K. Konno
Nagoya Math. J.
Vol. 129 (1993), 115–146

EVEN CANONICAL SURFACES WITH SMALL K^2, I

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Introduction

Let S be a minimal algebraic surface of general type defined over the complex number field $\mathbf{C}$, and let K denote the canonical bundle. According to [10], we call S a *canonical surface* if the rational map Φ_K associated with $|K|$ induces a birational map of S onto the image X. We denote by $Q(X)$ the intersection of all hyperquadrics through X.

One of the fundamental problems on canonical surfaces is a conjecture of Miles Reid [15, p. 541] which states that a canonical surface satisfies either (1) $K^2 \geq 4p_g - 12$, or (2) the irreducible component of $Q(X)$ containing the canonical image X is of dimension 3. In other words, canonical surfaces with $K^2 < 4p_g - 12$ should have a flavor similar to the exceptions of Enriques-Babbage-Petri's theorem, i.e., trigonal curves and plane quintic curves. Unfortunately, as of now, the conjecture is known to hold only for canonical surfaces with $K^2 = 3p_g - 7$, $3p_g - 6$ (see [4], [8], [1], [10] and [12]).

The present paper is an experiment for Reid's conjecture, considering regular canonical surfaces which are *even*. Here, a compact complex manifold of dimension 2 is called an even surface if the second Stiefel-Whitney class W_2 vanishes [10, §5]. Since they are closed under deformations, even surfaces have their own interest among surfaces of general type; Furthermore, as Horikawa stated in [10, §5], we can rediscover some important lines, e.g., $K^2 = 2p_g - 4$, by considering only even surfaces. This is why we choose them as an experimental material. Let S be an even surface. Since $W_2 = 0$, we can find a line bundle L on S which satisfies $K = 2L$. Such a line bundle L will be referred to as a semi-canonical bundle on S.

In Section 1, even canonical surfaces with $K^2 < 4p_g - 12$, $q = 0$ are classified into three types (I), (II) and (III) according as the nature of the semi-canonical map Φ_L. Namely, we call S a surface of type (I) (resp. type (II)) if Φ_L is a rational map of degree 1 (resp. 3) onto the image, whereas we call it a surface of type (III)

Received October 28, 1991.

if Φ_L is composed of a pencil.

In Section 2, we study surfaces of type (I). More generally, following an idea of Castelnuovo [4], we show that the inequality $L^2 \geq 4h^0(L) - 10$ holds whenever Φ_L induces a birational map onto the image, and we classify those which attain the lower bound $L^2 = 4h^0(L) - 10$. It will be turned out that they coincide with surfaces of type (I). We show that most of them have a pencil of plane quintic curves. A more precise statement can be found in Theorem 2.6 whose proof occupies Section 3.

In Sections 4–7, we study surfaces of type (II). The purpose here is to get an inequality similar to [17, (0.0)] in spirit (Theorem 7.4). Since the semi-canonical image is birationally a ruled surface, surfaces of type (II) may be considered as a 2-dimensional analogue of trigonal curves. By studying the structure of semi-canonical images, we can show that surfaces of type (II) have a pencil of trigonal curves of genus at most 10. When the semi-canonical image is ruled by lines, we apply a method in [18] in order to get such an upper bound on genus. The inequality (7.3) may suggest what we can expect on the slope of non-hyperelliptic fibrations, another problem in the geography of surfaces.

In Section 8, we show our main result, Theorem 8.4, which states that Reid's conjecture is true for regular even canonical surfaces. This essentially follows from a more general criterion (Theorem 8.3). We also give some examples. Among others, we exhibit a series of canonical surfaces with $K^2 = 4p_g - 12$ and $q = 0$, which may be called *bi-$K3$ surfaces,* whose canonical image is cut out by hyper-quadrics. This implies that, in Reid's conjecture, $K^2 = 4p_g - 12$ is the fatal line. In a future paper, we shall study even canonical surfaces on Reid's line.

The author thanks Professor E. Horikawa for sending him his recent papers [10] and [11] before publication, which inspired him very much and brought him to study even surfaces. He also thanks Professor M. Reid for his kind letters and for sending him an interesting paper [17]. Finally but not less deeply, he thanks Professors T. Ashikaga and S. Mukai for stimulating discussions, and the referee whose valuable suggestions improved the original arguments in Section 4.

§1. Semi-canonical map

In this section, we show the following proposition with several lemmas.

PROPOSITION 1.1. *Let S be an even canonical surface with $K^2 < 4p_g - 12$, $q = 0$, and let L denote a semi-canonical bundle. Then $L^2 \leq 4h^0(L) - 10$. Furthermore, the semi-canonical map Φ_L satisfies one of the following:*

(I) *Φ_L induces a birational map of S onto the image.*

(II) *Φ_L induces a generically finite map of degree 3 onto the image which is birationally equivalent to a rational ruled surface.*

(III) *Φ_L is composed of a pencil of non-hyperelliptic curves of genus 3 or 4.*

LEMMA 1.2. *Let S be as in Proposition 1.1. Then $L^2 \leq 4h^0(L) - 10$.*

Proof. Since L^2 is a positive even integer, there is an integer k such that $L^2 = 4h^0(L) - 2k$. It follows from $K^2 < 4p_g - 12$ that $L^2 \leq p_g - 4$, or equivalently that $p_g \geq 4h^0(L) - 2k + 4$. By the Riemann-Roch theorem, we have

$$2h^0(L) - h^1(L) = -L^2/2 + \chi(\mathcal{O}_S). \tag{1.1}$$

It follows that $2h^0(L) - h^1(L) = -2h^0(L) + k + 1 + p_g \geq 2h^0(L) - k + 5$. Therefore, we have $k \geq h^1(L) + 5 \geq 5$ and, hence, $L^2 \leq 4h^0(L) - 10$. Q.E.D.

We put $h^0(L) = n + 1$. Then we have $L^2 \leq 4n - 6$ by Lemma 1.2. Since L^2 is a positive even integer, we in particular have $n \geq 2$.

