Y. Komori
Nagoya Math. J.
Vol. 72 (1978), 127-133

SUPER-ŁUKASIEWICZ IMPLICATIONAL LOGICS

YUICHI KOMORI

§1. Introduction

In the traditional study of Łukasiewicz propositional logic, the finite-valued or infinite-valued linearly ordered model exists at the start, and then the axiomatization of the set of all formulas valid in its model are studied. On the other hand, we are in a point of view such that the set of provable formulas is important and models are no more than means to characterize the set.

It is known that the $\aleph_0$-valued Łukasiewicz implicational logic is axiomatizable by the four axioms and that the four axioms are valid in the m-valued Łukasiewicz implicational logic for any natural number m (cf. [1], [2], [3]). Hence, the question arises if there exist implicational logics (in which the four axioms are valid) other than the above logics. In this paper, we will give the answer that there exists no implicational logic other than the above logics.

§2. Super-Łukasiewicz implicational logics

We must give the accurate definition of implicational logics to answer the above question. By an implicational formula (C formula), we mean a propositional formula which contains no connective other than $\supset$.

DEFINITION 2.1. A set L of C formulas is *an implicational logic* if it satisfies the following two conditions:

1) L is closed with respect to modus ponens, that is, $P, P \supset Q \in L$ implies $Q \in L$,
2) L is closed with respect to substitution for propositional variables by C formulas.

An implicational logic L is *a super-Łukasiewicz implicational logic*

Received December 12, 1977.

(*SLIL*) if L contains the following four formulas.

2.1 $p \supset q \supset p$.

2.2 $(p \supset q) \supset (q \supset r) \supset p \supset r$.

2.3 $p \vee q \supset q \vee p$.

2.4 $(p \supset q) \vee (q \supset p)$.

Here we use $P \vee Q$ as the abbreviation of $(P \supset Q) \supset Q$. We associate to the right, and use the convention that $\supset$ binds less strongly than $\vee$. We denote the smallest SLIL by ***Lu***.

§3. *C* algebras as models

In our former paper [1], we gave the definition of C algebras and investigated some properties of C algebras. We borrow some definitions and results from [1].

A *C algebra* is an algebra $\langle A; 1, \rightarrow\rangle$ which satisfies the following five axioms, where A is a non-empty set and 1 and $\rightarrow$ are 0-ary and 2-ary functions on A, respectively.

3.1 $1 \rightarrow x = x$.

3.2 $x \rightarrow y \rightarrow x = 1$.

3.3 $(x \rightarrow y) \rightarrow (y \rightarrow z) \rightarrow x \rightarrow z = 1$.

3.4 $x \cup y = y \cup x$.

3.5 $(x \rightarrow y) \cup (y \rightarrow x) = 1$.

In the above, we abbreviate $(x \rightarrow y) \rightarrow y$ by $x \cup y$. We use the same convention as before. We say simply that A is a C algebra, when $\langle A; 1, \rightarrow\rangle$ is a C algebra. We denote $x \rightarrow y = 1$ by $x \leqq y$. Then, we can verify the following without 3.5:

3.6 $x \leqq 1$,

3.7 $x \leqq x$,

3.8 $x \leqq y$ and $y \leqq z \Rightarrow x \leqq z$,

3.9 $x \leqq y$ and $y \leqq x \Rightarrow x = y$,

3.10 $x \rightarrow y \rightarrow z = y \rightarrow x \rightarrow z$,

3.11 $x \leqq x \cup y$ and $y \leqq x \cup y$,

3.12 $x \leqq z$ and $y \leqq z \Rightarrow x \cup y \leqq z$,

3.13 $y \rightarrow z \leqq (x \rightarrow y) \rightarrow x \rightarrow z$.

By 3.6–3.9, the relation $\leqq$ is an order relation with the largest element 1. By 3.11 and 3.12, $x \cup y = \sup(x, y)$.

DEFINITION 3.1. Let A be a C algebra. A non-empty subset J of

A is *a filter* of A if it satisfies the following two conditions:

1) $1 \in J$,
2) $x \in J$ and $x \to y \in J \Rightarrow y \in J$.

