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RATIONAL DOUBLE POINTS ON A NORMAL OCTIC $K3$ SURFACE

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0. Introduction

Let S be a normal surface of degree n in $\mathbf{P}^k_{\mathbf{C}}$, where $(n,k) = (4,3)$, $(6,4)$ or $(8,5)$. People try to describe all possible combinations of singularities on such surfaces. The case $(4,3)$ is already very complicated. Using properties of $K3$ surface and elementary transformations of Dynkin Graphs effectively, Urabe [17] was able to solve the problem partially when all singularities are rational double points.

In the following, we consider the case when $(n,k) = (8,5)$. This paper only concerns normal octic $K3$ surfaces in $\mathbf{P}^5$. But such a surface may not be a complete intersection. In this paper, we use Urabe's method to obtain a result concerning some possible combinations of rational double points on the class of surfaces with isolated singularities as its only singularities. We also give a criterion concerning only a simple combinatorial condition determine when an octic $K3$ surface in $\mathbf{P}^5$ is a complete intersection.

We assume that every variety is algebraic and is defined over the complex number field $\mathbf{C}$.

DEFINITION 0.1. A disjoint finite union of connected Dynkin Graphs of type A, B, D or E is called a Dynkin Graph. The following procedure is called an elementary transformation of such a Dynkin Graph:

(1) Replace each connected component by the corresponding extended Dynkin Graph;

(2) Choose in an arbitrary manner at least one vertex from each component (of the extended Dynkin Graph) and then remove these vertices together with the edges issuing from them [4].

Note that any connected Dynkin Graph of type A, D or E corresponds to a

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singularity on a surface [6].

If a Dynkin Graph G contains a_k connected components of type A_k, b_l components of type D_l, c_m components of type E_m and d_n components of type B_n ($k \geq 1$, $l \geq 4$, $m = 6, 7, 8$, $n \geq 1$), we denote

$$G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m + \sum d_n B_n.$$

MAIN THEOREM 0.2. *Let $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ (a finite sum) be a Dynkin Graph with components of type A, D or E only. Set $r = \sum a_k k + \sum b_l l + \sum c_m m$. Then the following conditions* (A) *and* (B) *are equivalent.*

(A) *There exists a normal octic $K3$ surface in $\mathbf{P}^5$ whose combination of singularities corresponds to G, and moreover one of the following conditions $\langle 1 \rangle$, $\langle 2 \rangle$, $\langle 3 \rangle$, $\langle 4 \rangle$ holds for the root lattice $Q = Q(G)$ of type G.*

$\langle 1 \rangle$ $r = 17$, $2d(Q) \in \mathbf{Q}^{*2}$, *and* $\varepsilon_p(Q) = 1$ *for every prime number p,*

$\langle 2 \rangle$ $r = 16$, $\varepsilon_p(Q) = (-2, d(Q))_p$ *for every prime number p,*

$\langle 3 \rangle$ $r = 15$, $-2d(Q) \notin \mathbf{Q}_p^{*2}$ or $\varepsilon_p(Q) = (-1, -1)_p$ *for every prime number p,*

$\langle 4 \rangle$ $r \leq 14$.

(B) *G coincides with a Dynkin Graph which is obtained from one of the following* 19 *basic Dynkin Graphs by elementary transformation repeated twice.*

$2E_8 + A_1$, $D_{16} + A_1$, A_{17}, $D_{10} + E_7$, $2E_8$, D_{16}, $D_4 + D_{12}$, $2D_8$, $E_8 + D_8$, $2A_1 + 2E_7$, $2A_8$, $A_{12} + A_4$, A_{16}, $A_7 + 2D_5$, $A_9 + A_1 + D_6$, $A_7 + D_9$, A_{15}, $E_7 + A_9$, $E_6 + D_7 + A_3$.

(C) *In particular, if the Dynkin Graph G satisfies condition* (B), *then there is a normal octic $K3$ surface which is a complete intersection of three quadrics in $\mathbf{P}^5$ whose combination of singularities corresponds to G.*

Remark and explanations

1. $r = \operatorname{rank} Q =$ The number of vertices in G;

2. The symbol $\varepsilon_p(Q) \in \{+1, -1\}$ is the Hasse symbol of the inner product space $Q \otimes \mathbf{Q}$ over $\mathbf{Q}$. The symbol $(\ ,\ )_p$ is the Hilbert symbol. $\mathbf{Q}_p$ is the field of p-adic numbers and $\mathbf{Q}_p^{*2} = \{a^2 \mid a \in \mathbf{Q}_p,\ a \neq 0\}$ [14]

3. If there is a normal octic $K3$ surface in $\mathbf{P}^5$, whose combination of singularities corresponds to G, then $r \leq 19$.

In Section 1, some notation and terminology are stated. Section 2 is devoted to the geometry on $K3$ surfaces. We owe essential ideas in this section to Saint-Donat [13]. Theorem 2.15 is the goal of Section 2. To show it we used the theory of integral bilinear forms. In Sections 3, 4 and 5, we explain this theory. It is explained in Section 3 why the elementary transformation of Dynkin Graphs is essential. In Section 5 the conditions on isotropic elements written with the Hasse symbol and Hilbert symbol are discussed. The proof of the converse of the main Theorem 0.2 is given in Section 6.

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1. Notation and terminology

A free $\mathbf{Z}$-module L of finite rank is called a *quasi-lattice* if L is supplied with a symmetric bilinear form $L \times L \to \mathbf{Q}$. If all possible values of the bilinear form are integers, then L is called a *lattice*.

For a quasi-lattice L, set $R(L) = \{x \in L \mid x^2 = 2 \text{ or } 1\}$. We say that L is a *root module* and $R(L)$ is the *root system* of L if the following two conditions are satisfied

(R_1) For every $\alpha \in R(L)$ and every $x \in L$, $\dfrac{2(\alpha, x)}{(\alpha, \alpha)} \in \mathbf{Z}$;

(R_2) For every $\alpha, \beta \in R(L)$, $(\alpha, \beta) \in \mathbf{Z}$.

If L is a root module, then the sub-module $Q(R(L)) = \sum_{\alpha \in R(L)} \mathbf{Z}\alpha$ of L is a lattice, called the *root lattice* of the root system $R(L)$. The elements in $R(L)$ are called the *roots*. A root α is a *long root* if $\alpha^2 = 2$, a *short root* if $\alpha^2 = 1$.

If a root module L is positive definite or negative definite, then there are only finitely many roots. Every finite root system can be decomposed to a direct sum of irreducible root systems, and every irreducible positive definite root system without short roots is of type A, D, or E.

Let p be a rational double point on a surface X, $\pi : \tilde{X} \to X$ be the minimal resolution of p. The intersection matrix of the exceptional set $\pi^{-1}(p)$ is negative definite. If we reverse the sign of every entry of this matrix, the lattice generated by the irreducible components of $\pi^{-1}(p)$ is a root lattice, which is of type A_k $(k \geq 1)$,

D_k $(k \geq 4)$, or E_k $(k = 6, 7, 8)$. The sets are exactly the irreducible root lattices without short roots.

2. Geometry of $K3$ surface

In this section, we will reduce the problem of determining the combination of rational double points on a normal octic $K3$ surface to a purely combinatorial problem. The main result is Theorem 2.15. First, we make some preparation.

LEMMA 2.1 ([13] Proposition 3.2). *Let $|D|$ be a complete linear system on a $K3$ surface, then $|D|$ has no base point outside its fixed components.*

LEMMA 2.2 ([13] Proposition 2.6). *Let D be a divisor on a $K3$ surface S such that $|D|$ is non-empty. Assume that $|D|$ has no fixed components. Then either*

(1) *$D^2 > 0$ and the general member of $|D|$ is an irreducible curve of arithmetic genus $\frac{D^2}{2} + 1$. In this case $h^1(O_S(D)) = 0$; or*

(2) *$D^2 = 0$, then D is linearly equivalent to kE, where k is a positive integer and E is an irreducible curve of arithmetic genus 1. In this case, $h^1(O_S(D)) = k - 1$ and every member of $|D|$ can be written as a sum $E_1 + E_2 + \cdots + E_k$, where $E_i \in |E|$ for $i = 1, \ldots, k$. Thus $|E|$ is also base point free.*

LEMMA 2.3 ([18]). *Let D be a nef divisor with $D^2 = 2n$ on a $K3$ surface S where n is a positive integer. Then $h^0(O_S(D)) = n + 2$ and $h^1(O_S(D)) = h^2(O_S(D)) = 0$. Moreover, the following three conditions are equivalent.*

1. *The complete linear system $|D|$ has a base point.*
2. *$|D|$ has a fixed component.*
3. *There is a smooth elliptic curve E and a smooth rational curve A such that $(n+1)E + A \in |D|$, $EA = 1$, $E^2 = 0$ and $A^2 = -2$.*

Proof. Kawamata's vanishing theorem [9] implies that $h^1(O_S(-D)) = 0$. Meanwhile, $-DH \leq 0$ for every very ample divisor H, for D is nef. Thus $h^0(O_S(-D)) = 0$. Hence $h^1(O_S(D)) = h^2(O_S(D)) = 0$ by Serre duality. Then Riemann-Roch Theorem implies that $h^0(O_S(D)) = n + 2$ immediately. The equivalence of 1 and 2 is clear by Lemma 2.1.

$2 \Rightarrow 3$ Write $D = V + F$, in which F denotes the fixed part. Suppose that $V^2 > 0$. Then $D^2 = V^2 + VF + DF > 0$. Since $\dim|V| = \dim|D|$, the first part of this lemma implies $V^2 = D^2$, thus $VF + DF = 0$. Since both VF and DF

are non-negative, we have $VF = F^2 = 0$, which is a contradiction. Thus V^2 is zero. By Lemma 2.2 there is a smooth elliptic curve E and a positive integer k such that V is linearly equivalent to kE, since $h^0(O_S(kE)) = k+1 = h^0(O_S(D)) = n+2$, k must be equal to $n+1$. Note that $VF > 0$, otherwise $F^2 = 2n$, which would contradict Hodge index theorem. Since $2n = D^2 = VF + DF \geq VF = (n+1)EF$, we have $EF = 1$ and so $F^2 = -2$. Write $F = \sum_i F_i$, where F_i's are the irreducible curves. We may assume that $EF_1 = 1$ and $EF_i = 0$ for $i \neq 1$. Every F_i must be a smooth rational curve, otherwise $F_i^2 \geq 0$ and $\dim |F_i| \geq 1$ by Riemann-Roch Theorem. Denote F_1 by A and $F - A$ by B. It suffices to show that $B = 0$. Suppose that $B \neq 0$. Let A_i be an arbitrary component of B. Thus $FA_i = DA_i \geq 0$, since $F^2 = \sum_i FA_i = -2$, we must have $FA \leq -2$. Note that $AB \geq 0$, for A is not a component of B. Thus $FA = A^2 + AB \geq A^2 = -2$, hence $AB = 0$, which means that A does not meet B. Since the intersection matrix for the components of B is negative definite, there exists a component A_i of B such that $A_iB < 0$. So $A_iD < 0$, which contradicts the assumption that D is nef.

$3 \Rightarrow 2$ Assume that D is linearly equivalent to $(n+1)E + A$, where E is a smooth elliptic curve and A is a smooth rational curve on S with $EA = 1$, then $D^2 = 2n$. Hence $h^0(O_S(D)) = n+2$ by the first part of the Lemma. Easy calculation shows that $h^0(O_S((n+1)E)) = n+2$, which implies that A is a fixed component of $|D|$. Q.E.D.