LEMMA 1.3. *Let S be as above, and assume that Φ_L induces a generically finite map $f : S \to V$ onto the image. Then $\deg f = 1$ or 3. When $\deg f = 3$, V is birationally a rational ruled surface.*

Proof. Since V is an irreducible nondegenerate surface in $\mathbf{P}^n$, we have

$$L^2 \geq (\deg f)(\deg V) \geq (\deg f)(n-1). \tag{1.2}$$

Since $L^2 \leq 4n - 6$, we have $\deg f \leq 3$. Suppose that $\deg f = 2$. Then we have $\deg V \leq 2n - 3$ by (1.2). It follows from [2, Lemma 3] that V is birationally equivalent to a ruled surface. On the other hand, being a canonical surface, S cannot be birationally equivalent to a double covering of a ruled surface. Therefore, $\deg f \neq 2$. If $\deg f = 3$, then V is birationally a ruled surface by [2] again. Since $q(S) = 0$, V is rational. Q.E.D.

LEMMA 1.4. *Let S be as above. Assume that $|L|$ is composed of a pencil as $|L| = |nD| + Z$, where $|D|$ is an irreducible pencil and Z denotes the fixed part. Then $|D|$ is a pencil of non-hyperelliptic curves of genus 3 or 4 which is free from base points.*

Proof. We have $4n-6 \geq L^2 = nLD + LZ \geq nLD$. It follows $3 \geq LD = nD^2 + DZ \geq nD^2 \geq 2D^2$. Since S is an even surface, D^2 is a nonnegative even integer. Therefore, we get $D^2 = 0$. Since S is a canonical surface, a general member of $|D|$ must be a non-hyperelliptic curve. Since $KD = 2LD \leq 6$ and $D^2 = 0$, $|D|$ is a pencil of non-hyperelliptic curves of genus 3 or 4 which is free from base points. Q.E.D.

§2. Surfaces of type (I)

In this section, we study surfaces of type (I) following a classical idea of Castelnuovo. A similar computation can be found in [8].

LEMMA 2.1. *Let S be an even surface with a semi-canonical bundle L. Put $n = h^0(L) - 1$ and assume that the semi-canonical map Φ_L induces a birational map of S onto the image $V \subset \mathbf{P}^n$. Then $L^2 \geq \deg V \geq 4n-6$. If $\deg V = 4n-6$, then $L^2 = 4n-6$, $|L|$ is free from base points and $q(S) \leq h^1(L)$.*

Proof. Let $\sigma: \tilde{S} \rightarrow S$ be a composite of blowing-ups such that the variable part $|M|$ of $|\sigma^*L|$ is free from base points. We can assume that σ is the shortest among those with such a property. Let Z denote the fixed part of $|\sigma^*L|$. Then

$$L^2 = M^2 + (\sigma^*L + M)Z \geq M^2. \tag{2.1}$$

Let C be a general member of $|M|$. We can assume that it is an irreducible nonsingular curve. We denote by M_C the restriction of M to C. From the cohomology long exact sequence for

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(M) \rightarrow \mathcal{O}_C(M_C) \rightarrow 0,$$

we have

$$h^0(M_C) \geq h^0(M) - 1. \tag{2.2}$$

Since $\deg V = M^2$, it is sufficient for our purpose to show that $\deg M_C \geq 4h^0(M_C) - 6$.

Let $\tilde{K}$ denote a canonical divisor on $\tilde{S}$. We have an exceptional divisor E for σ such that $\tilde{K} = \sigma^*K + E = 2M + 2Z + E$. Putting $h^0(M_C) = r+1$, let m be the integer part of $(\deg M_C - 1)/(r-1)$ and $\varepsilon = \deg M_C - 1 - m(r-1)$. Since the canonical bundle K_C of C is induced by $\tilde{K} + M$, it follows that $3 \deg M_C \leq 2g(C) - 2$, i.e., $3m(r-1) + 3\varepsilon + 5 \leq 2g(C)$ with equality holding only if $M(2Z+E) = 0$. On the other hand, Castelnuovo's bound (see e.g., [7, Theorem (3.7)]) implies that $2g(C) \leq m(m-1)(r-1) + 2m\varepsilon$. Putting these together, we get

$$m(m-4)(r-1) + (2m-3)\varepsilon \geq 5.$$

If $r=2$, we have $\varepsilon = 0$ and $m \geq 5$. If $r \geq 3$, then we have $m \geq 4$ and, furthermore, $\varepsilon \geq 1$ when $m = 4$. Hence, in either case, we have $(m-4)(r-1) + \varepsilon - 1 \geq 0$. Since $\deg M_C = 4(r+1) - 6 + (m-4)(r-1) + \varepsilon - 1$, we get $\deg M_C \geq 4h^0(M_C) - 6$.

Assume that $M^2 = 4n - 6$. The above observations imply that $h^0(M_C) = n$ and $M(2Z+E) = 0$. Hence we have $MZ = ME = 0$. In particular, σ is the identity map. In order to show that $|L|$ is free from base points, it is sufficient to show that $Z = 0$. Since $MZ = 0$, we have $Z^2 \leq 0$ by Hodge's index theorem. On the other hand, we have $0 \leq LZ = MZ + Z^2 = Z^2$ since L is nef. Hence $Z^2 = 0$, and we have $Z = 0$ by Hodge's index theorem. Now, by (2.1), we get $L^2 = 4n - 6$. Since the equality holds in (2.2), the restriction map $H^0(L) \to H^0(L_C)$ is surjective. Therefore, we get $h^1(\mathcal{O}_S) \leq h^1(L)$. Q.E.D.

DEFINITION 2.2. For any nondegenerate subvariety W in $\mathbf{P}^N$, we denote by $Q(W)$ the intersection of all hyperquadrics through W and call it the *quadric hull* of W. When there are no hyperquadrics through W, we put $Q(W) = \mathbf{P}^N$.

