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GROUPS WITH A (B, N)-PAIR AND LOCALLY TRANSITIVE GRAPHS

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1. Introduction.

Let Γ be an undirected graph and G a subgroup of aut (Γ). We denote by $\partial(x, y)$ the distance between two vertices x and y, by $E(\Gamma)$ the edge set of Γ, by $V(\Gamma)$ the vertex set of Γ, by $\Gamma(x)$ the set of neighbors of the vertex x and by $G(x)^{\Gamma(x)}$ the permutation group induced by the stabilizer $G(x)$ on $\Gamma(x)$. For each $i \in N$, let $G_i(x) = \{a \mid a \in G(y)$ for every y with $\partial(x, y) \leqslant i\}$. An s-path is an ordered sequence $(x_0, \cdots, x_s)$ of $s + 1$ vertices x_i with $x_i \in \Gamma(x_{i-1})$ for $1 \leqslant i \leqslant s$ and $x_i \neq x_{i-2}$ for $2 \leqslant i \leqslant s$. For each vertex x, let $W_s(x)$ be the set of s-paths $(x_0, \cdots, x_s)$ with $x = x_0$. We say that the graph Γ is locally (G, s)-transitive if for every vertex x, $G(x)$ acts transitively on $W_s(x)$ but not on $W_{s+1}(x)$ (compare [1], [11]). If, in addition, G acts transitively on $V(\Gamma)$, then Γ is called (G, s)-transitive; otherwise Γ is bipartite with vertex blocks V_0 and V_1 and G acts transitively on both V_0 and V_1, assuming that Γ is connected and $s \geqslant 1$.

Now let G be a finite group with a (B, N)-pair whose Weyl group is a dihedral group D_{2n} of order $2n$ ($n \geqslant 2$) and Γ be the incidence graph of the associated coset geometry as defined in [3, p. 129] (or [2, (15. 5. 1)]). The graph Γ has the following properties:

(A) $V(\Gamma) = V_0 \cup V_1$ with $V_0 \cap V_1 = \varnothing$ and $\Gamma(x) \subseteq V_{1-i}$ for every vertex $x \in V_i$ ($i = 0$ and 1). For $i = 0$ and 1 there exists a $d_i \in N$ such that $|\Gamma(x)| = d_i + 1$ for every vertex $x \in V_i$. The diameter of Γ is n and the girth $2n$.

(B) Γ is locally $(G, n + 1)$-transitive.

A generalized n-gon of order (d_0, d_1) is, by definition, an incidence structure whose incidence graph has the properties listed in (A).

W. Feit and G. Higman have shown in [3] that finite generalized n-gons of order (d_0, d_1) with $d_0 d_1 > 1$ exist only for $n = 2, 3, 4, 6, 8$ and 12, that

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$n = 8$ is possible only when the squarefree part of $d_0 d_1$ is equal to two and that $n = 12$ is possible only when d_0 or $d_1 = 1$. The only known finite groups with a (B, N)-pair whose Weyl group is isomorphic to D_{2n} with $n = 3$ (resp. $n = 4$, $n = 6$) and whose generalized n-gon is of order (d_0, d_1) with $d_0 = d_1$ are (essentially) the Chevalley groups $A_2(q)$ (resp. $B_2(q)$, $G_2(q)$) (with $q = d_0$). Let $\Gamma_{n,q}$ denote the corresponding graph.

We prove here the following theorems:

(1.1) *Let p be a prime, r and $s \in N$ with $r \geqslant 1$ and $s \geqslant 2$ and $q = p^r$. Let Γ be a finite undirected connected graph regular of valency $q + 1$ and G a subgroup of* aut (Γ) *such that Γ is locally (G, s)-transitive and $PSL(2, q) \lneqq G(x)^{\Gamma(x)} \lneqq P\Gamma L(2, q)$ for every vertex x. Then $s \leqslant 5$ or $s = 7$. Let $(x_1, \cdots, x_s)$ be an arbitrary $(s-1)$-path. Then $G_1(x_1) = 1$ if $s = 2$ and $G_1(x_1) \cap G_1(x_2) \cap G(x_3) \cap \cdots \cap G(x_s) = 1$ otherwise.*

(1.2) *Let Γ, G, etc. be as is* (1.1) *with $q \geqslant 3$ and $s \in \{4, 5, 7\}$. In addition, suppose that $s \neq 5$ if $q = 3$. Let $H_{3,q} = A_2(q)$, $H_{4,q} = B_2(q)$, $H_{6,q} = G_2(q)$ and $G_{n,q} =$* aut $(\Gamma_{n,q}) \cong$ aut $(H_{n,q})$ *for $n = 3, 4, 6$; $H_{n,q}$ is to be considered as a subgroup of $G_{n,q}$. Let $k = \{x, y\}$ be an edge of Γ, $\Delta_i = \{w \in V(\Gamma) \mid \partial(i, w) \leqslant s - 2\}$ for $i = x$ and y and $\Delta = \Delta_x \cup \Delta_y$. Then there exists a bijective map $\varphi\colon \Delta \to V(\Gamma_{s-1,q})$ mapping edges onto edges such that*:

(a) *For $i = x$ and y and for each $g \in G(i)$ (resp. $g \in G(k)$, where $G(k)$ is the stabilizer in G of the unordered pair $\{x, y\}$), there exists a unique element $h \in G_{s-1,q}((i)\varphi)$ (resp. $h \in G_{s-1,q}((k)\varphi)$) such that $(w)h = (w)\varphi^{-1}g\varphi$ for every $w \in (\Delta_i)\varphi$ (resp. $w \in (\Delta)\varphi = V(\Gamma_{s-1,q})$).*

(b) *For $i = x$ and y and for each $h \in H_{s-1,q}((i)\varphi)$ (resp. $h \in H_{s-1,q}((k)\varphi)$), there exists a unique element $g \in G(i)$ (resp. $g \in G(k)$) such that $(w)h = (w)\varphi^{-1}g\varphi$ for each $w \in (\Delta_i)\varphi$ (resp. $w \in (\Delta)\varphi$).*

In particular, $H_{s-1,q}((i)\varphi) \lneqq G(i) \lneqq G_{s-1,q}((i)\varphi)$ for $i = x$ and y and $H_{s-1,q}((k)\varphi) \lneqq G(k) \lneqq G_{s-1,q}((k)\varphi)$.

In the following theorem, $\hat{G}_{4,2}$ denotes the unique subgroup of aut $(\Gamma_{4,2}) \cong P\Gamma L(2, 9)$ isomorphic to $PGL(2, 9)$. The reader can check that $\Gamma_{4,2}$ is $(\hat{G}_{4,2}, 4)$-transitive.

(1.3) *Let Γ, G, etc. be as in* (1.1) *with $q = 2$ and $s \in \{4, 5, 7\}$. Let (X, Y) be an arbitrary 1-path of $\Gamma_{s-1,2}$. Then there exists a map $\varphi\colon k = \{x, y\} \to \{X, Y\}$ such that $H_{s-1,2}((i)\varphi) \cong G(i)$ for $i = x$ and y. Either $H_{s-1,2}((k)\varphi) \lneqq G(k) \lneqq G_{s-1,2}((k)\varphi)$ or $s = 4$ and $G(k) \cong \hat{G}_{4,2}(K)$ where K is any edge of*

$\Gamma_{4,2}$.

In the first part of the proof of (1.1) we show that $s \leqslant 5$ or $s \in \{7, 9, 13\}$. Note the remarkable coincidence with the numbers $n = 2, 3, 4, 6, 8$ and 12 obtained in [3] as the solution to a completely different sort of problem. To exclude $s = 9$ when $p = 2$ and $q \geqslant 4$ we construct a generalized 8-gon of order (q, q), thus obtaining a contradiction from [3]. To proceed in the case $q \equiv 3 \pmod 4$, we require [6, (8.2.11)] in order to prove that $PGL(2, q) \lneqq G(x)^{\Gamma(x)}$ for some vertex x. In the proof of (1.2) we use the characterizations of the graphs $\Gamma_{n,q}$ given in [5, Theorem 1.8], [7, Theorem 2] and [12, (4.4)]. Otherwise, the arguments contained in this paper are elementary and self-contained.

When proving (1.2), we include the case that $q = 3$ and $s = 5$, making the additional assumption that $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ for every vertex x. The conclusion reached is that $G_{4,3}(X)$ induces $PGL(2, 3)$ on $\Gamma_{4,3}(X)$ for every vertex X of $\Gamma_{4,3}$. Since this is not so, it follows that $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ does not hold for all vertices x of Γ when $q = 3$ and $s = 5$; in particular, G cannot act transitively on $V(\Gamma)$.

Theorems (1.2) and (1.3) imply that $G_{s-1,q}((k)\varphi)$ contains an element exchanging $(x)\varphi$ and $(y)\varphi$ if $G(k)$ contains an element exchanging x and y. Thus $\Gamma_{s-1,q}$ is $(G_{s-1,q}, s)$-transitive if Γ is (G, s)-transitive. For $n = 4$ and 6, $\Gamma_{n,q}$ is $(G_{n,q}, n + 1)$-transitive if and only if $p = n/2$ (see [2]). Hence we have the following corollary:

(1.4) *Let Γ, G, etc. be as in* (1.1). *If G acts transitively on $V(\Gamma)$ (i.e., if Γ is (G, s)-transitive), then $p = 2$ if $s = 5$ and $p = 3$ if $s = 7$.*

For other relevant results consult [4] and [9] where, however, completely different methods are used from those developed here.

2. **Proof of** (1.1): $s \in \{2, 3, 4, 5, 7, 9, 13\}$

Let Γ and G satisfy the hypotheses of (1.1). If $W = (x_0, \cdots, x_t)$ is any t-path (t arbitrary), we set $G(W) = G(x_0, \cdots, x_t) = G(x_0) \cap \cdots \cap G(x_t)$ and $G_i(W) = G_i(x_0, \cdots, x_t) = G_i(x_0) \cap \cdots \cap G_i(x_t)$ for each $i \in N$. If $b \in G(x)$, x a vertex, we denote by $|b|_x$ the order of the permutation that b induces on $\Gamma(x)$. We will often use integers to denote vertices of Γ.

For each vertex x, let $\bar{G}(x)$ be the largest subgroup of $G(x)$ such that $\bar{G}(x)^{\Gamma(x)} \lneqq PGL(2,q)$ and $f_x = [\bar{G}(x) \cap G(y, x, z): G_1(x)]$ where y and z are any two neighbors of x. A t-path $(0, \cdots, t)$ will be called good if $[G(W) \cap \bar{G}(i): G(W) \cap G_1(i)] = f_i$ for each i with $1 \leqslant i \leqslant t-1$.

