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FUNCTIONS MEROMORPHIC ON SOME RIEMANN SURFACES

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Consider an open Riemann surface R and a single-valued meromorphic function $w = f(p)$ defined on R. A value w_0 in the extended complex plane is said to be a *cluster value* for $w = f(p)$ if there exists a sequence $\{p_n\}$ in R accumulating at the ideal boundary of R such that

$$\lim_{n\to\infty} f(p_n) = w_0 .$$

The totality of cluster values is referred as the *global cluster set* for $w = f(p)$ and denoted by $C_R(f)$.

In terms of global cluster sets, we define the next two classes of open Riemann surfaces as follows;

DEFINITION. *We say that* $R \in C_{AB}$ *(resp.* C_{HB}*) if the global cluster set for each meromorphic function on* R *is either total or AB-removable (resp. HB-removable).*

We have shown in our preceding paper [1] that the following strict inclusion relations hold;

$$O_G < O_{HB} \begin{array}{l} \angle\, O_L \;\; < C_{HB} \\ \qquad\qquad\;\; \wedge \\ \nwarrow\, O^\circ_{AB} < C_{AB} < O_{AB} \end{array}$$

in general, and

$$O_G = C_{HB} < O^\circ_{AB} \subset C_{AB} = O_{AB}$$

for surfaces of finite genus. Moreover we have shown that if there exists a non-constant meromorphic function on R whose global cluster set is HB-removable then $R \in O_G$. Since $O_G \subset C_{HB}$, we deduce the following

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PROPOSITION. *In order that $R \in C_{HB}$, it is necessary and sufficient that one of the following alternatives holds:*

(i) *the global cluster set for any non-constant meromorphic function on R is total;*

(ii) *there exists a non-constant meromorphic function on R whose global cluster set is HB-removable.*

We shall show that the counter-part to the above is valid for AB.

THEOREM 1. *Let R be an open Riemann surface. Suppose that there exists a non-constant meromorphic function $w = f(p)$ on R whose global cluster set is AB-removable. Then $R \in C_{AB}$.*

We start by proving the following

LEMMA. *Let R be an n-sheeted complete covering surface over the z-sphere and E be an AB-removable set in the z-plane. Let e be the set of points on R which lie over E. Then e is AB-removable.*

Proof. Suppose that e is not AB-removable. Since R is closed, there exists a non-constant bounded analytic function $\varphi(p)$ on $R - e$. Without loss of generality we may assume that $\varphi(p_0) = 0$ for some $p_0 \in R - e$, where p_0 lies over $z_0 \in \mathscr{C}E$ (throughout this paper, $\mathscr{C}$ denotes the complement). For every $z \in \mathscr{C}E$, let $\{p_1^z, p_2^z, \cdots, p_n^z\}$ be the set of points of R which lie over z. Consider the function

$$F(z) = \varphi(p_1^z)\varphi(p_2^z) \cdots \varphi(p_n^z) .$$

Then $F(z)$ is a single-valued bounded analytic function on $\mathscr{C}E$ such that $F(z_0) = 0$. On the other hand, since $\varphi(p)$ does not vanish identically, $F(z) \neq 0$ for some $z \in \mathscr{C}E$. Hence $F(z)$ is a non-constant single-valued bounded analytic function on $\mathscr{C}E$. This contradicts the fact that E is AB-removable.

Proof of Theorem 1. Since $w = f(p)$ has Iversen's property, $w = f(p)$ covers every point of the w-sphere except $C_R(f)$ the same finite number of times. We use the same notation R for the covering surface. Let $g(p)$ be an arbitrary non-constant meromorphic function on R. We have only to show that $C_R(g)$ is AB-removable if it is not total. Suppose that $C_R(g)$ is not total. Then, without loss of generality, we may assume that $g(p)$ is bounded on a set of points of R which lie over a neighbourhood of $C_R(f)$.