DEFINITION 3.2. Let A be a C algebra and J be a filter of A. We define a relation $\sim_J$ on A as follows:

$$x \sim_J y \Leftrightarrow x \to y \in J \quad \text{and} \quad y \to x \in J\,.$$

THEOREM 3.3. *For any C algebra and any filter J of A, the relation $\sim_J$ is a congruence relation and $A/\sim_J$ is naturally a C algebra. ($A/\sim_J$ is denoted by A/J.)*

DEFINITION 3.4. Let A be a C algebra, x be an element of A other than 1. A is *irreducible with respect to* x if x is contained within any filter of A which contains at least an element other than 1. A is *irreducible*, if there exists an element such that A is irreducible with respect to the element or A has only one element 1.

THEOREM 3.5. [1] *Any irreducible C algebra is linearly ordered.*

Now, we will prove the following lemma.

LEMMA 3.6.

3.14. *If $x \leqq y$, then $z \to x \leqq z \to y$ and $x \to z \leqq y \to z$.*

3.15. $(x \to y) \to z \to y = (y \to x) \to z \to x$.

3.16. $(y \to x) \to y \to z = (x \to y) \to x \to z$.

3.17. *If $y \leqq x$, then $(x \to y) \to z \to y = z \to x$.*

3.18. *If $x \leqq y$, then $(y \to x) \to y \to z = x \to z$.*

Proof. By 3.3 and 3.10, we have immediately 3.14. As for 3.15,

$$\begin{aligned}(x \to y) \to z \to y &= z \to (x \to y) \to y &&\text{(by 3.10)}\\ &= z \to (y \to x) \to x &&\text{(by 3.4)}\\ &= (y \to x) \to z \to x &&\text{(by 3.10).}\end{aligned}$$

By 3.5, we have $(x \to y) \to y \to x \leqq y \to x$. Hence, we have (*) $(x \to y) \to y \to x = y \to x$. In 3.15, we substitute $x \to z$ and $y \to z$ for x and y, respectively. Then, using 3.15, we have

$$(**) \qquad [(z \to x) \to y \to x] \to y \to z = [(z \to y) \to x \to y] \to x \to z\,.$$

Hence, we have

$$\begin{aligned}
(y \to x) \to y \to z &\leqq [(z \to x) \to y \to x] \to (z \to x) \to y \to z && \text{(by 3.13)} \\
&= (z \to x) \to [(z \to x) \to y \to x] \to y \to z && \text{(by 3.10)} \\
&= (z \to x) \to [(z \to y) \to x \to y] \to x \to z && \text{(by (**))} \\
&= [(z \to y) \to x \to y] \to (z \to x) \to x \to z && \text{(by 3.10)} \\
&= [(z \to y) \to x \to y] \to x \to z && \text{(by (*))} \\
&\leqq (x \to y) \to x \to z && \text{(by 3.14)} .
\end{aligned}$$

Because we get the converse inequality if we exchange x and y, we have 3.16. 3.17 and 3.18 are immediately obtained from 3.15 and 3.16, respectively. Q.E.D.

Let A be a C algebra. If $f(P) = 1$ for any assignment of A, we say that P is valid in A. We denote the set of C formulas valid in A by $L(A)$. Let L and W be a SLIL and the set of all C formulas, respectively. We define the Lindenbaum C algebra $\lambda(L)$. The elements of $\lambda(L)$ is equivalence classes of determined by the equivalence relation $\sim_L$ on W defined by ($P \sim_L Q \Leftrightarrow P \supset Q \in L$ and $Q \supset P \in L$). The functions 1 and $\to$ are defined as follows:

$$[P] \to [Q] = [P \supset Q] , \qquad 1 = [P \supset P] .$$

Here $[P]$ denotes the equivalence class containing the C formula P. $\lambda^n(L)$ denotes the subalgebra of $\lambda(L)$ generated by elements corresponding to propositional variables $p_1, p_2, \cdots, p_n$. The following theorems are easily shown.

THEOREM 3.7. *For any C algebra $A, L(A)$ is a SLIL. Conversely, for any SLIL L, $L = L(\lambda(L)) = \bigcap_{n<\omega} L(\lambda^n(L))$.*

THEOREM 3.8. *If a C algebra B is a subalgebra of a C algebra A, or $B = A/J$ for some filter J of A, then $L(B) \supseteq L(A)$.*

By the next theorem, it is sufficient in many cases only if we deal with irreducible and finitely generated C algebras.