(2.1) *If $W = (0, \cdots, t)$ is a good t-path, then there exists a vertex $t+1$ such that $(0, \cdots, t, t+1)$ is a good $(t+1)$-path.*

Proof. Clearly all 1- and 2-paths are good, so that we can assume $t \geqslant 2$. Let $W_1 = (1, \cdots, t)$. By induction, there exists an element $b_t \in G(W_1) \cap \bar{G}(t)$ with $|b_t|_t = f_t$ and $(p, |b_t|) = 1$. For $1 \leqslant i \leqslant t-1$ there exists an element $b_i \in G(W) \cap \bar{G}(i)$ with $|b_i|_i = f_i$ and $(p, |b_i|) = 1$. The subgroup $\langle b_1, b_t \rangle$ contains an element c with $c^{-1}b_t c = b_t^c \in G(0)$. Let $a_t = b_t^c$ and $t+1$ be a fixed point of a_t in $\Gamma(t) - \{t-1\}$. For each i with $1 \leqslant i \leqslant t - 1$ there exists an element $c_i \in \langle b_i, a_t \rangle$ with $a_i = c_i^{-1} b_i c_i \in G(t+1)$. For $1 \leqslant i \leqslant t$ we have $a_i \in G(0, \cdots, t, t+1) \cap \bar{G}(i)$ and $|a_i|_i = f_i$. □

(2.2) *Every s-path is good. If $(0, \cdots, s, s+1)$ is a good $(s+1)$-path, then $G(0, \cdots, s) \leqslant G(s+1)$. If $f_s \neq 1$, then $s+1$ is the only vertex in $\Gamma(s) - \{s-1\}$ such that $(0, \cdots, s, s+1)$ is good.*

Proof. For every vertex x there exists, according to (2.1), at least one good s-path beginning at x. Since $G(x)$ acts transitively on $W_s(x)$, the first claim follows. Let $a \in G(0, \cdots, s, s+1) \cap \bar{G}(s)$ be an element with $|a|_s = f_s$. Suppose there exists an element $b \in G(0, \cdots, s) - G(s+1)$. Then $\langle a, b \rangle \leqslant G(0, \cdots, s)$ acts transitively on $\Gamma(s) - \{s-1\}$ (for if $f_s = 1$, then $\langle b \rangle$ itself must act transitively on $\Gamma(s) - \{s-1\}$), contradicting the hypothesis that $G(0)$ acts intransitively on $W_{s+1}(0)$. In particular, if $(0, \cdots, s, y)$ is a good path and $f_s \neq 1$, then there exists an element in $G(0, \cdots, s)$ whose only fixed point in $\Gamma(s) - \{s-1\}$ is y; thus $y = s+1$. □

If we take any 1-path and start extending it to an arbitrarily long good path, the resulting path, since Γ is finite, contains, after a while, no new vertices. Thus we may choose, once and for all, an infinitely long path $W = (\cdots, -1, 0, 1, 2, \cdots)$ such that for each i there exists an element $h_i \in G(W) \cap \bar{G}(i)$ with $|h_i|_i = f_i$.

(2.3) $G_1(1) = 1$ *if* $s = 2$ *and* $G_1(1, 2) \cap G(0, \cdots, s) = 1$ *otherwise.*

Proof. Let $A = G_1(1, 2) \cap G(0, \cdots, s)$. Since $h_s \in G(1, \cdots, s)$, $G(1, \cdots, s)$ acts primitively on $\Gamma(s) - \{s-1\}$. Since $G_1(1) \cap G(1, \cdots, s) \trianglelefteq G(1, \cdots, s)$ and $G_1(1) \cap G(1, \cdots, s) \leqslant G(0, \cdots, s)$ acts intransitively on $\Gamma(s) - \{s-1\}$, we have $G_1(1) \cap G(1, \cdots, s) \leqslant G_1(s)$ and in particular $G_1(1) \leqslant A$ if $s = 2$.

Similarly, $G_1(s) \cap G(s, \cdots, 2, x_2) \leqslant G_1(x_2)$, $x_2 \in \Gamma(2) - \{1, 3\}$ arbitrary. By (2.2), $G(0, \cdots, s) \leqslant G(-s+4, \cdots, s)$ and thus $A \leqslant G_1(x_2) \cap G(x_2, 2, 1, 0, \cdots, -s+4) \leqslant G_1(-s+4)$, hence $A \leqslant G_1(-s+4) \cap G(-s+4, \cdots, 3) \leqslant G_1(3)$. Choose any $y \in \Gamma(s) - \{s-1\}$. Then $A \leqslant G_1(2) \cap G(2, \cdots, s, y) \leqslant G_1(y)$, $A \leqslant G_1(y) \cap G(y, s, \cdots, 3, x_3)$, $x_3 \in \Gamma(3) - \{2, 4\}$ arbitrary, thus $A \leqslant G_1(x_3) \cap G(x_3, 3, 2, \cdots, -s+5) \leqslant G_1(-s+5)$, $A \leqslant G_1(-s+5) \cap G(-s+5, \cdots, 4) \leqslant G_1(4)$. Also, $A \leqslant G_1(3) \cap G(3, \cdots, s, y) \cap G_1(y) \leqslant G_1(z)$, $z \in \Gamma(y) - \{s\}$ arbitrary. It should now be clear that $A \leqslant G_1(1, \cdots, s, y, z, \cdots, w)$ for every path $(1, \cdots, s, y, z, \cdots, w)$ of arbitrary length beginning with $(1, \cdots, s)$. Since Γ is connected, it follows that $A = 1$. □

To prove (1.1), we have only to show now that $s \leqslant 5$ or $s = 7$. From now on we assume that $s \geqslant 3$.

(2.4) *$G_1(1, 2)$ is a p-group. For each $t \geqslant 3$ and each i with $1 \leqslant i \leqslant t-2$, $G_1(i, i+1) \cap G(0, \cdots, t) = G_1(1, \cdots, t-1)$.*

Proof. By (2.3), $G_1(1, 2)$ acts semi-regularly on the set of s-paths beginning with $(0, \cdots, 3)$ and thus $|G_1(1, 2)| \,|\, q^{s-3}$. To prove the second claim, we note that $G_1(1, 2) \trianglelefteq G_1(2) \trianglelefteq G(2, 3)$ and thus $G_1(1, 2)^{\Gamma(3)} \leqslant O_p(G(2, 3)^{\Gamma(3)})$ so that $G_1(1, 2) \cap G(4) = G_1(1, 2, 3)$. □

(2.5) *If $2 \leqslant t \leqslant s-1$, then $G_1(1, \cdots, t-1)$ acts transitively on $\Gamma(t) - \{t-1\}$.*

Proof. Let x_1 and x_2 be any two vertices in $\Gamma(t) - \{t-1\}$. There exists an element $a_i \in G(0, \cdots, t, x_i) \cap \overline{G}(t)$ with $|a_i|_t = f_t$ $(i = 1, 2)$. If $f_t \neq 1$, the commutator group $\langle a_1, a_2 \rangle' \leqslant \overline{G}(0, \cdots, t)$ of $\langle a_1, a_2 \rangle \leqslant G(0, \cdots, t)$, therefore any p-Sylow group of $\langle a_1, a_2 \rangle'$ and therefore $G_1(1, \cdots, t-1)$ act transitively on $\Gamma(t) - \{t-1\}$. If $f_t = 1$, then $q \leqslant 3$ so that $G_1(1, \cdots, t-1) \in \mathrm{Syl}_p(G(0, \cdots, t))$ and the claim follows directly from the fact that $G(0, \cdots, t)$ acts transitively on $\Gamma(t) - \{t-1\}$. □

From now on, we set $m = (s/2) - 1$ when s is even and $m = (s-3)/2$ when s is odd.

(2.6) *If $s \geqslant 4$, then $ZO_p(G(0, 1)) \leqslant G_m(0, 1)$,* where *$ZO_p(G(0, 1))$ denotes the center of $O_p(G(0, 1))$.*

Proof. By (2.5), $G_1(0, 1) \neq 1$ and thus $O_p(G(0, 1)) \neq 1$. Let b be a nontrivial element in $ZO_p(G(0, 1))$. If $w \in \Gamma(1) - \{0\}$ is arbitrary, then $G_1(1, w) \leqslant O_p(G(0, 1))$ and thus $G_1(1, w) = G_1(1, (w)b)$, so that $b \in G(w)$ by (2.5).

Thus $b \in G_1(1)$ and similarly, $b \in G_1(0)$. Since $G_1(0,1) \cap G(0,\cdots,s-1) = 1$, there exists an $n < s$ such that $b \in G(0,\cdots,n) - G(n+1)$. By (2.5), there exists a nontrivial element $a \in G_1(1,\cdots,s-2) \leqslant O_p(G(0,1))$. Since $b \in ZO_p(G(0,1))$, we have $a \in G_1(s-2, s-3, \cdots, n, (n+1)b, \cdots, (s-2)b)$. By (2.3), the length of the path $(s-2, s-3,\cdots, n, (n+1)b, \cdots, (s-2)b)$ is at most $s-3$. Therefore $s-1 \leqslant 2n$. $\square$

(2.7) *Suppose* $s \notin \{2,3,4,5,7\}$. *Then* s *is odd,* $ZO_p(G(0,1)) \leqslant G_{m+1}(0)$ *or* $ZO_p(G(0,1)) \leqslant G_{m+1}(1)$ *and* G *operates intransitively on the vertex set* $V(\Gamma)$.

Proof. We assume first that there exists an element $b \in ZO_p(G(0,1)) - G_1(m+1)$. Then $[b, ZO_p(G(m+1, m+2))] \leqslant G_1(-m+2,\cdots,2m)$ because of (2.6). Since $s \notin \{2,3,4,5,7\}$, the length of $(-m+2,\cdots,2m)$ is at least $s-2$. By (2.3), it follows that $[b, ZO_p(G(m+1,m+2))] = 1$ and therefore $ZO_p(G(m+1,m+2)) = ZO_p(G(m+1,(m+2)b))$, so that $ZO_p(G(m+1, m+2)) \trianglelefteq \langle G(m+1, m+2), G(m+1, (m+2)b)\rangle = G(m+1)$. By (2.6), we have $ZO_p(G(m+1,m+2)) \leqslant G_{m+1}(m+1)$.

On the other hand, if $ZO_p(G(0,1)) \leqslant G_1(m+1)$, then $ZO_p(G(0,1)) \leqslant G_{m+1}(1)$ since $ZO_p(G(0,1)) \trianglelefteq G(0,1)$ and $G(0,1)$ acts transitively on the set of $(m+1)$-paths beginning with $(0,1)$. Therefore $ZO_p(G(0,1)) \leqslant G_{m+1}(u)$ for $u=0$ or 1.

Suppose that $ZO_p(G(0,1)) \leqslant G_{m+1}(0)$. Since $G_{m+1}(0) \leqslant G_1(-m,\cdots,m)$, we have $2m \leqslant s-3$ so that s is odd and $G_{m+1}(0) \cap G_1(m+1) = 1$. If G contains an element c which exchanges 0 and 1, then $ZO_p(G(0,1)) = ZO_p(G(0,1))^c \leqslant G_{m+1}(0)^c = G_{m+1}(1)$ and thus $ZO_p(G(0,1)) \leqslant G_{m+1}(0) \cap G_1(m+1) = 1$, a contradiction. Therefore G acts intransitively on $V(\Gamma)$. $\square$

(2.8) $s \in \{2,3,4,5,7,9,13\}$.

Proof. We may assume that s is odd, $s \geqslant 9$ and $G_{m+1}(0) \neq 1$. Since $G_{m+1}(0) \neq 1$, $G_{m+1}(i) \neq 1$ for every even i. There exists an element $c \in G_{m+1}(0) - G_1(m+1)$. Suppose first that $s \equiv 3 \pmod 4$ and thus $G_{m+1}(m+2) \neq 1$. Since $[c, G_{m+1}(m+2)] \leqslant G_1(-m+2,\cdots,2m) - G_1(2m+1)$, we have $3m-2 \leqslant s-3$, hence $s \leqslant 7$. Therefore $s \equiv 1 \pmod 4$. It follows that $G_{m+1}(m+3) \neq 1$ and thus $[c, G_{m+1}(m+3)] \leqslant G_1(-m+4,\cdots,2m-1) - G_1(2m)$ so that $3m-5 \leqslant s-3$, hence $s \leqslant 13$. $\square$

Before going on to § 3, we prove more lemmas needed later.

