1 K_2}$  is  $\prod_{\sigma \in \sum_{K_1, K_2}} e_\sigma$ , where  $e_\sigma$  is  $(x - x^\sigma : x \in \mathfrak{D}_K)$ . This can be proved by fundamental properties of elements. From this fact we know that  $\mathfrak{D}_{K_1 K_2} = \mathfrak{D}_{K_1} \mathfrak{D}_{K_2}$  holds if and only if for any prime ideal  $\mathfrak{P}$  of  $K$  and any  $\sigma \in \sum_{K_1, K_2}$ ,  $\sigma$  is not contained in the inertia group of  $\mathfrak{P}$ .

**3.2.** We give some examples.

1.  $G = \text{Gal}(K/k) = \langle \sigma, \tau : \sigma^2 = \tau^2 = (\sigma\tau)^2 = 1 \rangle$ .

Let  $K_1, K_2, K_3$  be the fixed fields of  $\langle \sigma \rangle, \langle \tau \rangle, \langle \sigma\tau \rangle$  respectively. Then we have

$$\begin{aligned} \sum_{K_1, K_2} &= \{\sigma\tau\}, \\ f_{K_1, K_2} &= e_{\sigma\tau} = \mathfrak{D}_{K/K_3}. \end{aligned}$$

$\mathfrak{D}_K = \mathfrak{D}_{K_1} \mathfrak{D}_{K_2}$  holds if and only if  $K/K_3$  is unramified.

2.  $G = \text{Gal}(K/k) = \langle \sigma, \tau : \sigma^3 = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$ .

Let  $K_i$  be the fixed field of  $\langle \tau\sigma^{i-1} \rangle$  ( $i = 1, 2, 3$ ) and  $M$  the fixed field of  $\langle \sigma \rangle$ .

3.2.1  $\sum_{K_1, K_2} = \{\sigma\}$ ,

$$f_{K_1, K_2} = e_\sigma = \mathfrak{D}_{K/M}^{1/2}.$$

$\mathfrak{D}_K = \mathfrak{D}_{K_1} \mathfrak{D}_{K_2}$  holds if and only if  $K/M$  is unramified.

3.2.2  $\sum_{K_1, M} = \{\tau\sigma, \tau\sigma^2\}$ ,

$$f_{K_1, M} = e_{\tau\sigma} e_{\tau\sigma^2} = \mathfrak{D}_{K/K_2} \mathfrak{D}_{K/K_3}.$$

$\mathfrak{D}_K = \mathfrak{D}_{K_1} \mathfrak{D}_M$  holds if and only if  $M/k$  is unramified.

3.  $G = \text{Gal}(K/k) = A_4 \longleftrightarrow \text{Aut}(\{a, b, c, d\})$ .

We put  $x = (a b)(c d)$ ,  $y = (a c)(b d)$ ,  $z = (a d)(b c)$ ,  $t = (a b c)$  and  $H = \{1, x, y, z\}$ . Let  $K_1, K_2, K_3, K_4$  be the fixed fields of  $\langle t \rangle, \langle tx \rangle, \langle ty \rangle, \langle tz \rangle$  respectively. Let  $L_1, L_2, L_3$  be the fixed fields of  $\langle x \rangle, \langle y \rangle, \langle z \rangle$  respectively. Let  $M$  be the fixed field of  $H$ .

3.3.1  $\sum_{K_1, M} = \{t^2 x, t^2 y, t^2 z, tx, ty, tz\}$ .

$$f_{K_1, M} = \mathfrak{D}_{K/K_2} \mathfrak{D}_{K/K_3} \mathfrak{D}_{K/K_4}.$$

$\mathfrak{D}_K = \mathfrak{D}_{K_1} \mathfrak{D}_M$  holds if and only if  $M/k$  is unramified.

3.3.2  $\sum_{K_1, K_2} = \{x, tz, t^2 x, z\}$ ,

$$\mathfrak{f}_{K_1, K_2} = \mathfrak{D}_{K/L_1} \mathfrak{D}_{K/L_3} \mathfrak{D}_{K/K_3}^{1/2} \mathfrak{D}_{K/K_4}^{1/2}.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1} \mathfrak{O}_{K_2}$  holds if and only if  $K/k$  is unramified.

$$3.3.3 \quad \sum_{K_1, L_2} = \{t^2 y, ty\},$$

$$\mathfrak{f}_{K_1, L_2} = \mathfrak{D}_{K/K_3}^{1/2} \mathfrak{D}_{K/K_4}^{1/2}.$$

$\mathfrak{O}_K = \mathfrak{O}_{K_1} \mathfrak{O}_{L_2}$  holds if and only if  $M/k$  is unramified.

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