Some facts (see [13] Sec. 4 p. 614)

Let L be an invertible sheaf on a $K3$ surface F such that $L^2 > 0$ and $|L|$ is base point free. By Lemma 2.2 $L \cong O_F(C)$ where C is an irreducible curve. We shall denote by ϕ_L the map $F \to \mathbf{P}^{p_a(L)}$ defined by $|L|$. Note that $\dim \phi_L(F) = 2$ and $\phi_L(F)$ is not contained in any hyperplane. The degree of $\phi_L(F)$ is at least $p_a(L) - 1$, so we see that only two cases can occur:

Either (i) ϕ_L is of degree 2 and its image has degree $p_a(L) - 1$;

or (ii) ϕ_L is birational and its image has degree $2p_a(L) - 2$.

In the first case, we shall say that L is *hyperelliptic*: In the second case, L is *non-hyperelliptic*.

LEMMA 2.4 ([13] Theorem 5.2). *Let $|L|$ be a complete linear system on a $K3$ surface F, without fixed components such that $L^2 \geq 4$. Then L is hyperelliptic only in the following cases.*

(i) *There exists an irreducible curve E such that $p_a(E) = 1$ and $EL = 2$;*

(ii) *There exists an irreducible curve B such that $p_a(B) = 2$ and $L \cong O_F(2B)$.*

LEMMA 2.5 ([13] Proposition 5.6). *Let B be an irreducible curve on a $K3$ surface F such that $p_a(B) = 2$ and let $L = O_F(2B)$. Then $\phi_L(F)$ is the Veronese surface in $\mathbf{P}^5$: In fact, $\phi_L = u_2 \circ \phi_B$, where $u_2 : \mathbf{P}^2 \to \mathbf{P}^5$ is the Veronese embedding.*

LEMMA 2.6 ([13] Theorem 7.2). *Let $|L|$ be a non-hyperelliptic complete linear system on a $K3$ surface F without base point and $L^2 \geq 8$. Let I be the kernel of the canonical surjective map*

$$S^*H^0(L) \to \oplus_{n\geq 0}H^0(L^n).$$

Then the graded ideal I is generated by its elements of degree 2 and 3. I is generated by its elements of degree 2 except in the following cases:

i) *There exists an irreducible curve E such that $p_a(E) = 1$ and $E\cdot L = 3$.*

ii) *$L \cong O_F(2B + \Gamma)$ where B is an irreducible curve of genus 2, and Γ is an irreducible rational curve such that $B\Gamma = 1$.*

We denote by ε_L be the set of irreducible curves Δ such that $L\Delta = 0$; It follows from Hodge index Theorem that such a curve is rational and non-singular. Moreover ε_L is finite. Let $(\varepsilon_L^\lambda)_{\lambda=1,\ldots,N}$ be the connected components of ε_L: Hodge index Theorem implies that the intersection matrix of ε_L^λ is negative definite. If $\varepsilon_L^\lambda = \Delta_1,\ldots,\Delta_r$, we defined the fundamental cycle E_L^λ having for support ε_L^λ by the following conditions: $E_L^\lambda = \sum_{i=1}^r m_i \Delta_j$, $E_L^\lambda\Delta_i \leq 0$ for all i, $m_i \geq 0$, the m_i are the smallest.

Then there exists (M. Artin [1] p.638), by contraction of the E_L^λ, a normal surface F_L having only rational double points and a map $\theta_L : F \to F_L$ such that

(i) $\theta_L(\varepsilon_L^\lambda)$ is a point p_λ,

(ii) $\theta_L^{-1}(p_\lambda) = E_L^\lambda$ as schemes for all λ,

(iii) $\theta_L : F \setminus \cup_\lambda \varepsilon_L^\lambda \to F \setminus \cup p_\lambda$ is an isomorphism.

Furthermore, using Zariski's main Theorem we can say that ϕ_L admits a factorization $\phi_L = u_L \circ \theta_L$ where $u_L : F_L \to \phi_L(F)$ is a finite morphism; Morever, F_L is the normalization of $\phi_L(F)$ in the function field $K(F)$.

Now assume that $L^2 \geq 8$, $|L|$ is a non-hyperelliptic complete linear system without base point. Then ϕ_L induces an isomorphism onto its image outside ε_L (see [13] (6.5.13) p.625). Thus u_L is an isomorphism, so $\phi_L(F)$ is normal.

LEMMA 2.7 ([17] Proposition 1.9)). *Let L be a numerically effective line bundle on a $K3$ surface Z with $\deg L = L^2 > 0$.*

(1) *For every $M \in \operatorname{Pic} Z$ with $M^2 = -2$, either M or its dual M^*, and only one*

of them is effective.

(2) *Set* $R = \{M \in \mathrm{Pic}\, Z \mid M^2 = -2,\ ML = 0\}$ *and*

$\Delta = \{O_Z(C) \in \mathrm{Pic}\, Z \mid C$ *is an irreducible smooth rational curve,* $CL = 0,\ C^2 = -2\}$.

Then R *is a finite root system whose fundamental root system is* Δ. *Any irreducible components of* R *is of type* A, D, *or* E.

LEMMA 2.8 ([17] Corollary 1.10). *Let* E_L *denote the union of curves* C *such that* $O_Z(C) \in \Delta$. *Every connected component of* E_L *coincides with the exceptional curve in the minimal resolution of a rational double point. Let* $\rho : Z \to X$ *be the contraction morphism sending each connected component of* E_L *to a normal singular point.* X *has only rational double points as singularities and their combinations is described by the number of components of each type* A, D, E *in the irreducible decomposition* $R = \oplus R_i$ *of the root system* R.

Next, we state further properties related to numerical effectiveness.

The bilinear form induced by the intersection form on

$$H^{1,1}(Z, \mathbf{R}) = H^2(Z, \mathbf{R}) \cap H^1(Z, \Omega_Z^1)$$

has signature (1,19) for a $K3$ surface Z. Thus the positive cone

$$\Sigma = \{x \in H^{1,1}(Z, \mathbf{R}) \mid x^2 > 0\}$$

has two connected components. Let Σ_+ denote the component containing the *Kähler* class k, the other one is $\Sigma_- = -\Sigma_+$. Let $M \in H^2(Z, \mathbf{Z})$ be an element with $M^2 = -2$. We can define a linear isomorphism S_M :

$$S_M : H^2(Z, \mathbf{Z}) \to H^2(Z, \mathbf{Z})$$

by $S_M(P) = P + (P \cdot M)M$, we call S_M the reflection with respect to M, S_M induces an isomorphism

$$S_M : \mathrm{Pic}\, Z \to \mathrm{Pic}\, Z$$

if $M \in \mathrm{Pic}\, Z$. By Proposition 3.9, in [2, Chap. VIII], we have the following.

PROPOSITION 2.9. *Let* L *be a line bundle on a* $K3$ *surface* Z *with* $\deg L = L^2 > 0$, *such that* L *belongs to* Σ_+. *Then there are a finite number of elements* $M_1, M_2, \ldots, M_r$ *in* $\mathrm{Pic}\, Z$ *with* $M_i^2 = -2$ *for* $1 \le i \le r$ *so that* $S_{M_1} \ldots S_{M_r}(L)$ *are numerically effective.*

Here we recall the theory of periods for $K3$ surfaces.

Let Λ be an even unimodular lattice isomorphic to the second cohomology group of a $K3$ surface with the intersection form. It is known that

$$\Lambda \cong Q(-E_8) \oplus Q(-E_8) \oplus H \oplus H \oplus H.$$

The above $Q(-E_8)$ is a free $\mathbf{Z}$-module of rank 8 with the bilinear form which is -1 times that on the root lattice $Q(E_8)$ of type E_8. $H = \mathbf{Z}u + \mathbf{Z}v$ is the hyperbolic plane. H is a free $\mathbf{Z}$-module of rank 2 and $u^2 = v^2 = 0$, $uv = vu = 1$. A pair (Z, α) where Z is a $K3$ surface and $\alpha : H^2(Z, \mathbf{Z}) \to \Lambda$ is a linear isomorphism which preserves bilinear form, is called a *marked* $K3$ *surface.*

For any marked $K3$ surface (Z, α),

$$H^2(Z, \mathbf{R}) \otimes \mathbf{C} = H^2(Z, \mathbf{C})$$

has the Hodge decomposition

$$H^2(Z, \mathbf{C}) = H^2(Z, O_Z) \oplus H^1(Z, \Omega_Z^1) \oplus H^0(Z, K_Z).$$

We have a nowhere vanishing holomorphic 2-form ψ in $H^0(Z, K_Z)$, since the canonical line bundle K_Z is trivial. The 2-form ψ is unique up to the multiple of non-zero complex numbers. Thus the point

$$[\alpha(\psi)] = \alpha(\psi) \bmod \mathbf{C}^* \in \mathbf{P}(\Lambda \otimes \mathbf{C})$$

is uniquely determined by the pair (Z, a). The point $[\alpha(\psi)]$ is called the *period* of (Z, α). Set

$$\Omega = \{[\omega] \in \mathbf{P} \otimes \mathbf{C} \mid 0 \neq \omega \in \Lambda \otimes \mathbf{C},\ \omega\omega = 0,\ \omega\bar{\omega} = 0\}.$$

Then the point $[\alpha(\psi)] \in \Omega$.

The 20-dimensional complex manifold Ω is called the *period domain.*

THEOREM 2.10 ([2]). *For every point* $[\omega]$ *in* Ω, *there is a marked* $K3$ *surface* (Z, α) *whose period agrees with* $[\omega]$.

LEMMA 2.11. $\operatorname{Pic} Z = \{x \in H^2(Z, \mathbf{Z}) \mid x \cdot \psi = 0\}$.

LEMMA 2.12 ([11] Corollary 5.13). *If* $X \subset \mathbf{P}^n$ *is an irreducible projective variety and* X *is not contained in any hyperplane, then*

$$\deg X \geq \operatorname{codim} X + 1.$$

PROPOSITION 2.13. *Let L be a numerically effective line bundle of degree 8 on a $K3$ surface Z. Then the following two conditions are equivalent.*

(1) *$|L|$ does not define a morphism $\phi_L \to X \subset \mathbf{P}^5$ to an octic surface X.*

(2) *There is an element $M \in \operatorname{Pic} Z$ either with $M^2 = 0$, $ML = 2$ or with $L = 2M$.*

Proof. (1) $\Rightarrow$ (2) First of all, we assume that $|L|$ has a fixed point. Then Lemma 2.3 (3) holds. Then $L = O_Z(5E + \Gamma)$, $E\Gamma = 1$, $E^2 = 0$ and $\Gamma^2 = -2$. Let $M = O_Z(2E)$, then M satisfies condition (2) above. Next assume that $|L|$ has no base points. Since $h^0(L) = 6$ by Lemma 2.1, we have a morphism $\phi_L : Z \to \mathbf{P}^5$. By (1) we know that $|L|$ must be hyperelliptic. Then by Lemma 2.4 either there exists an irreducible curve E such that $p_a(E) = 1$ and $EL = 2$, or there exists an irreducible curve B such that $p_a(B) = 2$ and $L \cong O_Z(2B)$. Then $M = O_Z(B)$ or $M = O_Z(E)$ satisfies the above condition (2).
(2) $\Rightarrow$ (1) We shall deduce a contradiction assuming that (2) holds but (1) does not hold.