(2.9) *If* $s \in \{5,7,9,13\}$, *then* $G_{m+1}(u) \leqslant ZO_p(G(0,1))$ *for* $u = 0$ *and* 1 *and*

$G_{m+1}(u) \neq 1$ *for* $u = 0$ *or* 1 (*or both*); *if* $G_{m+1}(u) \neq 1$, *then* $|G_{m+1}(u)| = q$.

Proof. Let $u = 0$ or 1. Since $G_{m+1}(u) \trianglelefteq O_p(G(0, 1))$, either $ZO_p(G(0, 1)) \cap G_{m+1}(u) \neq 1$ or $G_{m+1}(u) = 1$. Suppose that $ZO_p(G(0, 1)) \cap G_{m+1}(u)$ contains a nontrivial element b. Then $G_{m+1}(u) = \langle h_{m+u+1}^{-j} b h_{m+u+1}^{j} \,|\, 0 \leqslant j < f_{m+u+1} \rangle$ since $G_{m+1}(u) \cap G_1(m + u + 1) = 1$ and $G_{m+1}(u)^{\Gamma(m+u+1)} \leqslant O_p(G(m + u, m + u + 1)^{\Gamma(m+u+1)})$. It follows that $|G_{m+1}(u)| = q$ and $G_{m+1}(u) \leqslant ZO_p(G(0, 1))$ since h_{m+u+1} normalizes $ZO_p(G(0, 1))$.

It remains only to show that $G_{m+1}(u) \neq 1$ for $u = 0$ or 1. Thus we suppose instead that $G_{m+1}(x) = 1$ for every vertex x. By (2.7), $s = 5$ or 7. Let $s = 5$. Then $ZO_p(G(3, 4)) \leqslant G(2) - G_1(2)$ since otherwise $ZO_p(G(3, 4)) \leqslant G_2(3)$. Since h_2 normalizes $ZO_p(G(3, 4))$, $ZO_p(G(3, 4))$ acts transitively on $\Gamma(2) - \{3\}$. Since $ZO_p(G(3, 4))$ centralizes $G_1(1, 2, 3)$, we have $G_1(1, 2, 3) \leqslant G_2(2) = 1$, in contradiction to (2.5). Thus $s = 7$ and $ZO_p(G(i, i + 1))$ acts transitively on $\Gamma(i + 3) - \{i + 2\}$ for every i. Since $ZO_p(G(1, 2))$ centralizes $G_1(1, \cdots, 5)$, we have $G_1(1, \cdots, 5) \leqslant G_2(4)$. Since $ZO_p(G(0, 1))$ centralizes $G_1(1, \cdots, 5)$, it follows that $G_1(1, \cdots, 5) \leqslant G_3(3) = 1$, again a contradiction. □

Thus we may suppose, from now on, that $G_{m+1}(i) \neq 1$ for every even i whenever $s \in \{5, 7, 9, 13\}$.

(2.10) *Let* $s \in \{5, 7, 9, 13\}$ *and* $p = 2$. *Then there exists an element* $a \in G_1((s - 1)/2) \cap G(0, \cdots, 2(s - 1)) \cap G_1(3(s - 1)/2)$ with $|a|_{s-1} = q - 1$.

Proof. We may suppose that $q \neq 2$. Let x_1 and x_2 be any two vertices in $\Gamma(s - 1) - \{s - 2, s\}$. By (2.5), there exists for $j = 1, 2$ an element $g_j \in O_2(G(x_j, s - 1))$ such that $(i)g_j = 2(s - 1) - i$ for $s - 1 \leqslant i \leqslant 2(s - 1)$. Since $O_2(G(x_j, s - 1))$ induces an elementary abelian 2-group on $\Gamma(s - 1)$, we have $(s - 2)g_j = s$. Therefore both $(0, \cdots, 2(s - 1))$ and $(2(s - 1), \cdots, 0)g_j = (0, \cdots, s, (s - 3)g_j, \cdots, (0)g_j)$ are good paths. By (2.2), $(i)g_j = 2(s - 1) - i$ also for $0 \leqslant i \leqslant s - 3$. Let $a = g_1 g_2$. Then $|a|_{s-1} = q - 1$. By (2.9), $G_{m+1}(s - 1) \leqslant ZO_2(G(x_1, s - 1)) \cap ZO_2(G(x_2, s - 1))$ and thus $[a, G_{m+1}(s - 1)] = 1$. Since $s - 1$ is even, $G_{m+1}(s - 1)$ acts transitively on $\Gamma((s - 1)/2) - \{(s + 1)/2\}$. Since $a \in G((s - 3)/2)$, $a \in G_1((s - 1)/2)$. Similarly, $a \in G_1(3(s - 1)/2)$. □

It is in the proof of the next lemma that we require [6, (8.2.11)].

(2.11) *If* $p \neq 2$ *and* $s \geqslant 4$, *then* $|\bar{G}(W)| = |\bar{G}(\cdots, -1, 0, 1, 2, \cdots)|$ *is even.*

Proof. We first suppose that we can choose $u \in \{0, 1\}$ such that f_u is even. The reader should check the following simple fact:

(*) Let $q - 1 = 2^k w$ with w odd. If σ is an arbitrary element in the stabilizer $P\Gamma L(2, q)_\infty$ of $\infty \in PG(1, q)$ but not in $PGL(2, q)$ whose order is a power of two, then $|\sigma| \| 2^k$ and either $k = 2$, $|\sigma| = 4$ and $\sigma^2 \in PGL(2, q)$ or $k \geqslant 3$ and $\sigma^{2^{k-2}} \in PGL(2, q)$.

We choose an odd $n \in N$ such that $|h_u|/n$ is a power of two. It follows from (*) that $h_u^{nf_u/2}$ or $h_u^{nf_u} \in \bar{G}(W) - \{1\}$.

It remains to show that f_u is even for $u = 0$ or 1. To show this, it will be necessary to make only a few minor changes in the proof of [8, (6.3)]: Suppose that both f_0 and f_1 are odd. Then $q \equiv 3 \pmod 4$, $\bar{G}(u)^{\Gamma(u)} \cong PSL(2, q)$ for $u = 0$ and 1 and $|G(0, 1)|$ is odd. Thus a 2-Sylow group of $\bar{G}(u)$ is isomorphic to a 2-Sylow group of $PSL(2, q)$, so that $\bar{G}(u)$ is p-stable for $u = 0$ and 1 (see [6, (2.8.3), (8.1.2)]). Let $u = 0$ or 1 and $C = C_{\bar{G}(u)}(O_p(\bar{G}(u)))$, the centralizer of $O_p(\bar{G}(u))$ in $\bar{G}(u)$, and $c \in C$. Let $w \in \Gamma(u)$. Since $G_1(u, w) \leqslant O_p(\bar{G}(u))$, we have $G_1(u, w) = G_1(u, (w)c)$. By (2.5) and the hypothesis $s \geqslant 4$, $G_1(u, w) \not\leqslant G_1(z)$ for $z \in \Gamma(u) - \{w\}$. Therefore $c \in G_1(u)$, since w was arbitrary. Now let z and w be any two neighbors of u. Since $G_1(u, w)^{\Gamma(z)} = O_p(G(u, z)^{\Gamma(z)})$, we have $C^{\Gamma(z)} \leqslant O_p(G(u, z)^{\Gamma(z)})$. Therefore we can find elements $d \in G_1(u, w)$ and $e \in G_1(u, z)$ such that $cd = e$ and thus $c = ed^{-1} \in O_p(\bar{G}(u))$, so that $C \leqslant O_p(\bar{G}(u))$. Thus $O_{p'}(\bar{G}(u)) = 1$ and $\bar{G}(u)$ is p-constrained (see [6, p. 268]).

Let $S \in \mathrm{Syl}_p(\bar{G}(0))$. By [6, (8.2.11)], we have $J(S) \trianglelefteq \bar{G}(0)$. We may assume that $S \leqslant \bar{G}(1)$ and thus $S \in \mathrm{Syl}_p(\bar{G}(1))$. Therefore $J(S) \trianglelefteq \langle \bar{G}(0), \bar{G}(1) \rangle$. Since Γ is connected, $\langle \bar{G}(0), \bar{G}(1) \rangle$ acts transitively on the set of edges of Γ and thus $J(S) = 1$, a contradiction. □

(2.12) *If* $s = 3$, *then* $q(q-1)/(q-1, 2) \,|\, |G_1(u) \cap \bar{G}(1-u)|$ *for* $u = 0$ *and* 1.

Proof. Let $u = 0$ or 1 and $A = \langle G_1(w) \,|\, w \in \Gamma(u) \rangle$. Let $y \in \Gamma(u)$. Then $[G_1(u), G_1(y)] \leqslant G_1(u, y)$ and thus, by (2.3), $[A, G_1(u)] = 1$. By (2.5), $G_1(y)$ acts transitively on $\Gamma(u) - \{y\}$, so that $A^{\Gamma(u)} \gneqq PSL(2, q)$. Let a be an element in $A \cap \bar{G}(u) \cap G(y)$ such that $|a|_u = (q-1)/(q-1, 2)$ and $(|a|, p) = 1$. Since $[a, G_1(u)] \leqslant [A, G_1(u)] = 1$ and $G_1(u)$ acts transitively on $\Gamma(y) - \{u\}$, we have $a \in G_1(y)$. □

(2.13) *Let* $s = 3$, $q = 3$, $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ *and* $|G_1(x)| = 3$ *for every vertex* x. *Let* $u = 0$ *or* 1 *and* y_1 *and* y_2 *be vertices such that* $(u, u+1, u+2, y_1,$

$y_2)$ *is a good* 4-*path. Then* $(y_2, y_1, u+2, u+3, u+4)$ *is also good.*

Proof. Let $u = 0$ (the proof is the same when $u = 1$), $A = \langle G_1(w) \mid w \in \Gamma(2) \rangle$ and $B = \langle A, G_1(2) \rangle$; we have $[A, G_1(2)] = 1$ and $|B| = |B^{\Gamma(2)}| \cdot |G_1(2)| = 36$. Let $G_1(2) = \langle h \rangle$ and $g_1 = 1, g_2, \cdots, g_{12}$ be elements of B inducing different permutations on $\Gamma(2)$ which we may choose such that $|g_i| = 2$ for $2 \leqslant i \leqslant 4$. Then three divides the order of every element in $B = \{g_i h^j \mid 1 \leqslant i \leqslant 12;\ 0 \leqslant j \leqslant 2\}$ except g_i for $1 \leqslant i \leqslant 4$. Thus B contains just one 2-Sylow group S. It follows that $A = \langle S, G_1(1) \rangle$, therefore $|A| = |S| \cdot |G_1(1)| = 12$ and, in particular, $A \cap G_1(2) = 1$.