LEMMA 2.3. *Let S be an even surface with $L^2 = 4h^0(L) - 10$, and assume that Φ_L induces a birational map onto the image. Put $n = h^0(L) - 1$. Then S is a regular surface with $p_g = 4n - 2$, $K^2 = 4p_g - 16$. Furthermore, the semi-canonical image $V = \Phi_L(S)$ has only rational double points as its singularity and $Q(V)$ is an irreducible threefold of degree $n - 2$ in $\mathbf{P}^n$.*

Proof. Let C be a general member of $|L|$. We can assume that it is an irreducible nonsingular curve. We denote by L_C the restriction of L to C. Then we have $h^0(L_C) = n$. Since $3L$ induces the canonical bundle K_C, we have

$$h^0(mL_C) = \begin{cases} 3n-3 & \text{if } m = 2, \\ g(C) = 6n-8 & \text{if } m = 3, \\ (2m-3)(2n-3) & \text{if } m \geq 4. \end{cases} \tag{2.3}$$

We put $C_0 = \Phi_L(C)$. We regard C_0 as a general hyperplane section of V. Then it is a nondegenerate curve in $\mathbf{P}^{n-1}$ of degree $L^2 = 4n - 6$. Let Z_0 denote a general hyperplane section of C_0. Since it is a nondegenerate set of $4n - 6$ distinct points in uniform position, we have

$$
\begin{array}{ll}
h_{Z_0}(1) = n-1, & h_{Z_0}(2) \geq 2n-3, \\
h_{Z_0}(3) \geq 3n-5, & h_{Z_0}(4) \geq 4n-7, \\
h_{Z_0}(m) = 4n-6 \ (m \geq 5), &
\end{array}
$$

where h_{Z_0} denotes the Hilbert function of Z_0 (see [7, Ch. 3] for the properties). Note that we have $h_{C_0}(m) \leq h^0(mL_C)$. Since $h_{C_0}(1) = n$ and $h_{C_0}(m) \geq h_{C_0}(m-1) + h_{Z_0}(m)$, it follows from (2.3) that

$$
\begin{array}{ll}
h_{Z_0}(1) = n-1, & h_{C_0}(1) = n, \\
h_{Z_0}(2) = 2n-3, & h_{C_0}(2) = 3n-3, \\
h_{Z_0}(3) = 3n-5, & h_{C_0}(3) = 6n-8, \\
h_{Z_0}(4) = 4n-7, & h_{C_0}(m) = (2m-3)(2n-3) \text{ for } m \geq 4.
\end{array}
$$

Since we have $h_{C_0}(m) = h_{C_0}(m-1) + h_{Z_0}(m)$ for all positive integer m, C_0 is projectively normal. Hence, it is a nonsingular curve of genus $6n-8$.

We next consider V. By the Riemann-Roch theorem and Ramanujam's vanishing theorem, we have

$$
h^0(mL) = \begin{cases} p_g & \text{if } m = 2, \\ m(m-2)(2n-3) + \chi & \text{if } m \geq 3, \end{cases} \tag{2.4}
$$

where $\chi = \chi(\mathcal{O}_S)$. Since $h_V(2) \geq h_V(1) + h_{C_0}(2) = 4n-2$, we have $p_g \geq 4n-2$. Since $h_V(3) \geq h_V(2) + h_{C_0}(3) \geq 10n-10$, we have $\chi \geq 4n-1$. On the other hand, it follows from (1.1) that $\chi = 4n-1-h^1(L)$. Hence we have $h^1(L) = 0$ and $\chi = 4n-1$. By Lemma 2.1, we get $q = 0$ and $p_g = 4n-2$. Then it is easy to see that $h^0(mL) = h_V(m) = h_V(m-1) + h_{C_0}(m)$ holds for any positive integer m. This shows that V is projectively normal, that is, the multiplication map $\mathrm{Sym}^m H^0(L) \to H^0(mL)$ is surjective for any $m \geq 0$. Since $K = 2L$, it follows that V is isomorphic to the canonical model of S. Hence V has at most rational double points as its singularity.

We show the last assertion of Lemma 2.3. Since Z_0 is a nondegenerate set of $4n-6$ distinct points in $\mathbf{P}^{n-2}$ and since we have $h_{Z_0}(2) = 2n-3$, it follows from Castelnuovo's Lemma (see, e.g., [7, Lemma (3.9)]) that $Q(Z_0)$ is a rational normal curve. Since V is projectively normal, $Q(V)$ is an irreducible threefold of degree $n-2$ in $\mathbf{P}^n$. Q.E.D.

By this lemma and Lemma 1.2, we see that even surfaces with $L^2 = 4h^0(L) - 10$ for which Φ_L is birational onto the image coincide with surfaces of type (I) in the sense of Proposition 1.1. Hence we have the following:

LEMMA 2.4. *Let S be a surface of type* (I). *Then the canonical image X is not cut out by hyperquadrics. More precisely, the quadric hull of X is the Veronese transform of the quadric hull of the semi-canonical image which is of dimension* 3.

Proof. Since the semi-canonical image V is projectively normal, X is nothing but the Veronese transform of V. Since the homogeneous ideal of $Q(V)$ is generated in degree 2, $Q(X)$ is contained in the Veronese transform of $Q(V)$. We have $h^0(2K) = K^2 + \chi(\mathcal{O}_S) = 20n - 25$ and $h^0(Q(V), \mathcal{O}(4)) = 20n - 25$. It follows that any hyperquartic through V contains $Q(V)$. Hence $Q(X)$ coincides with the Veronese transform of $Q(V)$. Q.E.D.

2.5. In order to describe surfaces of type (I), we recall that an irreducible nondegenerate threefold of degree $n - 2$ in $\mathbf{P}^n$ is one of the following varieties (see [6] or [8]).

(A) $\mathbf{P}^3$ $(n = 3)$.

(B) A hyperquadric in $\mathbf{P}^4$ $(n = 4)$.

(C) A cone over $\mathbf{P}^2$ embedded into $\mathbf{P}^5$ by the holomorphic map associated with $|\mathcal{O}_{\mathbf{P}^2}(2)|$ $(n = 6)$, i.e., the weighted projective space $\mathbf{P}(1, 1, 1, 2)$.

(D) A rational normal scroll, that is, the image of the total space of the $\mathbf{P}^2$-bundle

$$\pi : \mathbf{P}_{a,b,c} = \mathbf{P}(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \to \mathbf{P}^1,$$

by the holomorphic map associated with $|T|$, where T denotes a tautological divisor and a, b, c are integers satisfying

$$0 \leq a \leq b \leq c, \quad a + b + c = n - 2 \quad (n \geq 5).$$

We have the following subcases:

(D.1) $a > 0 : \mathbf{P}_{a,b,c}$.

(D.2) $a = 0$, $b > 0$: a cone over the Hirzebruch surface Σ_{c-b} embedded by the holomorphic map associated with $|\Delta_0 + c\Gamma|$, where Δ_0 is a section with $\Delta_0^2 = b - c$ and Γ is a fiber.

(D.3) $a = b = 0$, $c > 0$: a generalized cone over a rational normal curve of degree $c = n - 2$ in $\mathbf{P}^{n-2}$ whose ridge is a line.

The proof of the following is much similar to [1], and we postpone it to the next section.