Case 1, $\exists\, M \in \operatorname{Pic} Z$ with $M^2 = 0$, $ML = 2$. Let M^* denote the dual line bundle of M. By Riemann-Roch Theorem,

$$h^0(M) + h^0(M^*) \geq \frac{M^2}{2} + 2 \geq 2.$$

Since $LM = 2$, $h^0(M) \geq 2$. Let $D + \Delta$ be a general member in the linear system $|M|$, Δ being the fixed part. Now $D^2 \geq 0$ and $\phi_L|_D$ is a generically one to one morphism, since the condition (1) does not hold. Note that every irreducible component of D has positive arithmetic genus by Lemma 2.2. If $LD = 0$ then $D^2 < 0$ by Hodge index Theorem. Thus $LD > 0$ since L is nef. By the same reason $L\Delta \geq 0$. Since $2 = LM = LD + L\Delta$, $\deg \phi_L(D) = LD \leq 2$. Let D' be an irreducible component of D. We have $\deg \phi_L(D') \leq 2$. By Lemma 2.12 we have $\phi_L(D')$ in $\mathbf{P}^2$. Hence $\phi_L(D') \cong \mathbf{P}^1$. Since $\phi_L|_{D'}$ is generically one to one, D' is isomorphic to $\mathbf{P}^1$, which is a contradiction.

Case 2, $\exists\, M \in \operatorname{Pic} Z$ with $L = 2M$. We claim that $|M|$ is base point free. Otherwise by Lemma 2.3, $|M| = |2E + A|$, where E is a smooth elliptic curve and A is a smooth rational curve, such that $EA = 1$, note that $h^0(E) = 2$. Since $LE = 2$, this implies $|L|$ is hyperelliptic by Lemma 2.4, which contradicts the above hypotheses. It follows that $|M|$ is base point free. By Lemma 2.5 $\phi_{|L|} = \phi_{|2M|}$ defines a double cover. Q.E.D.

THEOREM 2.14. *Let $|L|$ be a non-hyperelliptic complete linear system on a $K3$ surface Z without base point and $L^2 = 8$, and $\phi_L : Z \to \phi_L(Z) \subset \mathbf{P}^5$ be the associated morphism. Then the following two conditions are equivalent.*

(A) *$\phi_L(Z)$ is a complete intersection of three quadrics;*

(B) *There exists no $F \in \operatorname{Pic} Z$, $F^2 = 0$, $LF = 3$.*

Proof. Let $C \in |L|$ be a generic member of $|L|$. Then C is an irreducible curve.

(B) $\Rightarrow$ (A) is obvious by Lemma 2.6.

(A) $\Rightarrow$ (B) Let $F \in \operatorname{Pic} Z$, $F^2 = 0$, $LF = 3$. We will get a contradiction. Note that $h^0(F) \geq 2$.

Case 1, $|F|$ has no fixed part. By Lemma 2.2, there exists a positive integer k, so that $|F| = |kE|$, where E is a smooth elliptic curve. Hence $3 = LF = kLE$. If $k = 1$, then $LE = 3$. Then C has a g_3^1, which contradicts that $\phi_L(Z)$ is a complete intersection (see [3] Ex.11, Ch. Ⅷ). If $k = 3$, then $LE = 1$, $h^0(E) = 2$. $CE = 1$ implies $C \cong \mathbf{P}^1$ which contradicts $C^2 = 8$. Thus Case 1 can not occur.

Case 2. Let $F = M + T$, where T is the fixed part of $|F|$. Then we have the following subcases:

(i) $LM = 0$, then $M^2 < 0$, which contradicts $M^2 \geq 0$.

(ii) $LM = 1$ or 2. Note that every irreducible component D' of the general member D of $|M|$ has positive arithmetic genus by Lemma 2.2. Since

$$\phi_L|_{D'} : D' \to \phi_L(D') \subset \mathbf{P}^5,$$

is generically one to one and deg $\phi_L(D') \leq 2$, it implies that $\phi_L(D')$ is a curve in $\mathbf{P}^5$ with degree less than 3. So $\phi_L(D') \subset \mathbf{P}^2$ by Lemma 2.2. Thus $\phi_L(D')$ has an irreducible component which is isomorphic to $\mathbf{P}^1$. Since $\Phi_L|_{D'}$ is generically one to one, D' isomorphic to $\mathbf{P}^1$, which is a contradiction.

(iii) $LM = 3$. Since $L^2 > 0$, $L^2M^2 \leq (LM)^2$, by Hodge index theorem. So we have obtained $8M^2 \leq 9$. Thus $M^2 = 0$ since M^2 is neither negative nor odd. It returns to Case 1.

Finally, we have proved (A) $\Rightarrow$ (B). Q.E.D.

THEOREM 2.15. *Let $G = \sum a_k A_k + \sum b_l D_l + \sum e_l E_l$ be a Dynkin Graph with components of type A, D or E only. The following conditions are equivalent.*

(1) *There is a normal octic $K3$ surface in $\mathbf{P}^5$ with only rational double points as singularities, the combination of singularities corresponding to G;*

(2) *Let $Q = Q(G)$ be the root lattice of type G. Let Λ denote the unimodular even lattice with signature* (19,3). *The lattice $S = \mathbf{Z}\lambda \oplus Q$ ($\lambda^2 = -8$, orthogonal direct*

sum) has an embedding $S \subset \Lambda$ *satisfying the following conditions* (a) *and* (b). *Let* S *denote the primitive hull of* S *in* Λ.

(a) *If* $\eta \in \tilde{S}$, $\eta\lambda = 0$ *and* $\eta^2 = 2$, *then* $\eta \in Q$.

(b) $\tilde{S}$ *does not contain any element* μ *either with* $\mu^2 = 0$ *and* $\mu\lambda = -2$ *or with* $\lambda = 2\mu$.

Proof. (2) $\Rightarrow$ (1) First we reverse all the sign of bilinear forms in (2). Thus in the sequel Λ has signature (3,19). $\lambda^2 = 8$, $\eta^2 = -2$ in (a) and $\lambda = 2\mu$ or $\mu\lambda = 2$, $\mu^2 = 0$, in (b) Let T be the orthogonal complement of S in Λ. T has signature $(2, t-2)$ ($t =$ rank T). Choose a base $e_1, \ldots, e_t$ of T with $e_t^2 > 0$. Let $\varepsilon_1, \ldots, \varepsilon_{t-1}$ be real numbers so that $\varepsilon_1, \ldots, \varepsilon_{t-1}$, 1 are linearly independent over $\mathbf{Q}$. Let ε_t be a sufficiently large rational number. Set $\mu = \sum_{i=1}^{t} \varepsilon_i e_i \in T \otimes \mathbf{R}$. Since

$$\mu^2 = \left(\sum_{i=1}^{t-1} \varepsilon_i e_i\right)^2 + 2 \sum_{i=1}^{t-1} \varepsilon_i \varepsilon_t (e_i e_t) + \varepsilon_t^2 e_t^2,$$

we have $\mu^2 > 0$. Pick $x \in \Lambda$, since $(x, e_i) \in \mathbf{Z}$,

$$(x, \mu) = 0 \longleftrightarrow \sum_{i=1}^{t} \varepsilon_i (x, e_i) = 0 \longleftrightarrow (x, e_i) = 0 (i = 1,2,\ldots,t) \longleftrightarrow x \in \tilde{S}.$$

Set $T' = \{u \in T \otimes \mathbf{R} \mid u\mu = 0\}$. T' is an $\mathbf{R}$-vector space equipped with a bilinear form with signature $(1, t-2)$. Pick $u \in T'$ with $u^2 = \mu^2$. Set $\omega = \mu + \sqrt{-1}\, u \in \Lambda \otimes \mathbf{C}$. Now $\omega^2 = \mu^2 - u^2 + 2\sqrt{-1}\, \mu u = 0$ and $\bar{\omega}\,\omega = \mu^2 + u^2 = 2\mu^2 > 0$. Thus $[\omega] \in \Omega$. Here $\tilde{S} = \{x \in \Lambda \mid (x, \omega) = 0\}$ since

$$\tilde{S} \subset \{x \in \Lambda \mid (x, \omega) = 0\} = \{x \in \Lambda \mid (x, \mu) = (x, u) = 0\} \subset \tilde{S}.$$

Let (Z, α) be the marked $K3$ surface whose period is $[\omega]$. Let ψ be a non-zero holomorphic 2-form on Z. We identify ψ with the cohomology class defined by it. There is a non-zero complex number c with $\alpha(\psi) = c\omega$. By Lemma 2.11, α induces an isomorphism $\alpha : \mathrm{Pic}\, Z \rightarrow \tilde{S}$. We consider the line bundle $L = \alpha^{-1}(\lambda)$, $L^2 = \deg L = \lambda^2 = 8$. Note that (Z, α) and $(Z, -\alpha)$ defines the same period. Thus considering $(Z, -\alpha)$ instead of (Z, α) if necessary, we can assume that L and the Kähler class k belongs to the same connected component of the positive cone in $H^{1,1}(Z, \mathbf{R})$. Then by Proposition 2.9, there are finite elements $M_1, \ldots, M_r$ in Pic Z with $M_i^2 = -2$ $(1 \leq i \leq r)$ such that $s_{M_1} \ldots s_{M_r}(L)$ is numerically effective. Now for $M \in$ Pic Z with $M^2 = -2$, (Z, β) and $(Z, \beta s_M)$ defines the same period. Thus by considering $(Z, \alpha s_{M_1} s_{M_2} \ldots s_{M_r})$ instead of (Z, α), we can assume that $L = \alpha^{-1}(\lambda)$ is numerically effective. By condition (b) and Proposition 2.13, the morphism $\phi_L : Z \rightarrow \mathbf{P}^5$ to a normal octic $K3$ surface in $\mathbf{P}^5$ is defined. Let $\rho : Z \rightarrow X$ be the contraction morphism defined in Lemma 2.8, then $\phi_L(Z) \cong$

X. The singularities on X are described by the root system

$$R = \{M \in \mathrm{Pic}\, Z \mid ML = 0,\ M^2 = -2\}.$$

Condition (a) implies that R has the type corresponding to the original Dynkin Graph G. Hence the desired surface exists.

(1) $\Rightarrow$ (2) First, we will show that the assertion obtained by reversing all the sign of bilinear forms in (2) holds under (1). Now let $X \in \mathbf{P}^5$ be the normal octic $K3$ surface with singularities as G. Let $\rho : Z \to X$ be the minimal resolution of singularities. Then Z is a $K3$ surface. Set $\Delta = \{O_Z(C) \in \mathrm{Pic}\, Z \mid C$ is an irreducible component of an exceptional curve of $\rho\}$. Δ is a fundamental system of roots of type G. Let $\bar{Q} \subset \mathrm{Pic}\, Z$ be the free module generated by Δ, set $R = \{M \in \bar{Q} \mid M^2 = -2\}$. $\bar{Q}$ is a lattice isomorphic to $Q(-G)$ and R is a root system of type G. Let $L = \rho * O_Z(1)$. L is a numerically effective line bundle such that $L^2 = 8$ and $L\bar{Q} = 0$. Let $R' = \{M \in \mathrm{Pic}\, Z \mid ML = 0,\ M^2 = -2\}$. By definition $R \subset R'$, let $\rho' : Z \to X'$ be the contraction morphism defined as before associated with R'. By the preceding notation, we have $X = X'$, thus $R = R'$. Next take a suitable isomorphism $\alpha : H^2(Z, \mathbf{Z}) \to \Lambda$, where Λ is an even unimodular lattice with signature (3,19). The lattice $S = \mathbf{Z}\lambda \oplus Q(G)$ ($\lambda^2 = 8$, orthogonal direct sum) has an embedding $S \subset \Lambda$ such that $\lambda = \alpha(L)$ and $Q(-G) = \alpha(\bar{Q})$. We have only to check that the embedding satisfies the conditions corresponding to (a) and (b). Take $\eta \in \tilde{S}$ with $\eta^2 = -2$, $\eta\lambda = 0$, then $M = \alpha^{-1}(\eta)$ belongs to the primitive hull of $\mathbf{Z}L \oplus \bar{Q}$ in $H^2(Z, \mathbf{Z})$, since $H^2(Z, \mathbf{Z})/\mathrm{Pic}\, \mathrm{Z}$ has no torsion, $M \in \mathrm{Pic}\, Z$. Moreover since $M^2 = -2$ and $ML = 0$, one can conclude that $M \in R' = R \subset \bar{Q}$. Thus $\eta = \alpha(M) \in \alpha(\bar{Q}) = Q(-G)$. The condition corresponding to (a) is satisfied. By Proposition 2.13, the condition corresponding (b) is satisfied. Then reversing the sign, we have (2). Q.E.D.