Since $(0, \cdots, 4)$ is good, there exists an involution $b \in G(0, \cdots, 4)$. For $i = 1$ and 3, there exists an element $c_i \in G_1(i)$ mapping $4 - i$ to y_1. Since $(0, \cdots, 4)c_3 = ((0)c_3, y_1, 1, 2, 3)$ is good, we may assume that $(0)c_3 \neq y_2$. On the other hand, since both $(0, \cdots, 4)c_1 = (0, 1, 2, y_1, (4)c_1)$ and $(0, 1, 2, y_1, y_2)$ are good, we have $(4)c_1 = y_2$ by (2.2). Let c be the element in $G_1(y_1)$ mapping 1 onto 3 and $d = cc_1^{-1}(c_1cc_3^{-1})^b c_3 c$. Then $d \in A \cap G_1(2) = 1$. But $b^{c_1c^{-1}} \in G(2, y_1, y_2)$ and $b^{c_3c} \in G(2, y_1, (0)c_3)$ so that $d = b^{c_1c^{-1}} b^{c_3c} \notin G_1(y_1)$, a contradiction. □

3. The case $s = 9$

Since $G_4(2) \leqslant ZO_p(G(2, 3))$ and $G_4(2)$ acts transitively on $\Gamma(6) - \{5\}$, it follows that $G_1(2, \cdots, 8) \leqslant G_2(6)$. Choose an arbitrary element $b_{10} \in G_4(10)^* = G_4(10) - \{1\}$. For any $b_5 \in G_1(2, \cdots, 8)^*$, we have $[b_5, b_{10}] \in G_1(5, \cdots, 11) - G_1(12)$, therefore $[b_5, b_{10}] \notin G_1(4)$ and hence $b_5 \notin G_1((4)b_{10}^{-1})$. Let b_2 be the element in $G_4(2)$ with $(5)b_{10}^{-1}b_2 = 7$. Since $[G_4(2), G_1(2, \cdots, 8)] = 1$, $b_5 = b_5^{b_2} \in G_1(2, \cdots, 8) - G_1((4)b_{10}^{-1}b_2)$. Thus $G_1(2, \cdots, 8) \cap G_1((4)b_{10}^{-1}b_2) = 1$.

(3.1) a) *There exist elements* $b_i \in G_1(i-3, \cdots, i+3)^*$ *for* $i = 3, 4$ *and* 5 *such that* $[b_3, b_5] = b_4$.
b) *If* $b_4 \in G_4(4)^*$ *and* $b_9 \in G_1(6, \cdots, 12)^*$, *then there exists an element* $b_6 \in G_4(6)^*$ *such that* $[b_4, b_9] = b_6$.
c) *If* $b_4 \in G_4(4)^*$ *and* $b_{10} \in G_4(10)^*$, *then there exist elements* $b_6 \in G_4(6)^*$, $b_7 \in G_1(4, \cdots, 10)$ *and* $b_8 \in G_4(8)^*$ *such that* $[b_4, b_{10}] = b_6 b_7 b_8$.
d) *If* $b_7 \in G_1(4, \cdots, 10)^*$ *and* $b_{11} \in G_1(8, \cdots, 14)^*$, *then there exist elements* $b_8 \in G_4(8)^*$, $b_9 \in G_1(6, \cdots, 12)$ *and* $b_{10} \in G_4(10)^*$ *such that* $[b_7, b_{11}] = b_8 b_9 b_{10}$.

Proof. a) We have seen that there exists a vertex $x \in \Gamma(7)$ such that $G_1(2, \cdots, 8) \cap G_1(x) = 1$. Let b_3 be the element in $G_1(0, \cdots, 6)$ such that $(8)b_3^{-1} = x$. Then $[b_3, b_5] \in G_4(4)^*$ for every $b_5 \in G_1(2, \cdots, 8)^*$. b) is left to the reader. c) We have $[b_4, b_{10}] \in G_1(5, \cdots, 9) - G_4(4) - G_4(10)$. There exist elements $b_6 \in G_4(6)^*$ and $b_8 \in G_4(8)^*$ such that $[b_4, b_{10}]b_6^{-1}b_8^{-1} \in G_1(4, \cdots, 10)$. Since $[G_4(6), G_4(8)] = [G_4(6), G_1(4, \cdots, 10)] = 1$, the claim follows. d) is now clear. □

We now suppose that $p \neq 2$. By (2,11), $\bar{G}(W)$ contains an involution a. Let $\zeta(i) = (-1)^{|a|_i+1}$ for each i.

(3.2) *For every even* i:

A) $\zeta(i) = \zeta(i-1)\zeta(i+1)$
B) $\zeta(i) = \zeta(i-2)\zeta(i+3)$
C) $\zeta(i)\zeta(i+6) = \zeta(i+2) = \zeta(i+4)$

Proof. A) Choose b_3, b_4 and b_5 as in (3.1.a). Since $G_1(2, \cdots, 8)^{\Gamma(1)} = O_p(G(1, 2)^{\Gamma(1)})$, $G_1(2, \cdots, 8)^{\Gamma(9)} = O_p(G(8, 9)^{\Gamma(9)})$ and $G_1(1, \cdots, 8) = G_1(2, \cdots, 9) = 1$, we have $b_5^a = b_5^{\zeta(1)} = b_5^{\zeta(9)}$ and, in particular, $\zeta(1) = \zeta(9)$. Similarly, $b_3^a = b_3^{\zeta(-1)} = b_3^{\zeta(7)}$ and $b_4^a = b_4^{\zeta(0)} = b_4^{\zeta(8)}$. We have $[b_3^{\zeta(-1)}, b_5^{\zeta(1)}] = [b_3, b_5]^{\zeta(-1)\zeta(1)}$ because $[b_3, b_4] = [b_4, b_5] = 1$. Therefore $b_4^{\zeta(1)\zeta(-1)} = b_4^a = b_4^{\zeta(0)}$ and thus $\zeta(1)\zeta(-1) = \zeta(0)$. For arbitrary even i, we find, as in (3.1.a), elements $b_{i+j} \in G_1(i+j-3, \cdots, i+j+3)^*$ for $j = 3, 4$ and 5 such that $[b_{i+3}, b_{i+5}] = b_{i+4}$ and proceed as before. B) follows analogously from (3.1.b). C) Choose b_i for $i = 4, 6, 7, 8$ and 10 as in (3.1.c). Then $b_6^{\zeta(2)}b_7^{\zeta(3)}b_8^{\zeta(4)} = (b_6b_7b_8)^a = [b_4^a, b_{10}^a] = [b_4^{\zeta(0)}, b_{10}^{\zeta(6)}] = (b_6b_7b_8)^{\zeta(0)\zeta(6)} = b_6^{\zeta(0)\zeta(6)}b_7^{\zeta(0)\zeta(6)}b_8^{\zeta(0)\zeta(6)}$ since $[G_4(j), G_1(k-3, \cdots, k+3)] = 1$ whenever j is even and $|j-k| \leqslant 4$. Thus $b_6^{\zeta(0)\zeta(6)-\zeta(2)} \in G_1(10)$. It follows that $\zeta(0)\zeta(6) = \zeta(2)$. Similarly, $\zeta(0)\zeta(6) = \zeta(4)$. □

By (3.2.C), $\zeta(i) = \zeta(0)$ for every even i. By (3.2.B), it follows that $\zeta(i) = 1$ for i odd. Therefore, by (3.2.A), $\zeta(2) = 1$ and thus $a \in G_1(1, 2)$. By (2.4), it follows that $a = 1$, a contradiction.

Thus $p = 2$. First let $q = 2$. For each i let b_i be the nontrivial element in $G_1(i-3, \cdots, i+3)$. Since there exists a vertex $x \in \Gamma(7)$ such that $G_1(2, \cdots, 8) \cap G_1(x) = 1$ and $|\Gamma(7)| = 3$, it follows that $b_5 \notin G_1((6)b_{11})$. Similarly, $b_{11} \notin G_1((10)b_5)$. Thus $[b_5, b_{11}] \in G_1(7, 8, 9) - G_1(6) - G_1(10)$, so that $b_6b_{10}[b_5, b_{11}] \in G_1(6, \cdots, 10)$ and $(b_6b_{10}[b_5, b_{11}])^2 \in G_1(5, \cdots 11)$. Since $[G_4(i), G_1(7, 8, 9)] = 1$ for $i = 6$ and 10, $(b_6b_{10}[b_5, b_{11}])^2 = [b_5, b_{11}]^2$. If $[b_5, b_{11}]^2 = 1$, then $[b_5, [b_5, b_{11}]] = 1$ and therefore $b_5 \in G_1(2, \cdots, 6, (5)[b_5, b_{11}], \cdots, (2)[b_5, b_{11}])$, in contradiction to (2.3). Therefore $[b_5, b_{11}]^2 \neq 1$ and, in particular, $[b_5, b_{11}]^2 \notin G(3)$. Since $[G_4(2), G_1(2, \cdots, 8)] = 1$, we have $b_5 \in G_1((8)b_2)$. If $(8)b_2$

$= (4)[b_5, b_{11}]$, then $b_5 \in G((3)[b_5, b_{11}])$ and thus $(3)[b_5, b_{11}]^2 = (3)[b_5, b_{11}]b_{11}b_5b_{11} = 3$. It follows that $(8)b_2 \neq (4)[b_5, b_{11}]$. Since $[b_5, b_{10}] \in G(4, \cdots, 12) - G(13)$, we have $[b_5, b_{10}] \notin G(3)$, so that $b_5 \notin G_1((4)b_{10})$ and therefore $(4)b_{10} \neq (8)b_2$. Thus $(4)b_{10} = (4)[b_5, b_{11}]$. Hence $(b_{10}[b_5, b_{11}])^2 \in G(3)$. Since $[b_{10}, G_1(6, 7, 8)] = 1$, $[b_5, b_{11}]^2 \in G(3)$, a contradiction.

When $q > 2$, a different argument is required.

(3.3) *Let $p = 2$. For every i there exists an element $e_i \in G_1(i) \cap G(i, \cdots, i + 8) \cap G_1(i + 8)$ with $|e_i|_j = q - 1$ for $i < j < i + 8$.*

Proof. By (2.10), there exists an element $a \in G_1(4) \cap G(4, \cdots, 12) \cap G_1(12) \leqslant G(W)$ with $|a|_8 = q - 1$. Thus $q - 1 \,|\, |a|$. Since $a^{|a|_5} \in G_1(4, 5)$, we have $|a| = |a|_5$ by (2.3). If $\sigma \in P\Gamma L(2, q)_\infty$ and $q - 1 \,|\, |\sigma|$, then $|\sigma| = q - 1$. It follows that $|a|_5 = q - 1$. Similarly, $|a|_{11} = q - 1$.

For each i let $a_i = a^{|a|_i}$. Then $[a_i, G_1(i + 1, \cdots, i + 7)] = 1$ and thus $a_i \in G_1(i + 8)$. It follows that $a_i \in G_1(j)$ whenever $j \equiv i \pmod 8$.

By (3.1.c), we can find elements $b_i \in G_4(i)^*$ for $i = 0, 2, 4$ and 6 and an element $b_3 \in G_1(0, \cdots, 6)$ such that $[b_0, b_6] = b_2b_3b_4$. Since $[b_0, b_6] = [b_0^{a_{10}}, b_6^{a_{10}}] = b_2^{a_{10}}b_3^{a_{10}}b_4^{a_{10}}$, $b_4^{a_{10}}b_4^{-1} \in G_1(0)$ and thus $[b_4, a_{10}] = 1$. Since $[b_4, a^j] = 1$ implies $|a|_8 = q - 1 \,|\, j$, we conclude that $|a|_{10} = q - 1$. Similarly, $|a|_6 = q - 1$. By (3.1.b), we can find $b_i \in G_1(i - 3, \cdots, i + 3)^*$ for $i = 8$, 10 and 13 such that $[b_8, b_{13}] = b_{10}$. Then $b_{10}^{a_9} = [b_8^{a_9}, b_{13}^{a_9}] = [b_8, b_{13}] = b_{10}$ and therefore $|a|_6 = q - 1 \,|\, |a|_9$. It follows that $|a|_9 = q - 1$ and similarly $|a|_7 = q - 1$. Thus the claim is proven for i even.