THEOREM 2.6. *Let S be a surface of type* (I). *Then S is the minimal resolution of a surface S' with at most rational double points, where S' is one of the following*:

(1) *A sextic surface in* $\mathbf{P}^3$.

(2) *A complete intersection of a hyperquadric and a hyperquintic in* $\mathbf{P}^4$

(3) *A hypersurface of degree* 9 *defined in* $\mathbf{P}(1, 1, 1, 2)$ *by*

$$x_0u^4 + A_3u^3 + A_5u^2 + A_7u + A_9 = 0,$$

where (x_0, x_1, x_2, u) *is a system of coordinates with* $\deg x_i = 1$, $\deg u = 2$, *and the* A_k *are homogeneous forms of degree k in the* x_i.

(4) *A member of* $|5T - (n-4)F|$ *on* $\mathbf{P}_{a,b,c}$, *where F denotes a fiber of π and a, b, c are integers satisfying*

$$0 \leq a \leq b \leq c, \quad a+b+c = n-2, \quad a+c \leq 4b+2, \quad b \leq 3a+2.$$

§3. Proof of Theorem 2.6

In this section, we show Theorem 2.6. We first consider (A), (B) of 2.5.

LEMMA 3.1. *When $n = 3$, V is a sextic surface in* $\mathbf{P}^3$. *When $n = 4$, V is a complete intersection of a hyperquadric and a hyperquintic.*

Proof. Let Z_0 be, as before, a set of points obtained by cutting V twice by general hyperplanes. If $n = 3$, then Z_0 is a set of 6 distinct points on $\mathbf{P}^1$. It follows that V is a sextic surface. If $n = 4$, then it consists of 10 distinct points and $Q(Z_0)$ is a nonsingular conic curve. Since $h_{Z_0}(5) = 10$ and $h_{Q(Z_0)}(5) = h^0(\mathbf{P}^1, \mathcal{O}(10)) = 11$, there exists a plane quintic curve which contains Z_0 but does not contain $Q(Z_0)$. Since Z_0 is of degree 10, it follows that Z_0 is a complete intersection of the quintic curve and $Q(Z_0)$. Since V and $Q(V)$ are both projectively normal, V is a hyperquintic section of $Q(V)$. Q. E. D.

We next consider the case (C) of 2.5.

PROPOSITION 3.2. *Let S be a surface of type* (I) *with* $h^0(L) = 7$ *and* $L^2 = 18$.

Assume that $Q(V)$ is a cone over the Veronese surface. Then S is the minimal resolution of a surface S' in the $\mathbf{P}^1$-bundle $\varpi : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbf{P}^2$, which has at most rational double points, and which is linearly equivalent to $4T + F$, where T and F respectively denote a tautological divisor and a pull-back of a line in $\mathbf{P}^2$. Therefore, the semi-canonical image is a weighted hypersurface of degree 9 in the weighted projective space $\mathbf{P}(1, 1, 1, 2)$ defined by

$$x_0u^4 + A_3u^3 + A_5u^2 + A_7u + A_9 = 0,$$

where (x_0, x_1, x_2, u) is a system of coordinates with $\deg x_i = 1$, $\deg u = 2$, and the A_k are homogeneous forms of degree k in the x_i.

Proof. Let v be the vertex of $Q(V) \subset \mathbf{P}^6$. Blow $\mathbf{P}^6$ up at v and let W be the proper transform of $Q(V)$. Then W can be identified with the total space of the $\mathbf{P}^1$-bundle $\varpi : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathbf{P}^2$. Note that T and F generate the Picard group of W, and that we have $T^2 = 2TF$ in the Chow ring of W. Note also that $Q(V)$ is nothing but the image of W under the holomorphic map associated with $|T|$. We denote by $\tau : W \to Q(V)$ the induced holomorphic map. Let T_∞ be the section linearly equivalent to $T - 2F$. Then $v = \tau(T_\infty)$.

Let Λ_0 be the pull-back by Φ_L of the linear system of hyperplanes through v, and let G be the fixed part of Λ_0. Since $Q(V)$ is a cone over the Veronese surface, we have a net Λ such that $2H \in \Lambda_0 - G$ for $H \in \Lambda$. Note that we have $L = [2H + G]$ and $LG = 0$ since $|L|$ is free from base points. Therefore, we have $18 = L^2 = 2LH + LG = 4H^2 + 2GH$. Since S is an even surface, we have $H^2 = 2$ or 4. If $H^2 = 2$, then Λ induces a rational map of degree less than three onto $\mathbf{P}^2$. This contradicts that S is a canonical surface. It follows $H^2 = 4$, $GH = 1$ and $G^2 = -2$.

We remark that G is a (-2)-curve. To see this, note that G consists of (-2)-curves, because $KG = 2LG = 0$. Let G_0 be the irreducible component of G with $HG_0 > 0$. Since $HG_0 \leq HG = 1$, we have $HG_0 = 1$. Then we have $HG' = 0$, where $G' = G - G_0$. It follows from $LG_0 = 0$ that $G_0G' = 0$. Then, since $G^2 = G_0^2 = -2$, we get $(G')^2 = 0$. By Hodge's index theorem, we have $G' = 0$, that is, $G = G_0$.

We let S' denote the proper transform of V by $\tau : W \to Q(V)$. Since S' is obtained from V by blowing up at a point, S' has at most rational double points as its singularity (because so does V). Though we cannot directly conclude that S is the minimal resolution of S', we at least have $\chi(\mathcal{O}_{S'}) = \chi(\mathcal{O}_S) = 23$.

Suppose that S' is linearly equivalent to $\alpha T + \beta F$ on W. Since S' is irreducible, α and β must be nonnegative integers. Furthermore, since $H^2 = 4$, Λ induces a rational map of degree 3 or 4 onto $\mathbf{P}^2$. Hence we have $\alpha = 3$ or 4. Since $\deg V = L^2 = 18$, we have $18 = T^2(\alpha T + \beta F) = 4\alpha + 2\beta$, that is, $2\alpha + \beta = 9$. It follows $(\alpha, \beta) = (4, 1)$ or $(3, 3)$. We can exclude the last alternative as follows: Assume that S' is linearly equivalent to $3T + 3F$. From

$$0 \to \mathcal{O}_W(-S') \to \mathcal{O}_W \to \mathcal{O}_{S'} \to 0,$$

we have $\chi(\mathcal{O}_{S'}) = \chi(\mathcal{O}_W) - \chi(\mathcal{O}_W(-3T - 3F))$. Then an easy calculation shows $\chi(\mathcal{O}_{S'}) = 22$. This contradicts that $\chi(\mathcal{O}_{S'})$ must be 23. Therefore, S' is linearly equivalent to $4T + F$. Then the dualizing sheaf $\omega_{S'}$ of S' is induced by $2T$, and we have

$$\omega_{S'}^2 = (2T)^2(4T + F) = 72 = K^2.$$

It follows that S coincides with the minimal resolution of S'. Therefore, the natural map $S \to V$ factors through S'. We have shown that Λ induces a holomorphic map of degree 4 onto $\mathbf{P}^2$.