Remark. If $\tilde{S}$ in (2) satisfies another additional condition: there exists no $\mu \in \tilde{S}$ with $\mu^2 = 0$, $\mu\lambda = -3$, then the normal octic $K3$ surface obtained in $\mathbf{P}^5$ is a complete intersection.

3. Theory of bilinear forms and elementary transformation

By Theorem 2.1, describing possible combinations of singularities on octic $K3$ surfaces is reduced to the theory of integral symmetric bilinear forms. In this section we explain this theory. We free use the standard terminologies in [10], [12], [14]. Consider a quasi-lattice L of finite rank. Denote

$$O(L) = \{f \in \mathrm{Hom}(L, L) \mid f \text{ is bijective, for any } x, y \in L,\ (x, y) = (f(x), f(y))\}.$$

Now let L be a lattice. The correlation morphism $\phi : L \to \mathrm{Hom}\,(L, \mathbf{Z})$ is defined by $\phi(x) = (x, \cdot)$. This map is injective if and only if L is non-degenerate.

Next we explain the concept of discriminant quadratic forms due to *Nikulin* (see [12]).

Let L be an even non-degenerate lattice. The dual module $L^* = \mathrm{Hom}\,(L, \mathbf{Z})$ can be identified with $L^* = \{x \in L \otimes \mathbf{Q} \mid (x, y) \in \mathbf{Z}$ for every $y \in L\}$. The quotient L^*/L is called the discriminant group of L. The discriminant form $q_L : L^*/L \to \mathbf{Q}/2\mathbf{Z}$ of L is defined by $q_L(x \bmod L) = x^2 \bmod 2\mathbf{Z}$ for $x \in L^*$. Let L' be another even non-degenerate lattice. Then ([12, Corollary 6.2]): there is an isomorphism $\phi : L^*/L \to L'^*/L'$ of group such that $q_{L'} \circ \phi = -q_L$ if and only if there is an embedding $L \oplus L' \hookrightarrow \Gamma$ into some even unimodular lattice Γ such that L and L' are the orthogonal complement of each other in Γ.

LEMMA 3.1. (1) *Let L be a non-degenerate quasi-lattice and M be a primitive non-degenerate subquasi-lattice. Then $M^{\perp} = C(M, L)$ is a non-degenerate quasi-lattice, too. If we denote the composition of the natural morphisms $M^{\perp} \to L \to L/M$ by f, f is injective and we can define a non-degenerate bilinear form $(\,,\,)$ on L/M with values in $\mathbf{Q}$ such that $(x, y) = (f(x), f(y))$ for every $x, y \in M^{\perp}$.*

(2) *When L is a unimodular lattice, L/M and $M^{\perp *}$ are isomorphic as quasi-lattices.*

The proof is easy.

New let L be a root module. For a root $\alpha \in R(L)$, we set $\check{\alpha} = 2\alpha/\alpha^2$ and we call $\check{\alpha}$ the coroot of α. We define the reflection $s_\alpha : L \to L$ with respect to α by

$$s_\alpha(x) = x - 2(x, \alpha)\alpha/\alpha^2 = x - (x, \check{\alpha})\alpha = x - (x, \alpha)\check{\alpha}.$$

Then $s_\alpha \in \mathrm{Hom}\,(L, L)$. The subgroup of $O(L)$ generated by s_α's is denoted by $W(L)$ and is called the *Weyl group* of the root system $R(L)$ or the Weyl group of L. $W(L)$ is a normal subgroup of $O(L)$. We call

$$Q(\check{R}) = \sum_{\alpha \in R} \mathbf{Z}\check{\alpha} (\subset Q(R) \subset L)$$

the *coroot lattice*. Note that the reflection s_α $(\alpha \in R)$ defines an isomorphism of lattice $s_\alpha : Q(\check{R}) \to Q(\check{R})$.

LEMMA 3.2 ([17, Lemma 2.2]). *For any* $\alpha \in R$, $\mathbf{R}\check{\alpha} \cap Q(\check{R}) = \mathbf{Z}\check{\alpha}$.

We say L is of finite type, if $R(L)$ is a finite set.

Now we explain the notation of Dynkin Graphs. Let $\Delta = \{\alpha_1, \ldots, \alpha_k\}$ be the fundamental root system of a finite root system R. The system Δ is a basis of $Q(R)$, we can draw a graph according to the following rule.

(1) The set of vertices in the graph has one to one correspondence with Δ.

(2) If $i \neq j$, $\alpha_i\alpha_j \neq 0$ and $\alpha_i^2 = \alpha_j^2$, then the vertices corresponding to α_i and α_j are connected with a simple edge;

(3) If $i \neq j$, $\alpha_i\alpha_j = 0$, then the corresponding vertices to α_i and α_j are not connected;

(4) If $i \neq j$, $\alpha_i\alpha_j \neq 0$ and $\alpha_i^2 > \alpha_j^2$, then the vertices to α_i and α_j are connected with a double edge with an arrow directing from α_i to α_j.

The resulting graph is called the *Dynkin Graph* of R. Note that the irreducible root system under consideration is uniquely determined by its Dynkin Graph, if it is not of type A_1 or B_1.

DEFINITION 3.3. Let L be a root module, $R(L)$ be its finite root system. Let $\Delta = \{\alpha_1, \ldots, \alpha_k\}$ be its fundamental root system. The following procedure by which we can make a root system R' from $R(L)$ is called an elementary transformation of the root system R.

(1) Decompose $R = \bigoplus_{i=1}^m R_i$ into irreducible root systems;

(2) Choose a fundamental system of root $\Delta_i \subset R_i$ for $1 \leq i \leq m$, set $\tilde{\Delta}_i = \Delta_i \cup \{-\eta_i\}$ for $1 \leq i \leq m$, where η_i is the maximal root associated with Δ_i;

(3) Choose a proper subset $\Delta_i' \subset \tilde{\Delta}_i$ for $1 \leq i \leq m$;

(4) Set $R' = \bigoplus_{i=1}^m R_i'$ where R' is the root system generated by Δ_i'.

Assume that the above R is of type G and R' is of type G'. It is clear that G' is obtained by an elementary transformation of Dynkin Graph from the Dynkin Graph G.

PROPOSITION 3.4 ([17] Corollary 2.6). *Let L be a non-degenerate root module of finite type and R be the root system of L. Let $x \in L^* \otimes \mathbf{R}$ be a point. Then*

$$R' = \{\alpha \in R \mid s_\alpha(x) - x \in Q(\check{R})\}$$

is a root system which can be obtained from R by one elementary transformation. In particular, the Dynkin Graph of R' can be obtained from the Dynkin Graph of R by one elementary transformation.

We would like to apply the above notions to our problem.

LEMMA 3.5. *Let Λ be a non-degenerate even lattice and λ be a primitive element of Λ with $\lambda^2 = -8$. Let $M_\lambda = (\mathbf{Z}\lambda)^\perp = C(\mathbf{Z}\lambda, \Lambda)$*

(1) *$\Lambda/\mathbf{Z}\lambda$ is a free module;*

(2) *Let $\pi : \Lambda \to \Lambda/\mathbf{Z}\lambda$ denote the canonical map, and let $\bar{\mu} = \pi(\mu)$;*

(a) *There exists no μ in Λ, such that $\bar{\mu}^2 = 1$,*

(b) *If $\bar{\mu}^2 = 2$, then $\frac{\mu\lambda}{4} \in Z$,*

(c) *Let $x \in \Lambda/\mathbf{Z}\lambda$, $x^2 = 2$. Then $\exists\ \mu \in M_\lambda$, $\mu^2 = 2$ such that $\bar{\mu} = x$ if and only if $\frac{\mu\lambda}{8} \in \mathbf{Z}$ for some $\eta \in \pi^{-1}(x)$.*

(3) *Assume further that Λ is unimodular and denote q_K to be the discriminant form of the lattice K. Then*

(e) *$\Lambda/\mathbf{Z}\lambda = \bar{M}_\lambda + \mathbf{Z}\bar{\omega}$,*

where $\omega \in \Lambda$, $\omega\lambda = 1$, $M_\lambda = \pi(M_\lambda)$. Then $|\det M_\lambda| = 8$, $q_{M_\lambda} = -q_N$ where $N = \mathbf{Z}\lambda$, and $a\bar{\omega} \in \bar{M}_\lambda$, $a \in \mathbf{Z} \Leftrightarrow \frac{a}{8} \in \mathbf{Z}$.

(f) *Let $U_\lambda = \{x \in \Lambda/\mathbf{Z}\lambda \mid x^2 \in \mathbf{Z}\}$, then $U_\lambda = \bar{M}_\lambda + \mathbf{Z}4\bar{\omega}$ and U_λ is an even lattice with discriminant 2.*

(g) *If Λ has signature $(l, 1)$, $l \geq 1$, then $R(M_\lambda)$ can be obtained from $R(U_\lambda)$ by one elementary transformation.*

Proof. (1) Obvious.

(2)(a) If $\mu \in \Lambda$, $\bar{\mu}^2 = 1$, let $a = \lambda\mu$. Then $\bar{\mu}^2 = \left(\mu + \frac{a}{8}\lambda\right)^2 = 1$. Hence for some integer n, $-2n = \mu^2 = 1 - \frac{a^2}{8}$. We get a contradiction.

(b) If $\bar{\mu}^2 = 2$, then $2 = \left(\mu + \frac{a}{8}\lambda\right)^2$, where $a = \lambda\mu$. Thus for some integer n, $-2n = \mu^2 = 1 - \frac{a^2}{8}$. We have $\frac{a}{4} \in \mathbf{Z}$.

(c) If $\mu \in S$, $\bar{\mu} = x$, then $\mu \in \pi^{-1}(x)$ and $\frac{\mu\lambda}{8} = 0 \in \mathbf{Z}$. Conversely, if $\eta \in \pi^{-1}(x)$, $b = \frac{\eta\lambda}{8} \in \mathbf{Z}$, then $\mu = \eta + b\lambda$ satisfies $\mu\lambda = 0$, $\mu^2 = 2$ and $\bar{\mu} = \bar{\eta} = x$.

(3)(e) and (f) are easy.

(g) Since Λ has signature $(l, 1)$ $l \geq 1$. U_λ is an even positive definite lattice of rank l. Let $x_1, \ldots, x_l$ be in Λ such that $\bar{x}_1, \ldots, \bar{x}_l$ forms a basis of U_λ. We define $x : U_\lambda \to \mathbf{R}$, by $\bar{x}_i \mapsto \frac{x_i\lambda}{8}$, $i = 1, \ldots, l$. Then $x \in U_\lambda^* \otimes \mathbf{R}$. For $\alpha \in R(U_\lambda)$,

$s_\alpha(x) - x = -(x, \alpha)\alpha \in Q(R(U_\lambda)^\vee)$ if and only if $(x, \alpha) \in \mathbf{Z}$ by Lemma 3.2. Moreover by (2)(c) $(x, \alpha) \in \mathbf{Z}$ if and only if $\alpha \in M_\lambda$. Thus

$$R' = \{\alpha \in R(U_\lambda) \mid s_\alpha(x) - x \in Q(R(U_\lambda)^\vee)\} = R(M_\lambda).$$

Then the result follows from Proposition 3.4 ([17]). Q.E.D.