Let c be an element in $G_1(2) \cap G(2, \cdots, 10) \cap G_1(10)$ with $|c|_i = q - 1$ for $3 \leqslant i \leqslant 9$. We can choose c such that $d = ac \in G_1(3)$; let $d_i = d^{|d|_i}$ for each i. Since $[d, G_1(4, \cdots, 10)] = 1$, $d \in G_1(11)$. Since $a \in G_1(4)$ and $c \in G_1(10)$, we have $|d|_4 = |d|_{10} = q - 1$. By (3.1.a), we can find elements $b_i \in G(i - 3, \cdots, i + 3)^*$ for $i = 7$, 8 and 9 such that $[b_7, b_9] = b_8$. Then $b_8^{d_5} = [b_7^{d_5}, b_9^{d_5}] = [b_7, b_9] = b_8$ and thus $|d|_4 = q - 1 \,|\, |d|_5$ so that $|d|_5 = q - 1$. Similarly, $|d|_9 = q - 1$. By (3.1.b), we can find elements $b_i \in G_1(i - 3, \cdots, i + 3)^*$ for $i = 7, 10$ and 12 such that $[b_7, b_{12}] = b_{10}$. Then $b_{10}^{d_8} = [b_7^{d_8}, b_{12}^{d_8}] = [b_7, b_{12}] = b_{10}$ and so $|d|_6 \,|\, |d|_8$. Similarly, we have $|d|_8 \,|\, |d|_6$ and therefore $|d|_6 = |d|_8$. If we pick b_i ($i = 4, 6, 7, 8, 10$) as in (3.1.c), then $(b_6b_7b_8)^{d_8} = [b_4^{d_8}, b_{10}^{d_8}] = [b_4, b_{10}] = b_6b_7b_8$ and so $b_6^{d_8}b_6 \in G_4(6) \cap G_1(10) = 1$, thus $|d|_{10} = q - 1 \,|\, |d|_6 = |d|_8$. Finally, let b_i with $7 \leqslant i \leqslant 11$ be as in (3.1.d). Then $(b_8b_9b_{10})^{d_7} = [b_7^{d_7}, b_{11}^{d_7}] = [b_7, b_{11}] = b_8b_9b_{10}$ and therefore $b_8^{d_7}b_8 \in G_4(8) \cap G_1(12) = 1$, so that $|d|_4 = q - 1 \,|\, |d|_7$. □

We are now in a position to obtain a contradiction by constructing a generalized 8-gon of order (q, q). We will save space, however, by postponing this until later, where we include it as one case in the construction crucial to the proof of (1.2).

4. The case $s = 13$

This time we suppose first that $p = 2$. If $b_0 \in G_6(0)^*$ and $b_{10} \in G_6(10)^*$, then $[b_0, b_{10}] \in G_1(3, \cdots, 7) - G_1(2) - G_1(8)$. If $-2 \leqslant i \leqslant 6$, then $\partial(0, (i)[b_0, b_{10}]) \leqslant 6$, so that $(i)[b_0, b_{10}]b_0 = (i)[b_0, b_{10}]$ and thus $[b_0, b_{10}]^2 \in G(i)$. If $4 \leqslant i \leqslant 12$, then $\partial(10, (i)[b_0, b_{10}]) \leqslant 6$, so that $(i)b_0b_{10}b_0 = (i)[b_0, b_{10}]b_{10} = (i)[b_0, b_{10}]$ and thus $[b_0, b_{10}]^2 \in G(i)$. Therefore $[b_0, b_{10}]^2 \in G(-2, \cdots, 12) \cap G_1(3, \cdots, 7) = 1$. It follows that $[b_0, [b_0, b_{10}]] = 1$ and hence $b_0 \in G_1(-5, \cdots, 1, 2, (1)[b_0, b_{10}], \cdots, (-5)[b_0, b_{10}]) = 1$. Contradiction.

Thus $p \geqslant 3$.

(4.1) a) *If* $b_0 \in G_6(0)^*$ *and* $b_7 \in G_1(2, \cdots, 12)^*$, *then there exists an element* $b_2 \in G_6(2)^*$ *such that* $[b_0, b_7] = b_2$.
b) *If* $b_0 \in G_6(0)^*$ *and* $b_8 \in G_6(8)^*$, *then there exists an element* $b_4 \in G_6(4)^*$ *such that* $[b_0, b_8] = b_4$.
c) *If* $b_0 \in G_6(0)^*$ *and* $b_9 \in G_1(4, \cdots, 14)^*$, *then there exist elements* $b_i \in G_1(i - 5, \cdots, i + 5)$ *for* $i = 2, 3, 4, 5$ *and* 6 *with* $b_6 \neq 1$ *such that* $[b_0, b_9] = b_2b_3b_4b_5b_6$.

Proof. We leave a) and b) to the reader and turn to part c). Since $[G_1(4, \cdots, 14), G_6(12)] = 1$ and $G_6(12)$ acts transitively on $\Gamma(6) - \{7\}$, we have $G_1(4, \cdots, 14) \leqslant G_2(6)$. Thus $[b_0, b_9] \in G_1(1, \cdots, 7) - G_1(0)$. There exist $b_2 \in G_1(-3, \cdots, 7)$ and $b_6 \in G_1(1, \cdots, 11)^*$ such that $[b_0, b_9]b_2^{-1}b_6^{-1} \in G_1(0, \cdots, 8)$ and thus $b_i \in G_1(i - 5, \cdots, i + 5)$ for $i = 3, 4$ and 5 such that $[b_0, b_9] \cdot b_2^{-1}b_6^{-1}b_5^{-1}b_3^{-1} = b_4$. Since $[b_2, b_i] = [b_4, b_i] = 1$ for $2 \leqslant i \leqslant 6$, we have $[b_0, b_9] = b_2b_3b_4b_5b_6$. □

By (2.11), there exists an involution a in $\bar{G}(W)$. Let $\zeta(i) = (-1)^{|a|_i+1}$ for each i.

(4.2) *For every even* i:
A) $\zeta(i - 1) = \zeta(i + 4)\zeta(i + 6)$ *and* $\zeta(i + 7) = \zeta(i)\zeta(i + 2)$
B) $\zeta(i) = \zeta(i + 4)\zeta(i + 8)$
C) $\zeta(i) = \zeta(i + 6)$ *if* $\zeta(i + 3) = 1$.

Proof. A) We may take $i = 2$. If b_0, b_2 and b_7 are as in (4.1.a), then $b_2^{\zeta(8)} = b_2^a = [b_0^a, b_7^a] = [b_0^{\zeta(6)}, b_7^{\zeta(1)}] = [b_0, b_7]^{\zeta(1)\zeta(6)}$ since $[b_2, b_0] = [b_2, b_7] = 1$.

Thus $\zeta(8) = \zeta(1)\zeta(6)$. By (3.1.a), we can find elements $b_i \in G_1(i-5, \cdots, i+5)^*$ for $i = 3, 8$ and 10 such that $[b_3, b_{10}] = b_8$. Then $b_8^{\zeta(2)} = [b_3, b_{10}]^a = [b_3^{\zeta(9)}, b_{10}^{\zeta(4)}] = b_8^{\zeta(9)\zeta(4)}$. B) follows analogously from (4.1.b). For C) we assume $i = 0$ and $\zeta(3) = 1$. If b_i with $i = 0, 2, 3, 4, 5, 6$, and 9 are as in (4.1.c), then $(b_2b_3b_4b_5b_6)^a = [b_0^a, b_9^a] = [b_0^{\zeta(6)}, b_9] = [b_0, b_9]^{\zeta(6)}$ since $[b_0, b_i] = 1$ for $2 \leqslant i \leqslant 6$. Since $[b_i, b_6] = 1$ for $2 \leqslant i \leqslant 5$, we have $(b_2b_3b_4b_5b_6)^{\zeta(6)} = (b_2b_3b_4b_5)^{\zeta(6)}b_6^{\zeta(6)}$ and therefore $b_6^{\zeta(0)-\zeta(6)} = b_6^a b_6^{-\zeta(6)} \in \langle b_2, b_3, b_4, b_5\rangle \leqslant G_1(0)$, so that $\zeta(0) = \zeta(6)$. □

Suppose that $\zeta(3) = 1$. By (2.4), we have $\zeta(2) = \zeta(4) = -1$. By (4.2.C), $\zeta(0) = \zeta(6)$. By (4.2.B), $\zeta(8) = \zeta(0)\zeta(4) = -\zeta(0)$ and $\zeta(10) = \zeta(2)\zeta(6) = -\zeta(0)$. By (4.2.A), $\zeta(9) = \zeta(2)\zeta(4) = 1$, $\zeta(1) = \zeta(6)\zeta(8) = -1$ and $\zeta(11) = \zeta(4)\zeta(6) = -\zeta(0)$. Since $a \notin G_1(8, 9)$, we have $\zeta(8) = -\zeta(0) = -1$ and therefore $\zeta(6) = 1$. Since $a \notin G_1(5, 6)$ and $a \notin G_1(6, 7)$, we have $\zeta(5) = \zeta(7) = -1$.

We now choose elements b_i with $i = 0, 2, \cdots, 6, 9$ as in (4.1.c). Since $\zeta(3) = \zeta(6) = 1$, we have $b_2 \cdots b_6 = [b_0, b_9] = [b_0^a, b_9^a] = (b_2 \cdots b_6)^a = b_2^{\zeta(8)}b_3^{\zeta(9)} \cdot b_4^{\zeta(10)}b_5^{\zeta(11)}b_6^{\zeta(0)} = b_2^{-1}b_3b_4^{-1}b_5^{-1}b_6$. Thus $b_5^2 \in \langle b_2, b_3, b_4\rangle \leqslant G_1(-1)$, so that $b_5 = 1$. Therefore $b_4^2 \in \langle b_2, b_3\rangle \leqslant G_1(-2)$, so that $b_4 = 1$ and thus $b_2 = 1$. There exists an element $g \in G$ with $(0, \cdots, 13)g = (2, \cdots, 15)$. Since $\zeta(1) = \zeta(2) = -1, f_i > 1$ for every i and thus, by (2.2), $(i)g = i + 2$ for every i. If $c = gag^{-1}$, then $b_3^{-1}b_6^{-1} = b_3^c b_6^c = [b_0^c, b_9^c] = [b_0^{-1}, b_9^{-1}]$. From $[b_0, b_9] = b_3b_6$ it follows that $[b_0^{-1}, b_9^{-1}] = b_9b_0b_3b_6b_0^{-1}b_9^{-1}$. Since $[b_6, b_i] = 1$ for $i = 0$ and 9 and $[b_0, b_3] = 1$, we have $b_3^{-1}b_6^{-1} = [b_0^{-1}, b_9^{-1}] = b_9b_3b_9^{-1}b_6$ and thus $b_3^{-2}b_6^{-2} = b_3^{-1}b_9b_3b_9^{-1} \in G_1(9)$. Therefore $b_3^{-2} \in G_1(-2, \cdots, 9) = 1$, so that $b_3 = 1, b_6^{-2} = b_9b_9^{-1} = 1$ and thus $b_3 = 1$. Contradiction. It follows that $\zeta(3) = -1$ and thus $\zeta(i) = -1$ for every odd i.

From (4.2.A) we have that $\zeta(i) = -\zeta(i+2)$ for every even i. Thus either $\zeta(6) = \zeta(10) = \zeta(14) = -1$ or $\zeta(8) = \zeta(12) = \zeta(16) = -1$, in contradiction to (4.2.B).