Let X_0 and X_1 be sections of $[T]$ and $[T_\infty]$, respectively, such that they form a system of homogeneous fiber coordinates on W. Then the equation of any member of $|4T + F|$ can be written as

$$\alpha_1 X_0^4 + \alpha_3 X_0^3 X_1 + \alpha_5 X_0^2 X_1^2 + \alpha_7 X_0 X_1^3 + a_9 X_1^4 = 0,$$

where the α_i are homogeneous forms of degree i on $\mathbf{P}^2$. It follows that $|4T + F|$ is free from base points and contains an irreducible nonsingular member.

Since V is obtained from S' by contracting a (-2)-curve defined by $X_1 = \alpha_1 = 0$, we see that it is defined by the equation as in the statement of Proposition 3.2. Q.E.D.

Remark 3.3. It is shown in [5] that the moduli space is non-reduced for the above type of surfaces. The key is the presence of the (-2)-curve G.

We finally consider the case (D) of 2.5. We separately treat the three subcases (D.1), (D.2) and (D.3).

LEMMA 3.4. *Let S be a surface of type* (I). *Assume that $Q(V)$ is isomorphic to $\mathbf{P}_{a,b,c}$. Then the semi-canonical image is linearly equivalent to $5T - (n-4)F$, where F denotes a fiber of $\pi : \mathbf{P}_{a,b,c} \to \mathbf{P}^1$. Furthermore, a, b, c satisfy*

$$a + c \leq 4b + 2, \quad b \leq 3a + 2. \tag{3.1}$$

Proof. Assume that V is linearly equivalent to $\alpha T + \beta F$. Since $\deg V = 4n - 6$, we have $T^2(\alpha T + \beta F) = 4n - 6$. It follows $(\alpha - 4)(n - 2) + \beta = 2$. The dualizing sheaf ω_V is induced from $(\alpha - 3)T + (\beta + n - 4)F$. Since V has at most rational double points, and since $K = 2L$, ω_V and $2T$ are equivalent on V. Therefore, we have $((\alpha - 5)T + (\beta + n - 4)F)^2(\alpha T + \beta F) = 0$. It follows $(\alpha - 5)(\alpha(n - 2) + \beta + 2\alpha(\beta + n - 4)) = 0$. Since $n \geq 5$, we get $\alpha = 5$ and $\beta = 4 - n$.

The rest can be shown similarly as in the proof of [1, §2, Claim III], using the fact that V has at most rational double points. Q. E. D.

PROPOSITION 3.5. *Assume that $Q(V)$ is a cone over the Hirzebruch surface Σ_{c-b}. Then S is the minimal resolution of a surface S^* in the $\mathbf{P}^1$-bundle $\varpi : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + c\Gamma)) \to \Sigma_{c-b}$ which has at most rational double points and which is linearly equivalent to $4T + \varpi^*(\Delta_0 + (2 - b)\Gamma)$, where T denotes a tautological divisor. Furthermore,* (3.1) *is satisfied, i.e., $1 \leq b \leq 2$ and $c \leq 4b + 2$.*

Proof. Assume that $Q(V)$ is a cone over the Hirzebruch surface Σ_{c-b}. Recall that b and c are integers satisfying $0 < b \leq c$ and $b + c = n - 2$. Let v be the vertex of $Q(V) \subset \mathbf{P}^n$. Blow $\mathbf{P}^n$ up at v and let W be the proper transform of $Q(V)$. Then W can be identified with the total space of $\varpi : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\Delta_0 + c\Gamma)) \to \Sigma_{c-b}$. We have $T^2 = T\varpi^*(\Delta_0 + c\Gamma)$ in the Chow ring of W. We let T_∞ denote the section linearly equivalent to $T - \varpi^*(\Delta_0 + c\Gamma)$. Note that $Q(V)$ is nothing but the image of W under the holomorphic map associated with $|T|$, and T_∞ is contracted to v. We denote by $\tau : W \to Q(V)$ the induced holomorphic map.

Let Λ_0 be the pull-back by Φ_L of the linear system of hyperplanes through v, and let G be the fixed part of Λ_0. We put $\Lambda = \Lambda_0 - G$. Then it induces a rational map $\mu : S \to \Sigma_{c-b}$. We let H denote a general member of Λ. Then $L = [H + G]$. Note that we have $LG = 0$, since $|L|$ is free from base points. Let $\sigma : \tilde{S} \to S$ be the elimination of the base points of Λ, and let $|\tilde{H}|$ and $\tilde{E}$ be the variable and the fixed part of $\sigma^*\Lambda$. Then $\tilde{E}$ consists of exceptional curves for σ, and $|\tilde{H}|$ defines a holomorphic map $\tilde{\mu} : \tilde{S} \to \Sigma_{c-b}$. Note that $\tilde{\mu}$ is of degree not less than 3, and that $\tilde{G} := \tilde{E} + \sigma^*G$ is the inverse image of v. Since we have $4n - 6 = (\sigma^*L)^2 = (\sigma^*L)\tilde{H} = \tilde{H}^2 + \tilde{H}\tilde{G} = (\deg \tilde{\mu})(n - 2) + \tilde{H}\tilde{G}$, we see that $\deg \tilde{\mu} = 3$ or 4, and that $\tilde{G} \neq 0$. In particular, we have a holomorphic map $h : \tilde{S} \to W$ over μ such that $h^*T_\infty = \tilde{G}$.

We let S^* be the proper transform of V by $\tau : W \to Q(V)$. Then $S^* = h(\tilde{S})$, and it has at most rational double points since so does V. As we saw above, S may not be the minimal resolution of S^*. However, since S^* has only rational double points, we at least have $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_S) = 4n - 1$.