Now we would like to treat the situation in Theorem 2.15 (2). The following proposition is the key part in this article. It explains why the elementary transformation appears in our problem.

PROPOSITION 3.6. *Let Λ be an indefinite unimodular even lattice and S be a non-degenerate sub-lattice. We assume that the following condition $I(\Lambda, S)$ holds.*

$I(\Lambda, S)$: The orthogonal complement $S^\perp$ of S in Λ contains an isotropic element. Then the following hold.

(1) *There are element $\mu \in S^\perp$, $u \in \Lambda$ with $\mu^2 = u^2 = 0$ and $\mu u = 1$. Set $H = \mathbf{Z}\mu + \mathbf{Z}u$ and $\Lambda = \Lambda_1 \oplus H$ (orthogonal direct sum). The following inclusion relation holds:*

$$S \subset \tilde{S} = P(S, \Lambda) \subset \Lambda_1 + \mathbf{Z}\mu.$$

(2) *Let $\pi : \Lambda \to \Lambda_1$ denote the orthogonal projection to the Λ_1-factor. The restriction $\pi|_{\tilde{S}}$ and $\pi|_S$ are isomorphisms of lattices onto their images.*

(3) *Set $S_1 = \pi(S) \subset \Lambda_1$ and $\tilde{S}_1 = P(S_1, \Lambda_1)$. Then $\tilde{S}_1$ is a non-degenerate sub-lattice of Λ_1 and $\tilde{S}_1/\pi(\tilde{S})$ is a finite cyclic group.*

Now assume further that S has signature $(k, 1)$, $k \in \mathbf{Z}^+$. Let $\lambda \in S$ with $\lambda^2 = -8$, $M_\lambda = C(\mathbf{Z}\lambda, \Lambda)$. Assume that $M_\lambda \cap S$ is non-degenerate. Let $\pi(\lambda) = \lambda_1$. Then

(4) *$\tilde{\pi}(M_\lambda \cap S)/\pi(M_\lambda \cap \tilde{S})$ is a finite cyclic group, where $\tilde{\pi}(M_\lambda \cap S)$ is the primitive hull of $\pi(M_\lambda \cap S)$ in Λ_1.*

(5) *Let S' be a non-degenerate sublattice of M_{λ_1} satisfying* (a), (b), (c).

(a) *$\tilde{\pi}(M_\lambda \cap S) \subset S' \subset \Lambda_1$.*

(b) *S' is primitive in Λ_1.*

(c) *S' is positive definite.*

Then the finite root system $R(M_\lambda \cap \tilde{S})$ can be obtained from the finite root system $R(S')$ by one elementary transformation.

Remark. $S' = \tilde{\pi}(M_\lambda \cap S)$ satisfies (a), (b) and (c).

Proof. The proof is almost the same as that of Proposition 2.9 (4) ([17]).
Q.E.D.

Now here we consider the situation in Theorem 2.15 (2). By condition (a) and (b), the root system $R(\tilde{S} \cap M_\lambda)$ is of type G. Assume moreover that $I(\Lambda, S)$ is satisfied. Let $\lambda_1 = \pi(\lambda) \in \Lambda_1$, then $R(\tilde{S} \cap M_\lambda)$ is obtained by one elementary transformation from $R(\tilde{S}_1 \cap M_{\lambda_1})$.

Assume moreover $I(\Lambda_1, S_1)$ is satisfied.

We can apply the above proposition once more and we have an even unimodular lattice Λ_2 with signature (17,1). Let $\lambda_2 = \pi(\lambda_1)$. In this case M_{λ_2} itself satisfies conditions (a), (b) and (c) above. Thus we can conclude that the original Dynkin Graph G can be obtained from the Dynkin Graph of $R(M_{\lambda_2})$ by elementary transformations repeated twice.

Note that $\lambda_1^2 = \lambda_2^2 = -8$, but they may not be primitive in Λ_1 and Λ_2, respectively.

If λ_1 is not primitive in Λ_1 then $\frac{\lambda_1}{2} \in \Lambda_1$, $\frac{\lambda_2}{2} \in \Lambda_2$. If λ_1 is primitive in Λ_1, but λ_2 is not primitive in Λ_2, then $\frac{\lambda_2}{2} \in \Lambda_2$. So in both cases, M_{λ_2} is an even positive definite lattice of rank 17 with discriminant 2. By the classification of even positive definite lattice of rank 17 with discriminant 2 ([5]), there are only four kinds of such lattice as follows:

(1) $2K_8 \oplus I_1$, its root system is $2E_8 \oplus A_1$, where $K_8 = Q(E_8)$, $I_1 = Q(A_1)$;

(2) $K_{16} \oplus I_1$, its root system is $D_{16} \oplus A_1$;

(3) The even overlattice with index 3 of the root lattice of type A_{17}, its root system is A_{17};

(4) One of two isomorphic even overlattices with index 2 of the root lattice of type $D_{10} + E_7$, its root system is $D_{10} + E_7$.

If λ_2 is primitive in Λ_2, then M_{λ_2} is an even positive definite lattice of rank 17 with discriminate 8 and $q_{M_{\lambda_2}} = -q_N$, where $N = \mathbf{Z}\lambda$, $\lambda^2 = -8$ is a free module.

The remaining parts to show (A) $\Rightarrow$ (B) in our Theorem 0.2 are the following two.

(1) To write condition $I(\Lambda, S)$ and $I(\Lambda_1, S_1)$ with the Hasse symbol and the Hilbert symbol.

(2) To classify of $R(M_{\lambda_2})$ where Λ_2 is an even unimodular with signature (17,1), $\lambda_2 \in \Lambda_2$ is a primitive element with $\lambda_2^2 = -8$ and M_{λ_2} is the orthogonal complement of $\mathbf{Z}\lambda_2$ in Λ_2.

4. Positive definite even lattices of rank 17 with discriminant 8

There have been extensive lists of enumerations of positive definite lattices of small ranks and small determinants in literature (cf. [5]).

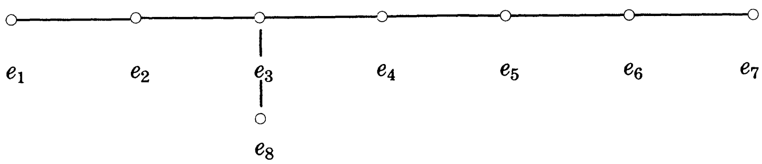
As far as we know, the complete classification of rank 17 positive definite lattices of discriminant 8 is still unknown. In this section we shall find the root systems for positive definite even lattices of rank 17 whose discriminant equal to 8 such that their discriminant quadratic form amounts to $-q_N$, where $N = \mathbf{Z}\lambda$ with $\lambda^2 = -8$.

Recall that the only positive definite even unimodular lattice of rank 8 is

$$Q(E_8) = \{(x_1,\ldots,x_8) \in \mathbf{R}^8\, \forall x_i \in \mathbf{Z}, \text{ or } \forall x_i \in \mathbf{Z} + \tfrac{1}{2}, \textstyle\sum x_i \equiv 0 \ (\mathrm{mod}\ 2)\}.$$

The fundamental root system of E_8 is

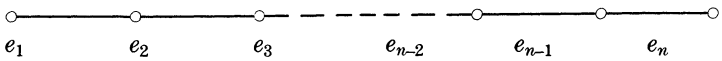
$$e_1 = (1,1,0^6),\ e_2 = (0,-1,1,0^5),\ e_3 = (0^2,-1,1,0^4),\ e_4 = (0^3,-1,1,0^3),$$
$$e_5 = (0^4,-1,1,0^2),\ e_6 = (0^5,-1,1,0),\ e_7 = (0^6,-1,1),$$
$$e_8 = \left(-\frac{1}{2},\frac{1}{2}^2,-\frac{1}{2}^5\right).$$



E_8

Recall that (i), $Q(A_n) = \{(x_0, x_1, \ldots, x_n) \in \mathbf{Z}^{n+1} : x_0 + x_1 + \cdots + x_n = 0\}$ for $n \geq 1$. The fundamental root system of A_n is

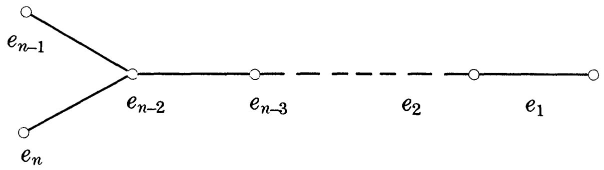
$$e_1 = (-1,1,0,\ldots,0),\ e_2 = (0,-1,1,0,\ldots,0),\ e_3 = (0,0,-1,1,0,\ldots,0),\ldots,$$
$$e_n = (0,0,\ldots,0,-1,1).$$



A_n

(ii) $Q(D_n) = \{(x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n : x_1 + x_2 + \cdots + x_n \text{ even}\}$, for $n \geq 4$. The fundamental root system D_n is

$e_1 = (0,\ldots,0,1,-1)$, $e_2 = (0,\ldots,0,1,-1,0)$, $e_3 = (0,\ldots,0,1,-1,0,0),\ldots,$
$e_{n-1} = (1,-1,0,\ldots,0)$, $e_n = (-1,-1,0,\ldots,0)$.



D_n

LEMMA 4.1. *Let $Q(A_7) \hookrightarrow Q(E_8)$ be a primitive embedding of the root-module of type A_7 into the root-module of type E_8. Then*

(1) *There exists an automorphism of the lattice $Q(E_8)$ such that the image of the seven vertices of A_7 in $Q(E_8)$ is $\{e_1, e_2,\ldots,e_7\}$;*

(2) *The root system of the orthogonal complement of $Q(A_7)$ in $Q(E_8)$ is empty.*

Proof. (1) By the transitive property ([7] Table 11, p.149) of the automorphisms group of E_8 acting on root sub-systems of E_8, we may assume that the images of the 7 vertices of A_7 in E_8 are $\{e_1, e_2,\ldots,e_7\}$.

(2) It is only due to a trivial calculation. Q.E.D.

Remark. The composition $Q(A_7) \hookrightarrow Q(E_7) \hookrightarrow Q(E_8)$ defines an embedding, but it is not primitive. Thus the above assumption that the embedding is primitive is essential.

LEMMA 4.2. (a) *There is an embedding: $Q(A_7) \hookrightarrow Q(A_n)$ if and only if $n \geq 7$;*

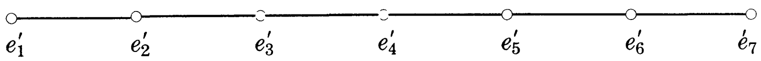
Now assume that $Q(A_7) \hookrightarrow Q(A_n)$ $(n \geq 7)$ is an embedding

(b) *There exists an automorphism of the lattice $Q(A_n)$ $(n \geq 7)$ so that the image of the seven vertices of A_7 is $\{e_1, e_2,\ldots,e_7\}$;*

(c) *The root system of the orthogonal complement of A_7 in $Q(A_n)$ is A_{n-8}.*

Proof. (a) Obvious.

(b) Let the image of the seven vertices of A_7 be e_1', e_2', e_3', e_4', e_5', e_6', e_7'.



A_7

Since $Q(A_n) = \{(x_0, x_1, \ldots, x_n) \in \mathbf{Z}^{n+1} : x_0 + x_1 + \ldots + x_n = 0\}$, we may assume that $e_1' = e_1 = (-1,1,0,\ldots,0)$.