Choose a point w_0 in $\mathscr{C}C_R(f)$ over which no branch points lie. In a sufficiently small neighbourhood of w_0, setting

$$g_i(w) = g(p_i^w) \qquad (i = 1, 2, \cdots, n) ,$$

we obtain n function elements $g_1(w), g_2(w), \cdots, g_n(w)$ with the center at w_0, where $\{p_1^w, p_2^w, \cdots, p_n^w\} = f^{-1}(w)$. Set $\mathscr{G} = \{g_1(w), g_2(w), \cdots, g_n(w)\}$. Here, we notice that every element of $\mathscr{G}$ can be continued everywhere on $\mathscr{C}C_R(f)$ by allowing algebraic elements and that some elements of $\mathscr{G}$ may coincide with each other. Hereafter, we partition the proof into two cases.

(i) Suppose that any two elements of $\mathscr{G}$ do not coincide with each other. Consider the symmetric functions of $g_1(w), g_2(w), \cdots, g_n(w)$:

$$\begin{aligned} \sigma_1(w) &= \sum_i g_i(w) , \\ \sigma_2(w) &= \sum_{i<j} g_i(w) g_j(w) , \\ &\vdots \\ \sigma_n(w) &= g_1(w) g_2(w) \cdots g_n(w) . \end{aligned}$$

Since any two elements of $\mathscr{G}$ are continued from each other on $\mathscr{C}C_R(f)$, each $\sigma_k(w)$ $(k = 1, 2, \cdots, n)$ can be continued everywhere on $\mathscr{C}C_R(f)$ and is single-valued meromorphic on $\mathscr{C}C_R(f)$. Moreover, by the assumption, each $\sigma_k(w)$ is bounded on a neighbourhood of $C_R(f)$. Since $C_R(f)$ is AB-removable, each $\sigma_k(w)$ is analytic throughout $C_R(f)$ and therefore rational. Hence, all elements of $\mathscr{G}$ are defined by the algebraic function

$$W^n - \sigma_1(w) W^{n-1} + \sigma_2(w) W^{n-2} + \cdots + (-1)^n \sigma_n(w) = 0 .$$

Obviously, by the assumption, the above equation is irreducible. Consequently we obtain the n-valued algebraic function $G(w)$ defined by the above equation. Denote by $C(G)$ the cluster set for $G(w)$ at $C_R(f)$. By the preceding lemma and the fact that $O_{AB} = C_{AB}$ for surfaces of finite genus, we see that $C(G)$ is AB-removable. For every cluster value ζ of $g(p)$, we can find a sequence $\{p_n\}$ in R accumulating at the ideal boundary such that $\lim_{n\to\infty} g(p_n) = \zeta$ and $w_n = f(p_n) \in \mathscr{C}C_R(f)$, for any n. Since $\{w_n\}$ accumulates to $C_R(f)$, $\zeta \in C(G)$, i.e. $C_R(g) \subset C(G)$. Therefore the conclusion follows.

(ii) Let $\mathscr{G}^{\sim} = \{g_{11}(w), g_{12}(w), \cdots, g_{1m}(w)\}$ be the totality of elements of