Suppose that S^* is linearly equivalent to $\alpha T + \varpi^*(\beta\Delta_0 + \gamma\Gamma)$. Then we have $\alpha \geq 3$. We also remark that β and γ are nonnegative, because $S^*|_{T_\infty}$ is effective. Since $\deg V = L^2 = 4n - 6$, we have $4n - 6 = T^2(\alpha T + \varpi^*(\beta\Delta_0 + \gamma\Gamma))$. It follows

$$(3.2) \qquad (n-2)\alpha + b\beta + \gamma = 4n - 6.$$

Let C' be a section of S^* by a general member of $|T|$. Then, we can assume that C' is a nonsingular curve of genus $6n - 8$. Since the canonical bundle of C' is induced by $(\alpha - 1)T + \varpi^*((\beta - 1)\Delta_0 + (\gamma + b - 2)\Gamma)$, we get $12n - 18 = T(\alpha T + \varpi^*(\beta\Delta_0 + \gamma\Gamma))((\alpha - 1)T + \varpi^*((\beta - 1)\Delta_0 + (\gamma + b - 2)\Gamma))$. By a calculation, we get

$$(3.3) \quad (4n-6)(\alpha - 4) + (\alpha + \beta)(b\beta + \gamma - 2) - (\beta - 1)(c\beta - \gamma) = 0.$$

It follows from (3.2) and (3.3) that $(\alpha, \beta, \gamma) = (4, 1, 2 - b)$ or $(3, 2, n - 2b)$. We can exclude the last alternative as follows: Assume that S^* is linearly equivalent to $3T + \varpi^*(2\Delta_0 + (c - b + 2)\Gamma)$. From

$$0 \to \mathcal{O}_W(-S^*) \to \mathcal{O}_W \to \mathcal{O}_{S^*} \to 0,$$

we have $\chi(\mathcal{O}_{S^*}) = \chi(\mathcal{O}_W) - \chi(\mathcal{O}_W(-S^*))$. Then an easy calculation shows $\chi(\mathcal{O}_{S^*}) = 4n - 2$. This contradicts that $\chi(\mathcal{O}_{S^*})$ must be $4n - 1$. Therefore, S^* is linearly equivalent to $4T + \varpi^*(\Delta_0 + (2 - b)\Gamma)$. Then the dualizing sheaf ω_{S^*} of S^* is induced by $2T$, and we have

$$\omega_{S^*}^2 = (2T)^2(4T + \varpi^*(\Delta_0 + (2 - b)\Gamma)) = 16n - 24 = K^2.$$

It follows that S coincides with the minimal resolution of S^*. Hence the natural map $S \to V$ factors through S^*. As a consequence, we have $H^2 = 4n - 8$. $G^2 = -2$ and see that μ is a holomorphic map of degree 4.

It is easy to see that $|4T + \varpi^*(\Delta_0 + (2 - b)\Gamma)|$ contains an irreducible surface with at most rational double points if and only if $b \leq 2$ and $c \leq 4b + 2$.

Q.E.D.

We remark that $\tau : W \to Q(V)$ factors through $\mathbf{P}_{0,b,c}$, and that T_∞ is contracted to a nonsingular rational curve on $\mathbf{P}_{0,b,c}$. The image S' of S^* is linearly equivalent to $5T - (n - 4)F$ on $\mathbf{P}_{0,b,c}$. Recall that G consists of (-2)-curves, and that it corresponds to the intersection $T_\infty \cap S^*$. Therefore, S' has at most rational double points as its singularity.

In order to complete the proof of Theorem 2.6, it is now sufficient to show the following:

LEMMA 3.6. *The case* (D.3) *is inadequate, that is, $Q(V)$ can never be a generalized cone over a rational normal curve.*

Proof. By considering the pull-back of the linear system of hyperplanes through the ridge of $Q(V)$, we see that $L = [(n-2)D + G]$, where $|D|$ is an irreducible pencil and G is an effective (possibly zero) divisor corresponding to the ridge (see [9, Lemma 1.5]). We have

$$4n - 6 = L^2 = (n-2)LD + LG \geq (n-2)LD.$$

Since $n \geq 5$, we get $4 \geq LD \geq (n-2)D^2 + DG \geq 3D^2$. Since S is an even surface, D^2 is a nonnegative even integer. Thus $D^2 = 0$. Note that a general member of $|D|$ is non-hyperelliptic, since S is a canonical surface. Then, Clifford's theorem shows that $h^0(D, \mathcal{O}(L|_D)) < LD/2 + 1 \leq 3$. It follows that the image of D under Φ_L is at most a line. This contradicts that Φ_L induces a birational map of S onto the image. Q.E.D.

§4. Semi-canonical images of surfaces of type (II)

We let S denote a surface of type (II) in the sense of Proposition 1.1. Put $n = h^0(L) - 1$ as usual. Let $\sigma : \tilde{S} \to S$ denote a succession of blowing-ups such that the variable part $|M|$ of $|\sigma^* L|$ is free from base points. We can assume that σ is the shortest among those enjoying such a property. Then $|M|$ induces a holomorphic map $f : \tilde{S} \to V$ of degree 3, where V is an irreducible non-degenerate surface in $\mathbf{P}^n$ which is birationally equivalent to a rational ruled surface. In this section, we study the structure of V more closely. See also [16] and [18].

First of all, we remark the following:

LEMMA 4.1. *Let S be an even surface, and let $h : S \to W$ be a holomorphic map of odd degree onto a surface W. Then W has no Cartier divisors with odd self-intersection number.*

Proof. If D_0 is a Cartier divisor on W with D_0^2 odd, then $(h^* D_0)^2 = (\deg h)\, D_0^2$ is odd. This is impossible, since S is an even surface. Q.E.D.

Let $\mu : \tilde{V} \to V$ denote the minimal resolution of V and let $\tilde{H}$ be the pull-back to $\tilde{V}$ of a hyperplane section of V.

LEMMA 4.2. *μ is induced by the complete linear system* $|\tilde{H}|$.

Proof. Let $\tau : V' \to V$ be the normalization of V and let H' be the pull-back of a hyperplane section of V. Then $f : \tilde{S} \to V$ can be lifted to a holomorphic map of $\tilde{S}$ onto V'. Hence we have $n + 1 = h^0(M) \geq h^0(H') \geq h^0(\mathcal{O}_V(1)) = n + 1$. It follows $h^0(H') = n + 1$. Since $\mu : \tilde{V} \to V$ also factors through V', we have $h^0(\tilde{H}) = n + 1$, which is what we want. Q.E.D.