By e_1', $e_2' = -1$, so either

Case (1).

$$e_2' = (\underbrace{0,-1,0,\ldots,0,1}_{3 \leq i \leq n+1},0,\ldots,0)$$

or

Case (2).

$$e_2' = (\underbrace{1,0,\ldots,0,-1}_{3 \leq i \leq n+1},0,\ldots,0)$$

Case (1). If $e_2' = (\underbrace{0,-1,0,\ldots,0,1}_{3 \leq i \leq n+1},0,\ldots,0)$, then there is a permutation ϕ of the coordinates of $Q(A_n)$ such that $e_1' = e_1$ is invariant under ϕ and

$$\phi e_2' = (0,-1,1,0,\ldots,0) = e_2.$$

Case (2). If $e_2' = (\underbrace{1,0,\ldots,0,-1}_{3 \leq i \leq n+1},0,\ldots,0)$ then there is a permutation ϕ of the coordinates of $Q(A_n)$ such that $\phi(e_1') = \phi(e_1) = (1,-1,0,\ldots,0)$ and

$$\phi(e_2') = (\underbrace{0,1,0,\ldots,0,-1}_{3 \leq i \leq n+1},0,\ldots,0).$$

Now let ψ be the automorphism of the lattice $Q(A_n)$ defined by the negative unit matrix $-I_{n+1}$.

Then $\psi\phi(e_1') = \psi\phi(e_1) = e_1$ and

$$\psi\phi(e_2') = (\underbrace{0,-1,0,\ldots,0,1}_{3 \leq i \leq n+1},0,\ldots,0).$$

Then Case (2) comes back to Case (1).

Thus in any cases, we may assume that $e_1' = e_1$, $e_2' = e_2$. Since $e_1' e_3' = 0$, $e_2' e_3' = -1$,

$$e_3' = (\underbrace{0,0,-1,0,\ldots,0,1}_{4\leq i\leq n+1},0,\ldots,0).$$

Obviously, there is a permutation ϕ of coordinates of $Q(A_n)$ so that e_1, e_2 are invariant under ϕ and $\phi(e_3') = (0,0,-1,1,0,\ldots,0) = e_3$.

By similar discussion, we conclude that $e_4' = e_4$, $e_5' = e_5$, $e_6' = e_6$, $e_7' = e_7$.

(c) By (b) we may assume that the image of the seven vertices of A_7 is $\{e_1, e_2,\ldots,e_7\}$. So

$$Q(A_7)^{\perp} = \{(x_0,\ldots,x_{n+1}) \in \mathbf{Z}^{n+1} : x_0 + \cdots + x_{n+1} = 0,\ x_0 = x_1 = \cdots = x_7\}$$

whose root system is obviously A_{n-8}, for $n > 9$ (we denote $A_0 = A_{-1} = \phi$). Q.E.D.

LEMMA 4.3. (1) *There exists an embedding*: $Q(A_7) \hookrightarrow Q(D_n)$ *if and only if* $n \geq 8$;

(2) *There is an automorphism of the lattice* $Q(D_n)$ $(n \geq 8)$ *such that the image of the seven vertices of* A_7 *in* D_n *is* $\{e_1,e_2,\ldots,e_7)$;

(3) *The root system of the orthogonal complement of* A_7 *in* $Q(D_n)$ *is empty if* $n = 8,9$; $2A_1$ *if* $n = 10$; A_3 *if* $n = 11$; D_{n-8} *if* $n \geq 12$.

Proof. (1) Obvious.

(2) Since $Q(D_n) = \{(x_1,\ldots,x_n) \in \mathbf{Z}^n : x_1 + \cdots + x_n \text{ even}\}$, we may assume that $e_1' = e_1 = (0,\ldots,0,1,-1)$. Since $e_1' e_2' = -1$, e_2' is either

Case (1).

$$(\underbrace{0,\ldots,0,\pm 1}_{1\leq i\leq n-2}.0,\ldots,0,-1,0)$$

or *Case* (2).

$$(\underbrace{0,\ldots,0,\pm 1}_{1\leq i\leq n-2},0,\ldots,0,1)$$

Case (1). Since the two elements $(\underbrace{0,\ldots,0,\pm 1}_{1\leq i\leq n-2},0,\ldots,0,-1,0)$ are different up an automorphism of the lattice $Q(D_n)$, we may assume that

$$e_2' = (\underbrace{0,\ldots,0,1}_{1\leq i\leq n-2},0,\ldots,0,-1,0).$$

Obviously there is a permutation ϕ of coordinates of $Q(D_n)$ so that $e_1' = e_1$ is invariant under ϕ and $\phi(e_2') = e_2 = (0,\ldots,0,1,-1,0)$.

Case (2). With the same reason as above, we may assume that

$$e_2' = (\underbrace{0,\ldots,0,1}_{1\le i\le n-2},0,\ldots,0,1).$$

Obviously, there is a permutation ψ of coordinates of $Q(D_n)$ so that $e_1' = e_1$ is invariant under ψ and

$$\psi(e_2') = (0,\ldots,0,1,0,1).$$

Then the permutation of the last two coordinates defines an automorphism α of the lattice $Q(D_n)$ so that $\alpha(e_1') = \alpha(e_1) = (0,\ldots,0,-1,1)$ and

$$\alpha \circ \psi(e_2') = (0,\ldots,0,1,1,0).$$

Then there exists another automorphism β of the lattice $Q(D_n)$ defined by the matrix

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}_{n\times n}$$

so that $\beta \circ \alpha(e_1') = \beta \circ \alpha(e_1) = e_1 = (0,\ldots,0,1,-1)$ and

$$\beta \circ \alpha \circ \psi(e_2') = e_2 = (0,\ldots,0,1,-1,0).$$

Hence in both cases, we may assume that $e_2' = e_2$. Since $e_3' e_1' = 0$, $e_3' e_2' = -1$, e_3' is either

Case (a). $(\underbrace{0,\ldots,0,\pm 1}_{1\le i\le n-3},0,\ldots,0,-1,0,0)$ or *Case* (b). $(0,\ldots,0,1,1)$.

Case (a). We may assume that $e_3' = (\underbrace{0,\ldots,0,1}_{1\le i\le n-3},0,\ldots,0,-1,0,0)$.

Obviously, there is a permutation ϕ of coordinates of $Q(D_n)$ so that e_1' and e_2' are invariant under ϕ and

$\phi(e_3') = e_3 = (0,\ldots,0,1,-1,0,0)$. Thus we may assume that $e_3' = e_3$.

Case (b). $e_3' = (0,\ldots,0,1,1)$. We shall prove that this case can not occur.

Since $e_4' e_3' = -1$, e_4' must be one of the following forms:

$$(\underbrace{0,\ldots,0,\pm 1}_{1\leq i\leq n-2},0,\ldots,0,-1,0) \text{ or } (\underbrace{0,\ldots,0,\pm 1}_{1\leq i\leq n-2},0,\ldots,0,-1).$$

But in both cases, we get $e_4' e_2' \neq 0$ or $e_4' e_2' \neq 0$, which is a contradiction. Thus Case (b) can not occur.

By a similar discussion, we may assume that $e_4' = e_4$, $e_5' = e_5$, $e_6' = e_6$, $e_7' = e_7$.

(3) $Q(A_7)^{\perp} = \{(x_1,\ldots,x_n) \in \mathbf{Z}^n \mid \sum_{i=1}^{n-8} x_i$ even and $x_{n-7} = \cdots = x_n\}$.

Then the assertion of (3) is obvious. Q.E.D.

Now suppose that a primitive embedding $\mathbf{Z}\lambda \hookrightarrow \Lambda_1$ is given. Let M denote the orthogonal complement of $\mathbf{Z}\lambda$ in Λ_1. Here consider the lattice $K = Q(A_7)$. Let $N = \mathbf{Z}\lambda$ with $\lambda^2 = -8$. Then K^*/K and N^*/N are both cyclic group of order 8. K^*/K is generated by the image of

$$\alpha = \frac{1}{8}(e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 + 7e_7) \in K^*$$

in K^*/K, N^*/N is generated by the image of $\frac{3}{8}\lambda \in N^*$ in N^*/N.

$$q_K(\alpha) = \frac{7}{8},\ q_N\left(\frac{3}{8}\lambda\right) = -\frac{9}{8} \equiv \frac{7}{8} \bmod (2\mathbf{Z}).$$

Let $\rho : K^*/K \to N^*/N$ be the isomorphism defined by $\rho(\alpha) = \frac{3}{8}\lambda$. Then $q_K = q_N \circ \rho$, where q denotes the discriminant quadratic form. Thus there is an isomorphism $\phi : K^*/K \to M^*/M$ with $q_M \circ \phi = -q_K$. We can conclude that for some positive definite even unimodular lattice Γ of rank 24 and for some primitive embedding $K \hookrightarrow \Gamma$, M is isomorphic to the orthogonal complement of K in Γ.

Conversely, associated with such an embedding of K in Γ, we can make a primitive embedding $\mathbf{Z}\lambda \hookrightarrow \Lambda_1$ with $C(K, \Gamma) \cong C(\mathbf{Z}\lambda, \Lambda_1)$. Each one Γ is uniquely determined by its root system. These root systems are listed as follows ([5] p.427):

ϕ, $24A_1$, $12A_2$, $8A_3$, $4A_6$, $3A_8$, $2A_{12}$, A_{24}, $6D_4$, $4D_6$, $3D_8$,
$2D_{12}$, D_{24}, $4E_6$, $3E_8$, $4A_5$, $+ D_4$, $2A_7 + 2D_5$, $2A_9 + D_6$, $A_{15} + D_9$,
$E_8 + D_{16}$, $2E_7 + D_{10}$, $E_7 + A_{17}$, $E_6 + D_7 + A_{11}$, $6A_4$.

The candidates of Γ have been listed in the above. Moreover, since the root system of Γ has to contain a root sub-system of type A_7, we can consider only the following of 14 isomorphism classes.

$3A_8$, $2A_{12}$, A_{24}, $3D_8$, $2D_{12}$, D_{24}, $3E_8$, $2A_7+2D_5$, $2A_9+D_6$,
$A_{15}+D_9$, E_8+D_{16}, $2E_7+D_{10}$, E_7+A_{17}, $E_6+D_7+A_{11}$.

PROPOSITION 4.4. *The root system of every positive definite even lattice of rank* 17 *and discriminant* 8 *such that their discriminant quadratic forms are equal to* $-q_N$ *as above is one of the following* 15:

$2A_8$, $A_{12}+A_4$, A_{16}, $2D_8$, $D_{12}+D_4$, D_{16}, $2E_8$, A_7+2D_5,
$A_9+A_1+D_6$, A_7+D_9, A_{15}, E_8+D_8, $2E_7+2A_1$, E_7+A_9, $E_6+D_7+A_3$.

Proof. By Lemmas 4.1, 4.2, 4.3 and the statement above, our results are obvious. Q.E.D.

COROLLARY 4.5. *There is a primitive element* $\lambda \in Q(2E_8)\oplus H=\Lambda$ *with* $\lambda^2=-8$, *such that the root system of* $M_\lambda = C(\mathbf{Z}\lambda, \Lambda)$ *is of type* G *if and only if* G *is of one of the following* 15:

$2E_8$, D_{16}, D_4+D_{12}, $2D_8$, E_8+D_8, $2A_1+2E_7$, $2A_8$, $A_{12}+A_4$, A_{16},
A_7+2D_5, $A_1+A_9+D_9$, A_7+D_9, A_{15}, A_9+E_7, $A_3+E_6+D_7$.