5. Proof of (1.2): Preliminaries

(5.1) *Let $q \neq 2$, $s \in \{4, 5, 7\}$, $p \neq 2$ if $s = 4$ and $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ for every vertex x when $s = 5$ and $q = 3$. Let $u = 0$ or 1. Then $G_1(u) \cap G(W) \cap \bar{G}(u+i) \not\leqslant G_1(u+i)$ for every i with $1 \leqslant i \leqslant s-2$ excluding $i = (s-1)/2$ if $q = 3$ and $s = 5$ or 7 and $i = 2$ and 4 if $q = 4$ and $s = 7$.*

Proof. Suppose $G_1(u) \cap G(W) \not\leqslant G_1(u+i)$ for some i. Since h_{u+i} normalizes $G_1(u) \cap G(W)$, it follows that $G_1(u) \cap G(W) \cap \bar{G}(u+i) \not\leqslant G_1(u$

$+ i)$. It thus suffices to prove $G_1(u) \cap G(W) \not\leq G_1(u + i)$ to conclude that $G_1(u) \cap G(W) \cap \bar{G}(u + i) \not\leq G_1(u + i)$. We choose, once and for all, an element $g \in G$ such that $(0, \cdots, s)g = (2, \cdots, s + 2)$ and, in case $p \neq 2$, an involution $a \in \bar{G}(W)$; let $\zeta(i) = (-1)^{|a|_i + 1}$ for every i.

Suppose first that $s = 4$ and $p \neq 2$. Then $b_i^a = b_i^{\zeta(i+2)}$ for every i and every $b_i \in G_1(i, i + 1)$. For each w there exist elements $b_i \in G_1(i, i + 1)^*$ for $i = w, w + 1$ and $w + 2$ such that $[b_w, b_{w+2}] = b_{w+1}$. Then $b_{w+1}^{\zeta(w+3)} = b_{w+1}^a = [b_w, b_{w+2}]^a = [b_w^{\zeta(w+2)}, b_{w+2}^{\zeta(w+4)}] = [b_w, b_{w+2}]^{\zeta(w+2)\zeta(w+4)}$ since $[b_{w+1}, b_w] = [b_{w+1}, b_{w+2}] = 1$. Thus $\zeta(w + 3) = \zeta(w + 2)\zeta(w + 4)$. Thus there exists a k such that $\zeta(i) = 1$ iff $i \equiv k \pmod 3$. In particular, $f_i > 1$ for every i so that, by (2.2), $(i)g = i + 2$ for every i. Therefore $ag^{-1}ag \in G_1(i)$ iff $i \equiv k + 1 \pmod 3$.

Now let $s = 5$. Since, by assumption, $f_i > 1$ for every i, we have $(i)g = i + 2$ for every i. We claim that it would suffice to show that $G(W) \cap G_1(u) \neq 1$ for $u = 0$ or 1 when $q > 3$ and for $u = 0$ and 1 when $q = 3$. Let, for instance, $H = G(W) \cap G_1(0)$ and suppose that $H \neq 1$. If $a \in H$, then $[a, G_1(1, 2, 3)] = 1$ and thus $a \in G_1(4)$. Thus $H \leqslant G_1(i)$ for every $i \equiv 0 \pmod 4$. By (2.4), we have $H \not\leq G_1(i)$ for every odd i. Let $\bar{H} = H \cap \bar{G}(1)$. By the remarks at the beginning of this proof, $\bar{H} \neq 1$. Since for each i, $[\bar{H}, h_i] \leqslant G_1(0, 1) \cap G(W) = 1$, $\bar{H} \leqslant \bar{G}(W)$ and thus $\bar{H} = H \cap \bar{G}(W) = H \cap \bar{G}(i)$ for each odd i. Suppose that $q > 3$ and $\bar{H} \leqslant G_1(2)$ so that $\bar{H} = \bar{G}(W) \cap G_1(i)$ for every even i. Let Σ be the graph with $V(\Sigma) = \{(0)n \mid n \in N_G(\bar{H})\}$ and $E(\Sigma) = \{\{x, y\} \mid x, y \in V(\Sigma)$ and $\partial(x, y) = 2\}$ and let S be the subgroup of aut (Σ) induced by $N_G(\bar{H})$. Since $G(i, \cdots, i + 4) \leqslant N_G(\bar{H})$ for every even i, Σ is $(S, 3)$-transitive and $PSL(2, q) \trianglelefteqq S(x)^{\Sigma(x)}$ for every $x \in V(\Sigma)$. By (2.12), $(q - 1)/(q - 1, 2)$ divides $|(S_1(0) \cap \bar{S}(2) \cap S(4))^{\Sigma(2)}|$ and hence $|(H \cap \bar{G}(2))^{\Gamma(2)}|$, too. Choose an element d in $H \cap \bar{G}(2)$ with $|d|_2 = (q - 1)/(q - 1, 2)$. Then $d^r \in H \cap \bar{G}(W)$ (where $q = p^r$) and, since $r < |d|_2$, $d^r \notin G_1(2)$. This contradicts the assumption that $\bar{H} \leqslant G_1(2)$. It follows that there exists an element $c \in \bar{H}$ not in $G_1(2)$. By (2.3), $|c| = |c|_{-1} = |c|_1$ and so $|c|_1 = |g^{-1}cg|_1$. Since $\bar{G}(W)^{\Gamma(1)}$ is cyclic, $\langle c \rangle$ and $\langle g^{-1}cg \rangle$ induce the same permutation group on $\Gamma(1)$. Hence there exists an integer j relatively prime to $|c|$ such that $c^j g^{-1}cg \in G_1(1)$. Since $g^{-1}cg \in G_1(2)$, $|c^j g^{-1}cg|_2 = |c^j|_2 \neq 1$. Hence $G_1(1) \cap G(W) \neq 1$ and we can proceed as before. If we start by assuming $G_1(1) \cap G(W) \neq 1$, the proof is the same.

When $p = 2$, $H \neq 1$ follows from (2.10). Let $p \neq 2$. There exist elements $b_i \in G_1(i - 1, i, i + 1)^*$ for $0 \leqslant i \leqslant 3$ such that $[b_0, b_3] = b_1 b_2$. Let

$c_2 = [b_1, b_3] \in G_1(1, 2, 3) = G_2(2)$. Suppose that $\zeta(i) = -1$ for every i. Then $c_2^{-1} = c_2^a = [b_1, b_3]^a = [b_1^{-1}, b_3^{-1}] = [b_1, b_3]$ since $[c_2, b_i] = 1$ for $i = 1$ and 3. Thus $c_2 = 1$. It follows that $[b_i, [b_0, b_3]] = 1$ for $i = 0$ and 3, so that $b_1^{-1}b_2^{-1} = [b_0, b_3]^a = [b_0^{-1}, b_3^{-1}] = [b_0, b_3] = b_1b_2$. Therefore $b_1b_2 = 1$, so that $b_1 \in G_1 \cdot(0, 1, 2, 3) = 1$, a contradiction. We are thus finished with the case $s = 5$ when $q > 3$. Let $q = 3$. If $\zeta(1) = 1$, then $b_1^{\zeta(3)}b_2^{\zeta(0)} = b_1^a b_2^a = [b_0, b_3]^a = [b_0^{\zeta(2)}, b_3] = [b_0, b_3]^{\zeta(2)} = b_1^{\zeta(2)}b_2^{\zeta(2)}$ since $[b_0, b_i] = 1$ for $i = 1$ and 2. Thus $\zeta(0) = \zeta(2) = \zeta(3)$. Since $a \notin G_1(0, 1), \zeta(0) = -1$. Therefore $ag^{-1}ag \in G_1(2) - G_1(3)$. Thus we may suppose that $G(W) \cap G_1(i) = 1$ for every odd i. Since W is good and, by assumption, $f_0 = 2$, we may, by replacing a if necessary, assume that $\zeta(0) = -1$. Then $c_2^{-1} = c_2^a = [b_1^a, b_3^a] = [b_1^{-1}, b_3^{-1}] = [b_1, b_3]$ so that $[b_1, b_3] = 1$. Thus $b_1^{-1}b_2^{-1} = [b_0^a, b_3^a] = [b_0^{\zeta(2)}, b_3^{-1}] = [b_0, b_3]^{-\zeta(2)} = b_1^{-\zeta(2)}b_2^{-\zeta(2)}$, so that $\zeta(2) = 1$. Therefore $ag^{-1}ag \in G_1(1) - G_1(2)$, a contradiction.

Now let $s = 7$. This time we claim that it suffices to show that $G_1(u) \cap G(W) \neq 1$ for $u = 0$ or 1 when $q = 3$ or $q \geqslant 5$ and for $u = 0$ and 1 when $q = 4$. Let, for instance, $H = G(W) \cap G_1(1)$ and suppose that $H \neq 1$. Since $[H, G_1(2, \cdots, 6)] = 1$, $H \leqslant G_1(7)$ and thus $H = G(W) \cap G_1(i)$ for every $i \equiv 1 \pmod 6$ and $H \not\leqslant G_1(i)$ for every $i \equiv 0$ or $2 \pmod 6$. If $H \leqslant G_1(4)$ and thus $H = G(W) \cap G_1(i)$ for every $i \equiv 1 \pmod 3$, we obtain a contradiction from (2.12) as in the case $s = 5$ (when $q \neq 3$). Let $\bar{H} = H \cap \bar{G}(2)$. As in the case $s = 5$, $\bar{H} = H \cap \bar{G}(W) = H \cap \bar{G}(i)$ for every $i \equiv 0$ or $2 \pmod 6$. Suppose that $\bar{H} \leqslant G_1(3)$. Let c be an element with $(i)c = 8 - i$ for $1 \leqslant i \leqslant 7$. Since $\bar{H} = \bar{G}(1, \cdots, 7) \cap G_1(1) = \bar{G}(1, \cdots, 7) \cap G_1(7)$, c normalizes $\bar{H}$. Thus $\bar{H} \leqslant G_1(5)$ and hence $\bar{H} = \bar{G}(W) \cap G_1(i)$ for every odd i. Let Σ be the graph with $V(\Sigma) = \{(1)n \mid n \in N_G(\bar{H})\}$ and $E(\Sigma) = \{\{x, y\} \mid x, y \in V(\Sigma)$ and $\partial(x, y) = 2\}$ and let S be the subgroup of $\operatorname{aut}(\Sigma)$ induced by $N_G(\bar{H})$. Then $PSL(2, q) \trianglelefteqq S(x)^{\Sigma(x)}$ for every $x \in V(\Sigma)$ and Σ is locally $(S, 4)$-transitive. We may thus conclude that $(q-1)/(q-1, 3)$ divides $|(S_1(1) \cap \bar{S}(3) \cap S(5))^{\Sigma(3)}|$ from the very theorem (i.e., (1.2)) we are busy proving, paying attention that we never use the case $s = 7$ in the proof of the case $s = 4$. This contradicts the assumption that $\bar{H} \leqslant G_1(3)$ as in the case $s = 5$ if $q \neq 4$. In particular, $f_i > 1$ for every i and thus $(i)g = i + 2$ for every i. Exactly as in the case $s = 5$, we can find an element $c \in \bar{H}$ and an integer j such that $c^j g^{-1}cg \in G_1(0) \cap G(W)^*$ (if $q \neq 4$). Thus we can proceed as before.