LEMMA 4.3. *V is not isomorphic to* $\mathbf{P}^2$.

Proof. Assume that $(V, \mathcal{O}_V(1)) \simeq (\mathbf{P}^2, \mathcal{O}(k))$ as polarized manifolds, where k is a positive integer. Then $\deg V = k^2$ and $n + 1 = h^0(V, \mathcal{O}_V(1)) = (k + 1)(k + 2)/2)$. Let $|L| = |M_0| + Z$ be the decomposition of $|L|$ into the variable and the fixed parts. Since $(V, \mathcal{O}_V(1)) \simeq (\mathbf{P}^2, \mathcal{O}(k))$, we have a net Λ such that $M_0 = kL_0$ for $L_0 \in \Lambda$ and which induces a rational map $\Phi_\Lambda : S \to \mathbf{P}^2$ of degree 3. By Lemma 4.1, Φ_Λ cannot be holomorphic. Hence we have $L_0^2 \geq 4$. Since $L^2 \leq 4n - 6 = 2k^2 + 6k - 6$ and

$$L^2 = M_0^2 + (L + M_0) Z \geq M_0^2 = k^2 L_0^2 \geq 4k^2,$$

we have no such an integer k. Q.E.D.

By this lemma, we know that $\tilde{V}$ is obtained from a Hirzebruch surface Σ_d by a succession of blowing-ups $\nu : \tilde{V} \to \Sigma_d$ at $x_1, \ldots, x_s$ on Σ_d. Put $H = \nu_* \tilde{H}$ and let r_i denote the multiplicity of H at x_i. We remark that, since $\mu : \tilde{V} \to V$ is the minimal resolution, there are no (-1)-curves E with $\tilde{H} E = 0$.

LEMMA 4.4. *Σ_d can be chosen so that all the* $r_i = 1$.

Proof. The following argument is suggested by the referee.

Let C be a general member of $|\tilde{H}|$. If $h^1(\tilde{H}|_C) > 0$, then we have $2h^0(\tilde{H}|_C) \leq \tilde{H}^2 + 2$ by Clifford's theorem, which contradicts $3\tilde{H}^2 \leq L^2 \leq 4n - 6$, $n = h^0(\tilde{H}|_C)$. Hence $h^1(\tilde{H}|_C) = 0$, $n = \tilde{H}^2 + 1 - g(C)$ and $\tilde{H}^2 \geq 4g(C) + 2$. In particular, we have $(2K_{\tilde{V}} + \tilde{H})\tilde{H} = 4g(C) - 4 - \tilde{H}^2 < 0$. Hence $K_{\tilde{V}} + \tilde{H}/2$ is not nef. If $\tilde{V}$ is not a Hirzebruch surface, Mori's cone theorem implies that there is a (-1)-curve E_1 with $(K_{\tilde{V}} + \tilde{H}/2)E_1 < 0$, i.e., $\tilde{H} E_1 \leq 1$. Then, by what we remarked above, $\tilde{H}E_1 = 1$. Let $\nu_1 : \tilde{V} \to V_1$ denote the contraction of E_1, and put $H_1 = (\nu_1)_* \tilde{H}$. On (V_1, H_1), we have $H_1^2 = \tilde{H}^2 + 1$ and $g(C_1) = g(C)$ for a general member C_1 of $|H_1|$, since $\tilde{H}E_1 = 1$. Hence, as above, we can find a (-1)-curve E_2 such that $H_1E_2 \leq 1$, provided that V_1 is not a Hirzebruch surface. If $H_1E_2 = 0$, then $\nu_1(E_1)$ is not on E_2 (since $\tilde{H}E_1 = 1$) and we see that $\nu_1^{-1}(E_2)$ is a (-1)-curve which does not meet $\tilde{H}$, a contradiction. Hence $H_1E_2 = 1$. Contracting E_2, we get another model (V_2, H_2) with $H_2^2 = H_1^2 + 1$ and $g(C_2) = g(C_1)$. In this way, we get a sequence of pairs (V_j, H_j) and (-1)-curves E_{j+1} with $H_jE_{j+1} = 1$ until we arrive at (Σ_d, H). Hence we can assume that all the $r_i = 1$. Q.E.D.

LEMMA 4.5 (cf. [16] and [18]). *If either* $\deg V < \frac{4}{3}(n-2)$ *or* $3 \leq n \leq 5$, *then* V *is ruled by straight lines. If* V *is not ruled by straight lines, then the possible* $(\tilde{V}, \tilde{H})$ *is one of the following*:

(1) $\deg V = \frac{4}{3}(n-2)$: $(\tilde{V}, \tilde{H}) = (\Sigma_d, 2\Delta_0 + \beta\Gamma)$, $n = 3(\beta - d) + 2$.

(2) $\deg V = (4n-7)/3$:

(2a) $\tilde{V}$ *is* Σ_d *blown up at a point* x *and* $\tilde{H}$ *is* $\nu^*(2\Delta_0 + \beta\Gamma) - \mathscr{E}$, *where* $\mathscr{E} = \nu^{-1}(x)$, $n = 3(\beta - d) + 1$, *or*

(2b) $(\tilde{V}, \tilde{H}) = (\Sigma_1, 3\Delta_0 + 4\Gamma)$, $n = 13$.

(3) $\deg V = (4n-6)/3$:

(3a) $\tilde{V}$ *is* Σ_d *blown up at two points* x_1, x_2 *which are possibly infinitely near, and* $\tilde{H} = \nu^*(2\Delta_0 + \beta\Gamma) - \mathscr{E}$, *where* $\mathscr{E} = \nu^{-1}(x_1) + \nu^{-1}(x_2)$, $n = 3(\beta - d)$, *or*

(3b) V *is the image of a quadric surface in* $\mathbf{P}^3$ *by* $|\mathscr{O}(3)|$, $n = 15$, *or*

(3c) $\tilde{V}$ *is* Σ_1 *blown up at a point* x, *and* $\tilde{H} = \nu^*(3\Delta_0 + 4\Gamma) - \mathscr{E}$, *where* $\mathscr{E} = \nu^{-1}(x)$, $n = 12$.

In (1), (2a) *and* (3a), β *is an integer satisfying* $\beta \geq \max(2d, d+2)$.