5. Theory of bilinear forms

In this section we denote by $d(L)\in\mathbf{Z}$ the discriminant of a lattice L.

Let V be a finite-dimensional vector space over $\mathbf{Q}$, equipped with a symmetric bilinear form $V\times V\to\mathbf{Q}$. We can define the discriminant $\bar{d}(V)$ and the Hasse symbol $\varepsilon_p(V)$. $\bar{d}(V)\in\mathbf{Q}/\mathbf{Q}^{*2}$. $\bar{d}(V)=0$ is equivalent to that V is degenerate. If V is non-degenerate, the Hasse symbol $\varepsilon_p(V)\in\{+1,-1\}$ is defined for every prime number p and $p=\infty$ (Serre [14]). Obviously for any lattice L,

$$\bar{d}(L\otimes\mathbf{Q})\equiv d(L)\ (\mathrm{mod}\,\mathbf{Q}^{*2}).$$

We use the Hilbert symbol $(\ .\ )_p$ in the sequel (Serre [13]).

LEMMA 5.1. *Let* Λ *be an even unimodular lattice with signature* (a, b) *and* S *be a non-degenerate sublattice of* Λ *with signature* $(r, 1)$. *Let* $\lambda\in S$ *with* $\lambda^2=-8$. *Set* $T=C(S,\Lambda)=S^\perp$, $Q=\cap\mathbf{Z}\lambda^\perp=C(\mathbf{Z}\lambda, S)$.

(1) $d(T)\equiv(-1)^b d(S)\ (\mathrm{mod}\,\mathbf{Q}^{*2})$,
(2) $\varepsilon_p(T)=\varepsilon_p(S)\ ((-1)^{b+1}, d(S))_p(-1,-1)_p^{b(b-1)/2}$,
(3) $d(T)\equiv(-1)^{b+1}2b(Q)\,(\mathrm{mod}\,\mathbf{Q}^{*2})$,

(4) $\varepsilon_p(T) = \varepsilon_p(Q)((-1)^b 2, d(Q))_p(-1, -1)_p^{(b-1)(b-2)/2}$.

(5) *The following three conditions are equivalent*

$$\varepsilon_p(T) = (-1, -d(T))_p,$$
$$\varepsilon_p(S) = ((-1)^b, d(S))_p(-1, -1)_p^{(b-1)(b-2)/2},$$
$$\varepsilon_p(Q) = ((-1)^{b+1}, d(Q))_p(-1, -1)_p^{(b+1)(b+2)/2}.$$

(6) *The following three conditions are equivalent*

$$\varepsilon_p(T) = (-1, -1)_p,$$
$$\varepsilon_p(S) = ((-1)^{b+1}, d(S))_p(-1, -1)_p^{(b+1)(b+2)/2},$$
$$\varepsilon_p(Q) = ((-1)^b 2, d(Q))_p(-1, -1)_p^{b(b+1)/2}.$$

Proof. It is known that $\varepsilon_p(\Lambda) = (-1, -1)_p^{b(b-1)/2}$ ([14]). The Lemma is an easy consequence of this equality. Q.E.D.

PROPOSITION 5.2. *Let Λ be an even unimodular lattice with signature (a, b) and S is a non-degenerate sublattice of Λ with signature $(r, 1)$. Set $T = C(S, \Lambda) = S^{\perp}$. The following 4 conditions are equivalent.*

(A) *$I(\Lambda, S)$ is satisfied, i.e. T contains an isotropic element.*

(B) *$T \otimes \mathbf{Q}$ contains an isotropic element.*

(C) *$a > r$, $b > 1$ and moreover one of the following* (1), (2), (3), (4) *is satisfied.*

(1) *$a + b = r + 3$ and $-d(T) \in \mathbf{Q}^{*2}$,*

(2) *$a + b = r + 4$ and $\varepsilon_p(T) = (-1, -d(T))_p$ for every prime number p,*

(3) *$a + b = r + 5$ and $d(T) \notin \mathbf{Q}^{*2}$ or $\varepsilon_p(T) = (-1, -1)_p$ for every prime number p,*

(4) $a + b \geq + 6$.

(D) *$a > r$, $b > 1$ and moreover one of the following* (1), (2), (3), (4) *is satisfied.*

(1) *$a + b = r + 3$ and $(-1)^{b+1} d(S) \in \mathbf{Q}^{*2}$,*

(2) *$a + b = r + 4$ and*

$$\varepsilon_p(S) = ((-1)^b, d(S))_p(-1, -1)_p^{(b-1)(b-2)/2}$$

for every prime number p,

(3) *$a + b = r + 5$ and $(-1)^b d(S) \notin \mathbf{Q}_p^{*2}$ or*

$$\varepsilon_p(S) = ((-1)^{b+1}, d(S))_p(-1, -1)_p^{(b+1)(b+2)/2}$$

for every prime number p,

(4) $a + b \geq r + 6$.

If $\exists\ \lambda \in S$ with $\lambda^2 = -8$ or -2, then the following condition (E) *is also equivalent to the above 4 conditions.*

(E) *$a > r$, $b > 1$ and moreover one of the following* (1), (2), (3), (4) *is satisfied.*

(1) $a+b=r+3$ *and* $(-1)^b 2d(Q) \in \mathbf{Q}^{*2}$,

(2) $a+b=r+4$ *and*

$$\varepsilon_p(Q) = ((-1)^{b+1}2, d(Q))_p(-1, -1)_p^{(b+1)(b+2)/2}$$

for every prime number p,

(3) $a+b=r+5$ *and* $(-1)^{b+1} 2d(Q) \notin \mathbf{Q}_p^{*2}$ *or*

$$\varepsilon_p(Q) = ((-1)^b 2, d(Q))_p(-1, -1)_p^{b(b+1)/2}$$

for every prime number p,

(4) $a+b \geq r+6$.

Proof. (A) $\Leftrightarrow$ (B) obvious. (B) $\Leftrightarrow$ (C) by Théorème 6 and Théorème 8 in ([14, Chap. IV]). (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (E) by Lemma 5.1. Q.E.D.

COROLLARY 5.3. *Let* Q *be a positive definite lattice of rank* r *and* $\lambda^2 = -8$ *or* -2. *Set* $S = \mathbf{Z}\lambda \oplus Q$ *(orthogonal direct sum). Assume that there is an embedding* $S \subset Q(2E_8) \oplus H \oplus H \oplus H$. *If* $I(Q(2E_8) \oplus H \oplus H \oplus H, S)$ *is satisfied, then there is an embedding* $S \subset Q(2E_8) \oplus H \oplus H$. *Moreover, the following conditions are equivalent.*

(a) $I(Q(2E_8) \oplus H \oplus H \oplus H, S)$ *and* $I(Q(2E_8) \oplus H \oplus H, S)$ *are satisfied,*

(b) *one of the following* (1), (2), (3), (4) *holds.*

(1) $r = 17$, $2d(Q) \in \mathbf{Q}^{*2}$ *and* $\varepsilon_p(Q) = 1$ *for every prime number* p,

(2) $r = 16$ *and* $\varepsilon_p(Q) = (-2, d(Q))_p$ *for every prime number,*

(3) $r = 15$, *and* $-2d(Q) \notin \mathbf{Q}_p^{*2}$ *or* $\varepsilon_p(Q) = (-1, -1)_p$ *for every prime number* p,

(4) $r \leq 14$.

By Theorem 2.15, Corollary 5.3 and Corollary 4.5, the implication (A) $\Rightarrow$ (B) in our main Theorem 0.2 is clear.

6. Proof of the converse

In this section we prove the converse part as well as (C) of the main Theorem 0.2. Let L be a non-degenerated lattice and L^* be its dual lattice of L.

LEMMA 6.1. *Let* $Q(G)$ *be the root lattice of a Dynkin Graph* G *with components of type* A, D *or* E *only.*

i) *If there is a nonzero element* $\xi \in Q(G)^*$ *with* $\xi^2 < 1$, *then* $\xi^2 = \frac{k}{k+1}$ *for some positive integer* k;

ii) *For every nonzero element* β *in* $Q(A_n)^*$ *of* $Q(A_n)$, *either* $\beta^2 = \frac{n}{n+1}$ *or* $\beta^2 \geq 1$, *and the lower bound* $\frac{n}{n+1}$ *can be reached;*

iii) *For every nonzero element* β *in the dual lattice of* $Q(D_n)$, $Q(E_8)$ *or* $Q(E_7)$, $\beta^2 \geq 1$;

iv) *For every nonzero element* β *in the dual lattice of* $Q(E_6)$, *either* $\beta^2 = \frac{4}{3}$ *or* $\beta^2 \geq 1$, *and the lower bound* $\frac{4}{3}$ *can be reached.*

v) *There exists an element* β *with* $\beta^2 = \frac{1}{2}$ *in* $Q(G)^*$ *if and only if* G *contains a component of type* A_1 *and* $\beta \in Q(A_1)^*$.

vi) *For any* $\xi \in Q(G)^*$ *with* $\xi^2 = \frac{9}{8}$, *there is a unique connected component* G_0 *of* G *with* $\xi \in Q(G_0)^* \subset Q(G)^*$. G_0 *must be of type* A_{8k-1} *for some positive integer* k.

vii) *Assume further rank* $Q(G) \leq 17$. *There does not exist an element* $\xi \in Q(G)^*$ *with* $\xi^2 = \frac{9}{8}$.

Proof. For i), ii), iii) and iv), we can use the standard theory of discriminant quadratic form on lattices ([16, Ch. I, p.19]).

v) It follows from the statements of i), ii), iii) and iv) above.

vi) Write $Q(G) = \bigoplus_i L_i$ where each L_i is the root lattice of a component of G. Accordingly ξ can be written as $\xi = \sum_i \xi_i$, where ξ_i belongs to L_i^*, the dual lattice of L_i for every i. Since $\xi^2 = \sum_i \xi_i^2$ and every ξ_i^2 is non-negative. By $\xi^2 = \frac{9}{8}$ and i), we conclude that only one ξ_i is nonzero.

Let $T = Q(G_0)$ be the root lattice of a component G_0 of G with $\xi \in T^*$.

If $T = Q(E_8)$, then $T^*/T \cong 0$. But $\xi^2 = \frac{9}{8}$, which is impossible.

If $T = Q(E_7)$, then $T^*/T \cong \mathbf{Z}/2$. Since $(2\xi)^2 = \frac{9}{2}$, this case can not occur.

If $T = Q(E_6)$, then $T^*/T \cong \mathbf{Z}/3$. Since $(3\xi)^2 = \frac{81}{8}$, this case still can not occur.

If $T = Q(D_r)$ $(r \geq 4)$, Then $T^*/T \cong \mathbf{Z}/2 + \mathbf{Z}/2$ if r is even and $T^*/T \cong$

$\mathbf{Z}/4$ if r is odd.

Since $(2\xi)^2 = \frac{9}{2}$, r must be odd and $T^*/T \cong \mathbf{Z}/4$. Let $\pi : T^* \to T^*/T$ be the canonical map. We assign e_i to be vertices of D_r as in Section 4. Set

$$\omega = \{e_1 + 2e_2 + \cdots + (r-2)e_{r-2} + (r-2)e_{r-1}/2 + re_r/2\}/2.$$

Note that $\omega^2 = \frac{r}{4}$. Then $\omega^2 \in T^*$ and $\pi(\omega)$ is a generator of the determinant group T^*/T (see [16, Ch. I, p.19 Ex. 3.3]). Note $\pi(\xi)$ is also a generator of T^*/T. So $\xi - k\omega = a$ where $\alpha \in T$ and $k = 1$ or 3. Thus

$$\frac{9}{8} = \xi^2 = k^2\omega^2 + 2ka\omega + a^2 = \frac{k^2 r}{4} + 2ka\omega + \alpha^2,$$

which is a contradiction since $2ka\omega + a^2 \in 2\mathbf{Z}$. Thus the case $T = Q(D_r)$ can not occur.