If $p = 2$, $G_1(1) \cap G(W) \neq 1$ follows from (2.10). Suppose $q = 4$. If $a = (h_1)^2$, then $a \in \bar{G}(W)$ and $a \notin G_1(1)$. There exists an element $b \in G_1(1) \cap G(W)$ such that $ab \in G_1(0)$. Hence $G_1(0) \cap G(W) \neq 1$. Finally, suppose that $p \neq 2$. Suppose $\zeta(i) = -1$ for every i. There exist elements $b_i \in G_3(i)^*$ for $i = 0$, 2 and 4 such that $[b_0, b_4] = b_2$. Thus $b_2^{\zeta(5)} = b_2^a = [b_0\, b_4]^a = [b_0^{\zeta(3)}, b_4^{\zeta(7)}] = [b_0, b_4]^{\zeta(3)\zeta(7)}$ since $[b_2, b_i] = 1$ for $i = 0$ and 4. Thus $-1 = \zeta(5) = \zeta(3) \cdot \zeta(7) = +1$. Contradiction. □

In the next lemma, we include the case $s = 9, p = 2$ and $q \geq 4$, continuing from where we left off in § 3.

(5.2) *Let $q > 2$, $s \in \{4, 5, 7\}$ or $s = 9$ and $p = 2$ and $G(x)^{\Gamma(x)} \cong PGL(2, 3)$ for every vertex x when $q = 3$ and $s = 5$. Let $u = 0$ or 1 and $y_1, \cdots, y_{s-1}$ be vertices with $y_1 \neq u + s$ such that $(u, u + 1, \cdots, u + s - 1, y_1, \cdots, y_{s-1})$ is a good $2(s - 1)$-path. Then $(y_{s-1}, \cdots, y_1, u + s - 1, u + s, \cdots, u + 2(s - 1))$ is a good $2(s - 1)$-path.*

Proof. By (2.1), there exist vertices $y_2', \cdots, y_{s-1}'$ such that $(y_{s-1}', \cdots, y_2', y_1, u + s - 1, u + s, \cdots, u + 2(s - 1))$ is a good $2(s - 1)$-path.

We first assume that $s = 4$ and $p = 2$. By (2.5), $\langle G_1(y_1, y_2), G_1(3 + u, y_1)\rangle$ contains an element a with $(1 + u, 2 + u)a = (5 + u, 4 + u)$. Since $[G_1(y_1, y_2), G_1(3 + u, y_1)] \leqslant G_1(3 + u, y_1, y_2) = 1$, a is an involution. By (2.2), a exchanges u and $6 + u$. Thus a exchanges y_2 and y_2'. But $a \in G_1(y_1)$ so that $y_2 = y_2'$. Now taking $(6 + u, 5 + u, 4 + u, 3 + u, y_1, y_2, y_3')$ in place of $(u, u + 1, \cdots, u + 6)$, $2 + u$ in place of y_1 and $1 + u$ in place of y_2, we conclude that $(1 + u, 2 + u, 3 + u, y_1, y_2\, y_3')$ is good. Since $(1 + u, 2 + u, 3 + u, y_1, y_2, y_3)$ is also good, it follows from (2.2) that $y_3 = y_3'$.

We may thus assume that $p \neq 2$ if $s = 4$. By (3.3) and (5.1), there exists an element $a \in G_1(u + s - 1) \cap \bar{G}(y_1) \cap G(y_2) - G_1(y_1)$ with $(|a|, p) = 1$. Since $(|a|, p) = 1$, there exists an $(s - 1)$-path $(x_u, x_{u+1}, \cdots, x_{u+s-2}, x_{u+s-1})$ with $x_{u+s-1} = u + s - 1$, $x_{u+s-2} \neq y_1$ and $a \in G(x_u, x_{u+1}, \cdots, x_{u+s-2}, x_{u+s-1})$. Since Γ is locally (G, s)-transitive, we may assume that $x_i = i$ for $u + 1 \leqslant i \leqslant u + s - 2$. By (2.2), $x_u = u$ since $f_x > 1$ for every vertex x, by assumption when $s = 5$ and by (5.1) when $s \in \{4, 7\}$. Since $a \in G(u, \cdots, u + s)$, $a \in G(u, \cdots, u + 2(s - 1))$ and thus $a \in G(y_2')$. But y_2 is the only fixed point of a in $\Gamma(y_1) - \{u + s - 1\}$. Thus $y_2 = y_2'$. Again using (3.3) and (5.1), we can find an element in $G(u, \cdots, u + s - 1) \cap G_1(u + s - 1) \cap G(y_1, y_2, y_3) \cap \bar{G}(y_2) - G_1(y_2)$, so that $y_3 = y_3'$. Continuing, we obtain $y_i = y_i'$ for $1 \leqslant i \leqslant s - 1$ except when $q = 3$ and $s \in \{5, 7\}$ or $q = 4$ and $s = 7$.

If $q=3$ and $s \in \{5, 7\}$, we have only $y_i = y'_i$ for $1 \leqslant i \leqslant v$ where $v = (s-1)/2$ from (5.1). If we knew that $G_1(u) \nleq G(W) \cap G_1(u+v)$ (which, however, a posteriori is not the case), we would be finished as before. Thus we may assume that $G_1(u) \cap G(W) = G_1(i) \cap G(W)$ for every $i \equiv u \pmod{v}$. Let $H = G_1(u) \cap G(W)$, $S = N_G(H)/H$ and Σ be the graph with $V(\Sigma) = \{(u)n \mid n \in N_G(H)\}$ and $E(\Sigma) = \{\{x, y\} \mid x, y \in V(\Sigma)$ and $\partial(x, y) = v\}$. The graph Σ is locally $(S, 3)$-transitive. Since $S(x)^{\Sigma(x)} \cong PGL(2, 3)$ and $|S_1(x)| = 3$ for every vertex, there exists, by (2.13), an involution in $S(y_{2v}, y_v, u+2v, u+3v, u+4v)$. Thus there exists an element in $G(y_v, \cdots, y_1, u+s-1, u+s, \cdots, u+2(s-1))$ whose only fixed point in $\Gamma(y_v) - \{y_{v-1}\}$ is y_{v+1}. Thus $y_{v+1} = y'_{v+1}$. Using (5.1), we can then conclude that $y_i = y'_i$ for $v+2 \leqslant i \leqslant s-1$.

If $q=4$ and $s=7$, we may assume that $G_1(u) \cap G(W) = G_1(i) \cap G(W)$ for every $i \equiv u \pmod{2}$. Let $H = G_1(u) \cap G(W)$, $S = N_G(H)/H$ and Σ be the graph with $V(\Sigma) = \{(u)n \mid n \in N_G(H)\}$ and $E(\Sigma) = \{\{x, y\} \mid x, y \in V(\Sigma)$ and $\partial(x, y) = 2\}$. The graph Σ is $(S, 4)$-transitive. By the case $s=4$ of the lemma we are busy proving, $(y_6, y_4, y_2, u+6, u+8, u+10, u+12)$ is a good 6-path in Σ. It follows that $(y_6, y_5, \cdots, y_2, y_1, u+6, u+7, \cdots, u+12)$ is a good 12-path in Γ. □

6. Proof of (1.2): The construction

We assume that $q \neq 2$, $f_x = 2$ for every vertex x when $s=5$ and $q=3$ and $s \in \{4, 5, 7\}$ or $s=9$ and $p=2$. For each $i \in N$ and each vertex x, let $\Gamma_i(x) = \{y \mid \partial(x, y) \leqslant i\}$. We point out that the girth of Γ is at least $2(s-1)$ (see, for instance, [10, p. 61]). Let $F = \Gamma_{s-2}(0) \cup \Gamma_{s-2}(1)$ and Π be the undirected graph with vertex set $V(\Pi) = F$ and $\{x, y\} \in E(\Pi)$ iff x or y or both are in $\Gamma_{s-3}(0) \cup \Gamma_{s-3}(1)$ and $x \in \Gamma(y)$ or there exists a good $(2s-3)$-path $(x_0, \cdots, x_{2s-3})$ with $x_{s-2} = 0$, $x_{s-1} = 1$ and either $x_0 = x$ and $x_{2s-3} = y$ or $x_0 = y$ and $x_{2s-3} = x$. By (2.2), Π is regular of valency $q+1$. Let $P = \operatorname{aut}(\Pi)$.

Let a be any element in $G(1) - G(0)$. We define a permutation $\hat{a}$ of F as follows: If $x \in \Gamma_{s-2}(1)$, we set $(x)a = (x)\hat{a}$. If $x \in F - \Gamma_{s-2}(1)$, we set $(x)\hat{a} = (x_{2(s-1)})a$, where $(x_0, \cdots, x_{2(s-1)})$ is the uniquely determined $2(s-1)$-path with $x_0 = x$, $x_{s-2} = 0$, $x_{s-1} = 1$ and $x_s = (0)a^{-1}$. It is straightforward to check, using (5.2), that $\hat{a}$ is an element of P. Thus $P(1) \nleq P(0)$. Similarly, $P(0) \nleq P(1)$.

If $a \in G(\{0, 1\})$, then clearly the permutation which a induces on F is an element of P. Since, for $u = 0$ and 1, $P(u) \not\leq P(1-u)$, it follows that $P(u)$ acts transitively on $\Pi(u)$. Since Π is connected, P acts transitively on $E(\Pi)$. Thus the girth of Π is $2(s-1)$ and Π is the incidence graph of a generalized $(s-1)$-gon of order (q, q). By [3], $s \in \{4, 5, 7\}$. Since, by (2.5) and (2.9), P contains sufficiently many "generalized elations", it follows from [5, Theorem 1.8], [7, Theorem 2] and [12, (4.4)] that $\Pi \cong \Gamma_{s-1,q}$ and $P \cong G_{s-1,q}$.

Let $u = 0$ or 1. We have seen that for each $a \in G(u)$ there exists an element $\hat{a} \in P(u)$ such that a and $\hat{a}$ agree on $\Gamma_{s-2}(u)$. The map τ mapping a onto $\hat{a}$ is an injective homomorphism from $G(u)$ into $P(u)$. For each $w \in \Gamma(u)$, an element $a \in G(u, w)$ lies in $O_p(G(u, w))$ iff for $i = u$ and w, a induces a permutation on $\Gamma(i)$ contained in $O_p(G(u, w)^{\Gamma(i)})$. Thus τ maps $O_p(G(u, w))$ into $O_p(P(u, w))$. But, by (2.3) and (2.5), $|O_p(G(u, w))| = q^{s-1} = |O_p(P(u, w))|$. Theorem (1.2) follows now from the next lemma whose proof is left to the reader:

(6.1) *Let $n = s - 1$ and (X, Y) be a 1-path in $\Gamma_{n,q}$. For $U = X$ and Y, let $\tilde{G}_{n,q}(U) = \langle O_p(G_{n,q}(U, W)) \mid W \in \Gamma_{n,q}(U)\rangle$. Then $\tilde{G}_{n,q}(U) \leqslant H_{n,q}(U)$ for $U = X$ and Y and $H_{n,q}(X, Y) = \langle \tilde{G}_{n,q}(X) \cap G_{n,q}(Y), G_{n,q}(X) \cap \tilde{G}_{n,q}(Y)\rangle$.*

7. Proof of (1.3)

When $q = 2$, we are in the unfortunate situation that every path is a good path, so that the construction used in the proof of (1.2) does not work. We leave undecided the question whether (1.2)—with an appropriate clause for the exceptional case $s = 4$ and $G(k) \cong \hat{G}_{4,2}(K)$—nevertheless remains true when $q = 2$.