THEOREM 3.9. *For any SLIL L, there exists a set $\{A_\lambda\}_{\lambda \in \Lambda}$ of irreducible and finitely generated C algebras such that $L = \bigcap_{\lambda \in \Lambda} L(A_\lambda)$.*

Proof. Let P be a C formula such that $P \notin L$. Let n be the number of different propositional variables appearing in P. Then, $P \notin L(\lambda^n(L))$. Let f be an assignment of $\lambda^n(L)$ such that $f(P) \neq 1$. Let

J be a filter of $\lambda^n(L)$ not containing $f(P)$ such that for any filter $K \supsetneqq J$ $f(P) \in K$. By Zorn's lemma, such a filter exists. Then, we can show that $\lambda^n(L)/J$ is an irreducible and n-generated C algebra and $\boldsymbol{L} \subseteq L(\lambda^n(L)/J)$. Therefore, for any C formula P such that $P \notin \boldsymbol{L}$, there exists an irreducible and finitely generated C algebra A such that $P \notin L(A)$ and $\boldsymbol{L} \subseteq L(A)$. Hence, we have the theorem. Q.E.D.

We denote the set $\{0, 1/m, 2/m, \cdots, (m-1)/m, 1\}$ and the set of all rationals in the interval $[0,1]$ by S_m $(m \geqq 1)$ and S_ω, respectively. We define the function on S_m $(1 \leqq m \leqq \omega)$ by $x \to y = \min(1, 1 - x + y)$. Then, we can regard S_m as a C algebra. S_m is the well-known Łukasiewicz $(m+1)$-valued (or $\aleph_0$-valued if $m = \omega$) model. We denote also the C algebra with only one element by S_0.

S_m is isomorphic to a subalgebra of S_n if $m \leqq n$. A finitely generated subalgebra of S_ω is isomorphic to S_m for some $m < \omega$. The result by Rose and Rosser [3] is that $L(S_\omega) = \boldsymbol{Lu}$. Hence, the following theorem is immediate.

THEOREM 3.10.

$$W = L(S_0) \supsetneqq L(S_1) \supsetneqq L(S_2) \supsetneqq \cdots \supsetneqq L(S_n) \supsetneqq \cdots \supsetneqq L(S_\omega) = \bigcap_{n<\omega} L(S_n) = \boldsymbol{Lu} .$$

In the above theorem, the inequalities due to the fact that $(\mathrm{p}\supset)^m q \vee \mathrm{p} \in L(S_m)$ and $\notin L(S_{m+1})$, where we define $(P\supset)^n Q$ as $(P\supset)^0 Q = Q$ and $(P\supset)^{n+1} Q = P \supset (P\supset)^n Q$.

DEFINITION 3.11. Let A be a C algebra and x be an element of A. *The order of* x, denoted by $\mathrm{ord}(x)$, is the least integer n such that $x \cup (x\to)^n y = 1$ for any element y of A. If no such integer n exists, then $\mathrm{ord}(x) = \omega$. $\mathrm{ord}(A)$ defined by

$$\mathrm{ord}(A) = \sup\{\mathrm{ord}(x) \mid x \in A\} .$$

It should be noticed that even if $\mathrm{ord}(x) < \omega$ for any x, it is possible that $\mathrm{ord}(A) = \omega$.

LEMMA 3.12. (1) *If* $x < y \leqq z$, *then* $z \to x < z \to y$.
(2) *If* $z \leqq x < y$, *then* $y \to z < x \to z$.

Proof. Suppose $(z \to y) \to z \to x = 1$. By 3.16, $(y \to z) \to y \to x = 1$. By $y \leqq z$, $y \to x = 1$. This is contradictory to $x < y$. Proof of (2) is similar. Q.E.D.

THEOREM 3.13. *If* $\operatorname{ord}(A) = m$ *and* A *is irreducible, then* A *is isomorphic to* S_m.

Proof. When m equals 0 or 1, the proof is easier than others. So we will prove it only for $m \geqq 2$.

Since A is irreducible, there exist a and b such that $(a\to)^m b = 1$ and $(a\to)^{m-1} b \neq 1$. We show that a is the second element from the top element 1 and b is the least element. Suppose x is an element such that $a < x < 1$. Suppose that $x \to x \to b \nleqq a \to b$. Since A is linearly ordered, $a \to b < x \to x \to b$ and $1 = (a \to b) \to x \to x \to b = x \to (a \to b) \to x \to b = x \to (b \to a) \to x \to a = x \to x \to a$, by 3.10, 3.15 and $b < a$. Hence, $x \leqq x \to a$. Then, we have, by 3.14, 3.16 and $a < x$,

$$\begin{aligned}[(x \to a) \to (x \to a) \to b] \to a \to b &\geqq [(x \to a) \to x \to b] \to a \to b \\ &= [(a \to x) \to a \to b] \to a \to b = 1 .\end{aligned}$$

Therefore, we have $w \to w \to b \leqq a \to b$ if we put $w = x$ or $w = x \to a$. By 3.10 and 3.14, we have

$$(w\to)^{2(m-1)} b \leqq a \to (w\to)^{2(m-2)} b \leqq \cdots \leqq (a\to)^{m-1} b < 1 .$$

This is contradictory to $\operatorname{ord}(A) = m$.