If $T = Q(A_n)$, then the determinant group $T^*/T \cong \mathbf{Z}/(n+1)$. Note that there exists an element ω in T^* such that the image of ω in T^*/T is a generator and $\omega^2 = \frac{n}{n+1}$ (see [16, Ch. I, p.9, Ex. 3.3]). Write $\xi = t\omega + a$, where $t \in \mathbf{Z}$ and $a \in T$. Thus $\frac{9}{8} = \xi^2 = \frac{nt^2}{n+1} + 2ta\omega + a^2$. Note that $2ta\omega + a^2$ is an even integer. Thus $n+1$ is a multiple of 8.

vii) By vi) we can assume $\xi \in Q(A_n)^*$ for $n = 7$ or 15 and $\xi^2 = \frac{9}{8}$. If $n = 7$, then $\frac{9}{8} - \frac{7}{8}t^2$ is an even integer for some integer t with $-4 < t \leq 4$. By trivial calculation, we find that the above equation has no solutions. If $n = 15$, then $\frac{9}{8} - \frac{15}{16}t^2$ is an even integer for some integer t with $-8 < t \leq 8$. The equation still has no solutions. All together, we have proved that there is no element $\xi \in Q(G)^*$ with $\xi^2 = \frac{9}{8}$. Q.E.D.

LEMMA 6.2. *Let $G = \sum a_k A_k + \sum b_l D_l + \sum c_m E_m$ be a Dynkin Graph with components of type A, D or E only, where a_k, b_l and c_m are positive integers. Set $r = \sum a_k k + \sum b_l l + \sum c_m m$. Let $Q = Q(G)$ be the root lattice of type G. Let $\mathbf{Z}\lambda$ be a lattice with $\lambda^2 = -8$. Assume that the orthogonal sum of $\mathbf{Z}\lambda$ and $Q(G)$ is embedded in the unimodular even lattice Λ of signature* (19,3) *such that λ is primitive in Λ.*

i) *There does not exist an element ξ in primitive hull of $\mathbf{Z}\lambda \oplus Q$ in Λ so that $\xi^2 = 0$ and $\xi\lambda = -2$.*

ii) *Suppose further that* $r \leq 17$. *There does not exist an element* ξ *in the primitive hull of* $\mathbf{Z}\lambda \oplus Q$ *in* Λ *such that* $\xi^2 = 0$ *and* $\xi\lambda = -3$.

Proof. i) Suppose such an element ξ exists. Since $\xi^2 = 0$ and $\xi\lambda = -2$. Write $\xi = \dfrac{\lambda}{4} + q$ in $(\mathbf{Z}\lambda \oplus Q) \otimes \mathbf{Q}$, where $q \in Q(G) \otimes \mathbf{Q}$, and $q^2 = \dfrac{1}{2}$. For every $\alpha \in Q(G)$, $q\alpha = \xi\alpha \in \mathbf{Z}$. By Lemma 6.1, G must contain a component of type A_1 and $q \in Q(A_1)^*$, the dual lattice of $Q(A_1)$. Since $Q(A_1)^*/Q(A_1)$ is a cyclic group of order 2, $2q \in Q(A_1) \subset \Lambda$. Since ξ lies in the primitive hull $\mathbf{Z}\lambda \oplus Q(G)$ in Λ, $\xi \in \Lambda$. $\dfrac{\lambda}{2} = 2\xi - 2q \in \Lambda$, which contradicts that λ is primitive in Λ.

ii) Suppose that such ξ does exist. Write $\xi = \dfrac{3}{8}\lambda + q$ in $(\mathbf{Z}\lambda \oplus Q) \otimes \mathbf{Q}$, where $q \in Q(G) \otimes \mathbf{Q}$ and $q^2 = \dfrac{9}{8}$. For every α in $Q(G)$, $q\alpha = \xi\alpha \in \mathbf{Z}$. Hence q lies in the dual lattice of $Q(G)$, which contradicts Lemma 6.1 vii). Q.E.D.

Let L be a root module and S be a submodule. We say that the embedding $S \subset L$ is full if $R(S) = R(\tilde{S})$ for root systems. Here $\tilde{S}$ is the primitive hull of S in L.

LEMMA 6.3 ([17] Proposition 4.2). *Let* R' *be a root system which is obtained by one elementary transformation from a finite root system* R *of some root module. Let* L *be another root module. Assume that a full embedding* $Q(R) \subset L$ *is given. Then there is a full embedding* $Q(R') \hookrightarrow L \oplus H$ *such that* u *orthogonal to* $Q(R')$, *where* $H = \mathbf{Z}u + \mathbf{Z}v$ *with* $u^2 = v^2 = 0$, $uv = vu = 1$ *If* $Q(R)$ *is orthogonal to* $\omega \in L$, *Then* $Q(R')$ *is orthogonal to* $\omega = \omega \oplus 0 \in L \oplus H$.

Next, we shall prove the converse of the main Theorem 0.2.

First, let R be a root system whose type is one of the following 15:

$2A_8$, $A_{12} + A_4$, A_{15}, A_{16}, $2D_8$, $D_{12} + D_4$, D_{16}, $2E_8$, $A_7 + 2D_5$, $A_7 + D_9$,
$A_1 + A_9 + D_6$, $E_8 + D_8$, $2E_7 + 2A_1$, $E_7 + A_9$, $E_6 + D_7 + A_3$.

There is a primitive element $\lambda \in Q(2E_8) \oplus H$ with $\lambda^2 = -8$ such that $R = R(M_\lambda)$, where M_λ is the orthogonal complement of $\mathbf{Z}\lambda$ in $Q(2E_8) \oplus H$. Let R' be a root system obtained from R by one elementary transformation. By Lemma 6.3, there is a full embedding $Q(R') \hookrightarrow M_\lambda \oplus H_1$ where $H_1 = \mathbf{Z}u_1 + \mathbf{Z}v_1$ ($u_1^2 = v_1^2 = 0$, $u_1 v_1 = 1$) is a hyperbolic plane. We can assume that u_1 is orthogonal to

$Q(R')$. Let R'' be a root system obtained from R' by one elementary transformation. There is a full embedding $Q(R'') \hookrightarrow M_\lambda \oplus H_1 \oplus H_2$ where $H_2 = \mathbf{Z}u_2 + \mathbf{Z}v_2$ ($u_2^2 = v_2^2 = 0$, $u_2 v_2 = 1$) is another hyperbolic plane. By Lemma 6.3, $Q(R'')$ is orthogonal to u_1 and u_2. Let i:

$$M_\lambda \oplus H_1 \oplus H_2 \hookrightarrow 2Q(E_8) \oplus H \oplus H_1 \oplus H_2$$

be the natural embedding. Set $S = \mathbf{Z}\lambda \oplus Q(R'')$ (orthogonal direct sum). The lattice S satisfies condition (b) in Corollary 5.3, since u_1 and u_2 are orthogonal to it. Let $\tilde{S}$ denote the primitive hull of S in $Q(2E_8) \oplus H \oplus H_1 \oplus H_2$.

Next we will check S and $\tilde{S}$ satisfy condition (a) and (b) in Theorem 2.15 (2).

(1) Let $\eta \in \tilde{S}$, $\eta\lambda = 0$ and $\eta^2 = 2$. We can write $\eta = m \oplus au_1 \oplus bu_2$, where $m \in 2Q(E_8) \oplus H$ and $a, b \in \mathbf{Z}$. Then $m^2 = 2$. Since $\eta\lambda = 0$, $m\lambda = 0$. Hence $m \in M_\lambda$, which means $\eta \in M_\lambda \oplus H_1 \oplus H_2$. So $\eta \in Q(R'')$ by fullness, which implies condition (a) in Theorem 2.15 (2) is satisfied.

(2) Let $\mu \in \tilde{S}$, $\mu^2 = 0$ and $\mu\lambda = -2$. By Lemma 6.2, such an element μ does not exist, which implies condition (b) in Theorem 2.15 (2) is satisfied. By Theorem 2.15 one knows that condition (A) in Theorem 0.2 holds. By Lemma 6.2 ii) and the Remark of Theorem 2.15, the surface obtained in $\mathbf{P}^5$ is in fact a complete intersection of three quadrics.

Second, let R be a root system whose type is one of the following 4: $2E_8 + A_1$, $D_{16} + A_1$, A_{17}, $D_{10} + E_7$.

There is $\lambda_1 \in Q(2E_8) \oplus H$ with $\lambda_1^2 = -2$ such that $R = R(U_{\lambda_1})$, where U_{λ_1} is the orthogonal complement of λ_1 in $Q(2E_8) \oplus H$. Let R' be a root system obtained from R by one elementary transformation. By Lemma 6.3 there is a full embedding $Q(R') \hookrightarrow U_{\lambda_1} \oplus H_1$ where $H_1 = \mathbf{Z}u_1 + \mathbf{Z}v_1$ ($u_1^2 = v_1^2 = 0$, $u_1 v_1 = 0$) is a hyperbolic plane. We can assume that u_1 is orthogonal to $Q(R')$. Let R'' be a root system obtained from R' by one elementary transformation. There is a full embedding $Q(R'') \hookrightarrow U_{\lambda_1} \oplus H_1 \oplus H_2$, where $H_2 = \mathbf{Z}u_2 + \mathbf{Z}v_2$ ($u_2^2 = v_2^2 = 0$, $u_2 v_2 = 1$) is another hyperbolic plane. By Lemma 6.3, $Q(R'')$ is orthogonal to u_1 and u_2. Let i:

$$U_{\lambda_1} \oplus H_1 \oplus H_2 \hookrightarrow 2Q(E_8) \oplus H \oplus H_1 \oplus H_2$$

be the natural embedding. Let $\lambda = 2\lambda_1 + u_2$. Then λ is primitive in $2Q(E_8) \oplus H \oplus H_1 \oplus H_2$. Set $S = \mathbf{Z}\lambda \oplus Q(R'')$ (orthogonal direct sum). Such S satisfies condition (b) in Corollary 5.3. Let $\tilde{S}$ denote the primitive hull of S in $2Q(E_8) \oplus H \oplus H_1 \oplus H_2$. Now let $\eta \in \tilde{S}$, $\eta\lambda = 0$, $\eta^2 = 2$. We can write $\eta = m \oplus au_1 \oplus bu_2$, where $m \in 2Q(E_8) \oplus H$, and $a, b \in \mathbf{Z}$. Thus $m^2 = 2$. Since $\eta\lambda = 0$, $m\lambda_1 = 0$.

Therefore $m \in R(U_{\lambda_1})$. By $\eta \in \tilde{S}$, η lies in the primitive hull of $Q(R'')$ in $U_{\lambda_1} \oplus H_1 \oplus H_2$. But $Q(R'')$ is full in $U_\lambda \oplus H_1 \oplus H_2$, thus $\eta \in Q(R'')$, which implies condition (a) in Theorem 2.15 (2) is satisfied. By Lemma 6.2, condition (b) in Theorem 2.15 (2) is also satisfied. By Theorem 2.15, one knows that condition (A) in Theorem 0.2 holds. By Lemma 6.2 ii) and the Remark of Theorem 2.15, the surface obtained in $\mathbf{P}^5$ is in fact a complete intersection of three quadrics.

We have completed all the proof of our main Theorem 0.2.

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