First let $s = 4$ and, for every i, b_i be the nontrivial element in $G_1(i, i+1)$. Then $[b_i, b_{i+2}] = b_{i+1}$ for every i. We have $|b_0b_3|_2 = 3$. Thus $(b_0b_3)^3 \in G_1(2) = \langle b_1, b_2\rangle$ and therefore $(b_0b_3)^6 = 1$. Suppose $(b_0b_3)^3 \neq 1$. Let $a \in G$ be an element with $(0, \cdots, 4)a = (0, 1, 2, (1)b_3, (0)b_3)$. Then $b_3^a = b_0^{b_3}$ and hence $((b_0b_3)^3)^a = (b_0b_3b_0b_3)^3 = (b_0b_3)^6 = 1$, a contradiction. Thus $G(x) \cong \langle t_0, t_1, t_2, t_3 \mid t_i^2 = 1$ for $0 \leqslant i \leqslant 3$; $[t_i, t_j] = 1$ if $|i-j| = 1$; $[t_i, t_{i+2}] = t_{i+1}$ for $i = 0$ and 1; $(t_0t_3)^3 = 1\rangle$ for every vertex x. If Γ is $(G, 4)$-transitive, then there exists an element $c \in G$ with $(0, 1, \cdots, 4)c = (5, 4, \cdots, 1)$. Thus $c^2 \in G(1, \cdots, 4) = \langle b_2\rangle$ and $cb_ic = b_{4-i}$ for $1 \leqslant i \leqslant 3$. We have $G(\{2, 3\}) \cong G_{3,2}(\{X, Y\})$ if $c^2 = 1$ and $G(\{2, 3\}) \cong \hat{G}_{4,2}(K)$ otherwise.

Let $s = 5$ and, for every i, b_i be the nontrivial element in $G_1(i-1,$

$i, i+1)$. Then $[b_i, b_{i+3}] = b_{i+1}b_{i+2}$. Since $G_1(i-1, i, i+1) = G_2(i)$ for every even i, we have $[b_i, b_j] = 1$ when $|i-j| \leqslant 2$, i even. Suppose $[b_1, b_3] = b_2$. Then $[b_0, b_3] = (b_0b_3)^2 = b_1b_2 = b_1(b_1b_3)^2 = b_3b_1b_3$ and thus $b_0b_3b_0 = b_3b_1$. Squaring both sides, we have $1 = (b_3b_1)^2 = b_2$, a contradiction. Thus $[b_i, b_j] = 1$ when $|i-j| \leqslant 2$, i arbitrary. If Γ is $(G, 5)$-transitive, then there exists an element $c \in G$ with $(0, \cdots, 5)c = (5, \cdots, 0)$ and thus $c^2 = 1$ and $cb_ic = b_{4-i}$ for $1 \leqslant i \leqslant 4$. Thus the structure of $G(\{2, 3\})$ is completely determined. We have $(b_1b_5)^3 \in G_1(3) = \langle b_2, b_3, b_4\rangle$ and thus $(b_1b_5)^6 = 1$. Suppose that $(b_1b_5)^3 \neq 1$. Let $b_5' = b_5b_1b_5$. Then $(b_1b_5')^3 = 1$. There exists, however, an element $a \in G$ with $(1, \cdots, 5)a = (1, 2, 3, (2)b_5, (1)b_5)$ and thus $b_1^a = b_1$ and $b_5^a = b_5'$ since $[b_i, b_{i+2}] = 1$ and thus $G_2(i) = \langle b_i\rangle$ for every i. Thus $(b_1b_5)^a = b_1b_5'$. Contradiction. It follows that $(b_1b_5)^3 = 1$. Similarly, $(b_0b_4)^3 = 1$. Thus $G(x) \cong \langle t_0, \cdots, t_4 \mid t_i^2 = 1$ for $0 \leqslant i \leqslant 4$; $[t_i, t_j] = 1$ if $|i-j| \leqslant 2$, $[t_i, t_{i+3}] = t_{i+1}t_{i+2}$ for $i = 0$ and 1; $(t_0t_4)^3 = 1\rangle$ for every vertex x.

Now let $s = 7$ and, for every i, b_i be the nontrivial element in $G_1(i-2, \cdots, i+2)$. For every even i, $G_1(i-2, \cdots, i+2) = G_3(i)$ and thus $[b_i, b_j] = 1$ when $|i-j| \leqslant 3$ and $[b_i, b_{i+4}] = b_{i+2}$. Also, there exist u and $v \in \{0, 1\}$ such that $[b_0, b_5] = b_1b_2^ub_3^vb_4$. Hence $b_5b_0b_5 = b_0b_1b_2^ub_3^vb_4$. Squaring both sides, we have $(b_0b_1b_2^ub_3^vb_4)^2 = 1$. Since $(b_0b_4)^2 = b_2, [b_2, b_i] = 1$ for $0 \leqslant i \leqslant 4$ and $[b_i, b_j] = 1$ for $i \in \{1, 3\}$ and $j \in \{0, 4\}$, we have $b_2(b_1b_3^v)^2 = 1$. Thus $v = 1$ and $(b_1b_3)^2 = b_2$. Therefore $(b_ib_{i+2})^2 = b_{i+1}$ for every odd i. In particular, $b_i \notin G_2(i-1)$ and $b_i \notin G_2(i+1)$ whenever i is odd. It follows that $[b_1, b_5] \in G(1, \cdots, 5) - G_1(1) - G_1(5)$ and thus $[b_1, b_5] = b_2b_3^wb_4$ with $w \in \{0, 1\}$. Therefore $b_5b_1b_5 = b_1b_2b_3^wb_4$. Squaring both sides, we have $1 = (b_1b_3^w)^2$ and thus $w = 0$.

Suppose $(b_0b_6)^3 \in G_1(3) = \langle b_1, \cdots, b_5\rangle$ has even order. Let $b_6' = b_6b_0b_6$. Then $|b_0b_6'| = |b_0b_6|/2$. There exists, however, an element $a \in G$ with $(0, \cdots, 6)a = (0, \cdots, 3, (2)b_6, \cdots, (0)b_6)$ and thus $(b_0b_6)^a = b_0b_6'$. Contradiction.

Let $(x_0, \cdots, x_8)$ be an arbitrary 8-path. Since $|G(x_1, \cdots, x_7)| = 2$, there exist exactly two elements g_1 and g_2 such that $(x_1, \cdots, x_7)g_i = (x_7, \cdots, x_1)$ for $i = 1$ and 2. If d is any involution in $G(x_4) - G_1(x_4)$, then there exists a 6-path $(y_1, \cdots, y_7)$ with $y_4 = x_4$ such that $(y_i)d = y_{8-i}$ for $1 \leqslant i \leqslant 7$. Since G contains an element mapping $(y_1, \cdots, y_7)$ onto $(x_1, \cdots, x_7)$, g_1 and g_2 must be involutions. If $(x_8)g_1 = (x_8)g_2$, then $g_1g_2 \in G(x_0, \cdots, x_8) = 1$, a contradiction. Thus $(x_0, \cdots, x_8)g_i = (x_8, \cdots, x_0)$ for $i = 1$ or 2.

Thus there exists an element g mapping $(-1, \cdots, 7)$ onto $(7, \cdots, -1)$. Since $G_1(i-2, \cdots, i+2) = G_3(i)$ for even i, $G_1(i-2, \cdots, i+2)^g = G_1(4-i,$

$\cdots, 8-i)$ for $0 \leqslant i \leqslant 6$ and thus $[b_1, b_6] = [b_5, b_0]^g = (b_4 b_3 b_2^u b_1)^g = b_2 b_3 b_4^u b_5$. Therefore, for $u = 0$ or 1, $G(3) = \langle b_0, \cdots, b_6 \rangle \cong H_u$ where $H_u = \langle t_0, \cdots, t_6 | t_i^2 = 1$ for $0 \leqslant i \leqslant 6$; $[t_i, t_j] = 1$ for $|i-j| \leqslant 3$, i even; $[t_i, t_{i+4}] = t_{i+2}$ for $i = 0$ and 2; $[t_i, t_{i+2}] = t_{i+1}$ for $i = 1$ and 3; $[t_1, t_5] = t_2 t_4$; $[t_0, t_5] = t_1 t_2^u t_3 t_4$; $[t_1, t_6] = t_2 t_3 t_4^u t_5$; $(t_0 t_6)^3 = 1\rangle$. The map $\xi: \{t_0, \cdots, t_6\} \to H_0$ given by $(t_1)\xi = t_1 t_2$ and $(t_i)\xi = t_i$ for $0 \leqslant i \leqslant 6, i \neq 1$, induces an isomorphism from H_1 onto H_0. Thus the structure of $G(3)$ (and therefore also that of $G(2, 3) = \langle b_0, \cdots, b_5 \rangle$) is uniquely determined. Since $b_1 \notin G_2(0)$ and $G_3(1) \leqslant G_1(-1, \cdots, 3) = \langle b_1 \rangle$, $G_3(1) = 1$. Thus Γ cannot be $(G, 7)$-transitive.

Let h be the involution mapping $(0, \cdots, 8)$ onto $(8, \cdots, 0)$. Let $x_i = i$ for $0 \leqslant i \leqslant 9$, $x_{-1} = (9)h$ and c_i be the nontrivial element in $G_1(x_{i-2}, \cdots, x_{i+2})$ for $i = 1$ and 7. Suppose that $|c_1 c_7|$ is even. If we set $y_i = x_i$ for $-1 \leqslant i \leqslant 4$, $y_i = (x_{8-i})c_7$ for $5 \leqslant i \leqslant 9$ and let d_i be the nontrivial element in $G_1(y_{i-2}, \cdots, y_{i+2})$ for $i = 1$ and 7, then $|d_1 d_7| = |c_1 c_1^{c_7}| = |c_1 c_7|/2$. In addition, $(y_i)c_7 = y_{8-i}$ for $-1 \leqslant i \leqslant 9$. Repeating, if necessary, we obtain a 10-path $(z_{-1}, \cdots, z_g)$ with $z_4 = 4$ such that there exists an involution a with $(z_i)a = z_{8-i}$ for $-1 \leqslant i \leqslant 9$ and $|e_1 e_7| = 3$ where e_i is the nontrivial element in $G_1(z_{i-2}, \cdots, z_{i+2})$ for $1 \leqslant i \leqslant 7$. There exists a $w \in \{0, 1\}$ such that $[e_2, e_7] = e_3 e_4^w e_5 e_6$. Thus $[e_1, e_6] = [e_7, e_2]^a = (e_6 e_5 e_4^w e_3)^a = e_2 e_3 e_4^w e_5$. Therefore $G(4) = \langle e_1, \cdots, e_7 \rangle \cong J_w$ where $J_w = \langle t_1, \cdots, t_7 | t_i^2 = 1$ for $1 \leqslant i \leqslant 7$; $[t_i, t_j] = 1$ for $|i-j| \leqslant 3$, i even; $[t_2, t_6] = t_4$; $[t_i, t_{i+2}] = t_{i+i}$ for $i = 1, 3$ and 5; $[t_i, t_{i+4}] = t_{i+1} t_{i+3}$ for $i = 1$ and 3; $[t_1, t_6] = t_2 t_3 t_4^w t_5$; $[t_2, t_7] = t_3 t_4^w t_5 t_6$; $(t_1 t_7)^3 = 1\rangle$. The map $\theta: \{t_1, \cdots, t_7\} \to J_0$ given by $(t_5)\theta = t_4 t_5$ and $(t_i)\theta = t_i$ for $1 \leqslant i \leqslant 7, i \neq 5$, induces an isomorphism from J_1 onto J_0. Thus the structure of $G(4)$ is uniquely determined.

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