$\mathscr{G}$ which coincide with $g_1(w) = g_{11}(w)$ containing $g_1(w)$ itself $(m \leq n)$. If there exists no element which does not coincide with $g_1(w)$, $g_1(w) = \cdots = g_n(w)$ is single-valued on $\mathscr{C}C_R(f)$. Hence, we have nothing to prove since $O_{AB} = C_{AB}$ for plane regions. Suppose that there exists at least one element $g_{21}(w)$ of $\mathscr{G}$ which does not coincide with $g_1(w)$. Choose a path L in $\mathscr{C}C_R(f)$ which starts from w_0 and terminates at w_0 and along which $g_{21}(w)$ is continued from $g_{11}(w)$. Without loss of generality, we may assume that there exist no branch points of R over L. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ be the totality of paths on R which lie over L and such that each λ_i starts from $p_{1i}^{w_0}$, where $p_{1i}^{w_0}$ is the point of R corresponding to the element $g_{1i}(w)$. Let $p_{2i}^{w_0}$ be the terminal point of λ_i. Then any terminal point does not coincide with the initial point, and does not coincide with any other terminal point. Hence, there exist m function elements $g_{21}(w)$, $g_{22}(w), \cdots, g_{2m}(w)$ such that each $g_{2i}(w)$ is continued from $g_{1i}(w)$ along λ_i and $g_{21}(w) = g_{22}(w) = \cdots = g_{2m}(w)$. Consequently, we see that if there exist l distinct elements $g_{11}(w), g_{21}(w), \cdots, g_{l1}(w)$ in $\mathscr{G}$, then $n = ml$, we can classify $\mathscr{G}$ into the following m classes

$$\begin{aligned}
\mathscr{G}_1 &= \{g_{11}(w), g_{21}(w), \cdots, g_{l1}(w)\}\,, \\
\mathscr{G}_2 &= \{g_{12}(w), g_{22}(w), \cdots, g_{l2}(w)\}\,, \\
&\vdots \\
\mathscr{G}_m &= \{g_{1m}(w), g_{2m}(w), \cdots, g_{lm}(w)\}\,,
\end{aligned}$$

where $g_{11}(w) = g_{12}(w) = \cdots = g_{1m}(w), g_{21}(w) = g_{22}(w) = \cdots = g_{2m}(w), \cdots,$ $g_{l1}(w) = g_{l2}(w) = \cdots = g_{lm}(w)$. Hence, using the same argument as in the first case, we see that every $\mathscr{G}_i$ defines the same l-valued algebraic function $G(w)$ and that the global cluster set for $G(w)$ at $C_R(f)$ is AB-removable. Therefore $C_R(g)$ is AB-removable. This completes the proof.

COROLLARY 1. *In order that $R \in C_{AB}$, it is necessary and sufficient that one of the following alternatives holds:*

(i) *the global cluster set for any non-constant meromorphic function on R is total;*

(ii) *there exists a non-constant meromorphic function on R whose global cluster set is AB-removable.*

COROLLARY 2. *Let R be an n-sheeted complete covering surface over the sphere less an AB-removable set. Then $R \in O_{AB}$.*

Proof. By Theorem 1, $R \in C_{AB}$. Hence, $R \in O_{AB}$ since $C_{AB} \subset O_{AB}$.

Related to the preceding theorem, we append the following result

THEOREM 2*. *Let R be an n-sheeted complete covering surface over the z-sphere less an AB-removable set E. Then there exists a meromorphic function on R which is bounded on the set of points of R lying over a neighbourhood of E (or whose global cluster set is not total) and which does not assume identically equal values on any two distinct sheets if and only if R is a subsurface of a closed surface.*

Proof. If R is a subsurface of a closed surface, as is well-known, there exists a meromorphic function on R which has a pole only at a non-branch point of R and is regular elsewhere. It is immediate that such a function does not assume identically equal values on any two distinct sheets.

Next, suppose that there exsits a meromorphic function $w = f(p)$ on R satisfying the condition stated above. On $\mathscr{C}E$, we consider the function

$$F(z) = \left[\prod_{i<j} \{f(p_i^z) - f(p_j^z)\}\right]^2 ,$$

where $\{p_1^z, p_2^z, \cdots, p_n^z\}$ is the set of points of R which lie over z. Then we can see that $F(z)$ is single-valued on $\mathscr{C}E$. By the assumption, $F(z)$ is bounded on a neighbourhood of E and does not vanish identically on $\mathscr{C}E$. Since E is AB-removable, $F(z)$ is analytic throughout E and hence rational. On the other hand, $F(z)$ vanishes on the points over which branch points lie. Therefore, R has at most finitely many branch points.

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