Suppose x is an element such that $x < b$. When $a \to x \leqq b$, we have $(a\to)^m x \leqq (a\to)^{m-1} b < 1$. This is contradictory to $\operatorname{ord}(A) = m$. When $b < a \to x < a \to b$, we have $1 > (a \to x) \to b > (a \to b) \to b = a$. This contradicts that a is the second element from the top.

By Lemma 3.12, we can show that

$$b = (a\to)^m b \to b < (a\to)^{m-1} b \to b < \cdots < (a \to b) \to b = a < b \to b = 1 .$$

Let x be an element such that $(a\to)^{n+1} b \to b < x < (a\to)^n b \to b$ $(n < m)$. Then, by 3.17 and 3.18, we have

$$\begin{aligned}1 > x \to (a\to)^{n+1} b \to b &> [(a\to)^n b \to b] \to (a\to)^{n+1} b \to b \\ &= (a\to)^{n+1} b \to (a\to)^n b = \cdots = (a \to b) \to b = a .\end{aligned}$$

This contradicts that a is the second element. Hence, any element is of the form $(a\to)^n b \to b$ for some $n \leqq m$. And we have

$$\begin{aligned}&[(a\to)^n b \to b] \to (a\to)^k b \to b \\ &\quad = (a\to)^k b \to (a\to)^n b = \begin{cases} 1 & (k \leqq n) \\ (a\to)^{k-n} b \to b & (k > n) . \end{cases}\end{aligned}$$

Hence, it is obvious that the function f defined by $f((a\to)^n b \to b) = (m-n)/m$ is an isomorphism from A onto S_m. Q.E.D.

THEOREM 3.14. *If* $\mathrm{ord}(A) = \omega$ *and* A *is irreducible, then* A *has a subalgebra isomorphic to* S_n *for any non-negative integer* n.

Proof. By the hypothesis, there exist a and b in A such that $(a\to)^n b \neq 1$ and $a \neq 1$. We can, similarly to the proof of Theorem 3.13, show that

$$(a\to)^n b \to b < (a\to)^{n-1} b \to b < \cdots < (a\to)^2 b \to b < a < 1\,,$$

and this series is a subalgebra of A isomorphic to S_n. Q.E.D.

§4. Main result

Now, we can prove the main theorem.

THEOREM 4.1. *For any SLIL* $\boldsymbol{L}$, $\boldsymbol{L} = L(S_n)$ *for some* $n \leqq \omega$.

Proof. By Theorem 3.9, there exists a set $\{A_\lambda\}_{\lambda\in\Lambda}$ of irreducible C algebras such that $\boldsymbol{L} = \bigcap_{\lambda\in\Lambda} L(A_\lambda)$. Let n be $\sup\{\mathrm{ord}(A_\lambda) \mid \lambda\in\Lambda\}$. By Theorem 3.10, Theorem 3.13 and Theorem 3.14, we have $\boldsymbol{L} = L(S_n)$.
Q.E.D.

$\boldsymbol{Lu} + P$ denotes the SLIL obtained by adding a C formula P as an axiom schema.

COROLLARY 4.2. *For any* C *formula* P *such that* $P \in L(S_n)$ *and* $P \notin L(S_{n+1})$ $(n < \omega)$, $\boldsymbol{Lu} + P = L(S_n)$.

REFERENCES

[1] Y. Komori, The separation theorem of the $\aleph_0$-valued Lukasiewicz propositional logic, Rep. Fac. Sci., Shizuoka Univ., **12** (1978), 1–5.
[2] A. Rose, Formalisation du calcul propositionnel implicatif a $\aleph_0$ valeurs de Lukasiewicz, C. R. Acad. Sc. Paris, **243** (1956), 1183–1185.
[3] A. Rose and J. B. Rosser, Fragments of many-valued statement calculi, Trans. Amer. Math. Soc., **87** (1957), 1–53.

Department of Mathematics
Faculty of Science
Shizuoka University