Proof. Since $3(n-1) \leq L^2 \leq 4n - 6$ and since L^2 is even, we have $n \geq 3$ and $\deg V = n - 1$ for $3 \leq n \leq 5$. Therefore, if $n \leq 5$, V is ruled by lines as is well-known.

Assume that H is linearly equivalent to $\alpha\Delta_0 + \beta\Gamma$ on Σ_d. Since $|H|$ contains an irreducible member and since it induces a birational map, we can assume that $\beta \geq d\alpha > 0$ if $d > 0$, and $\beta \geq \alpha > 0$ if $d = 0$. Note that, when $d = 1$, we can further assume $\beta > \alpha$, since, if $\beta = \alpha$, then $\Delta_0 H = 0$ and Δ_0 induces a (-1)-curve on $\tilde{V}$ which does not meet $\tilde{H}$. We have $n = \chi(\tilde{H}) - 1 = \chi(H) - 1 - \sum r_i(r_i + 1)/2$ and $\deg V = \tilde{H}^2 = H^2 - \sum r_i^2$. Since we can assume that all the

$r_t = 1$ by Lemma 4.4, we have

$$n = (\alpha+1)(\beta+1-d\alpha/2)-1-s, \ \deg V = 2\alpha(\beta - d\alpha/2) - s,$$

where s is the number of the blowing-ups appearing in $\nu : \tilde{V} \to \Sigma_d$. Hence

$$3 \deg V = 4n - 8 + (\alpha - 2)(2\beta - d\alpha - 4) + s.$$

In particular, we get $3 \deg V \geq 4n - 8$ when $\alpha \geq 2$. Now, the assertion follows from an easy calculation. Q.E.D.

As a consequence, we get the following:

PROPOSITION 4.6. *The semi-canonical image V of a surface of type* (II) *is ruled by rational curves of degree at most* 3. *Therefore, surfaces of type* (II) *can be further classified into the following three types.*

(IIa) *V is ruled by straight lines.*

(IIb) *V is ruled by rational curves of degree* 2.

(IIc) *V is ruled by rational curves of degree* 3.

§5. Surfaces of type (IIa)

In this section, we study surfaces of type (IIa) and show the following:

THEOREM 5.1. *Let S be a surface of type* (IIa). *Then the ruling of V induces on S a pencil $|D|$ of trigonal curves of genus g without base points, $4 \leq g \leq 6$. Furthermore,*

$$L^2 \geq \frac{g-1}{2(g-2)}(gn - 3g + 8). \tag{5.1}$$

Proof. We first show the inequality (5.1). For $g = 4$, (5.1) clearly holds, since we always have $L^2 \geq 3(n-1)$. In order to show (5.1) for $g \geq 5$, we need some preparations.

Let Λ be an irreducible pencil of curves on S such that Φ_L maps it onto a pencil of straight lines on V. Let $\rho : \hat{S} \to S$ be the blowing-up of the base points of Λ, and let $\hat{\Lambda}$ denote the strict transform of Λ. Then $\hat{\Lambda}$ induces a surjective holomorphic map $\lambda : \hat{S} \to \mathbf{P}^1$. We let $\hat{D}$ denote a general fiber of λ, and put $g = g(\hat{D})$. We denote by $\hat{K}$ the canonical bundle of $\hat{S}$, and let E be an exceptional divisor such that $\hat{K} = \rho^*K + [E]$. We have $\hat{D}E = \sum m_i$, where the m_i are the multiplicity of the base points appearing in ρ. Note that we have $D^2 = \sum m_i^2$ for a member D of Λ.

Since $\hat{K}^2 \leq K^2 = 4L^2$, (5.1) is a consequence of the following:

LEMMA 5.2. *If $g \geq 5$, then*

$$\hat{K}^2 \geq \frac{2(g-1)}{g-2}(gn - 3g + 8). \tag{5.2}$$

Proof. This is a mimic of [18, §4]. Put $|\rho^*L| = |\hat{M}| + Z$ and $\hat{Z} = 2Z + E$. Then we have $\hat{K} = 2\hat{M} + \hat{Z}$. Since $|\hat{M}|$ induces a rational map of degree 3 onto V, and since it maps $\hat{D}$ onto a straight line, we have $\hat{M}\hat{D} = 3$. We have $h^0(\hat{D}, \mathcal{O}_{\hat{D}}(\hat{M})) \leq 2$ by Clifford's theorem. Therefore, the restriction map $H^0(\hat{M}) \to H^0(\hat{M}|_{\hat{D}})$ is surjective, since $\Phi_{\hat{M}}$ maps $\hat{D}$ onto a line. Let n_1 be the greatest integer such that the restriction map $H^0(\hat{M} - n_1\hat{D}) \to H^0(\hat{M}|_{\hat{D}})$ is surjective. Then $n_1 \geq 0$ as we saw above. Let n_2 be the greatest integer such that $H^0(\hat{M} - n_2\hat{D}) \neq 0$. It is easy to see that we have $n_1 + n_2 = n - 1$. Furthermore, we can find $x_1 \in H^0(\hat{M} - n_1\hat{D})$ and $x_2 \in H^0(\hat{M} - n_2\hat{D})$ such that the pair (x_1, x_2) defines a rational map $h : \hat{S} \to \Sigma_{n_2 - n_1}$ of degree 3. Let t_0, t_1 be a basis for $H^0(\hat{D})$. Then the products $t_0^i t_1^{n_1 - i} x_1$, $t_0^j t_1^{n_2 - j} x_2 (0 \leq i \leq n_1, 0 \leq j \leq n_2)$ form a basis for $H^0(\hat{M})$. Since $|\hat{D}|$ is free from base points and since $|\hat{M}|$ has at most base points, we can assume that the divisors (x_1), (x_2) have no commom components. Then it is easy to see that $L_1 = [\hat{M} - n_1\hat{D}]$ is nef, since $\hat{M} - n_1\hat{D} \sim (x_1) \sim (x_2) + (n_2 - n_1)\hat{D}$. We also put $L_2 = [\hat{M} - n_2\hat{D}]$. Note that $K_{\hat{S}/\mathbf{P}^1} = [2\hat{M} + 2\hat{D} + \hat{Z}]$ is nef by Arakelov's theorem (cf. [3]).

Put $\delta = \hat{Z}\hat{D}$. Since $2g - 2 = \hat{K}\hat{D} = 6 + \delta$, we have $\delta = 2g - 8$. By the assumption $g \geq 5$, we have $\delta > 0$. We can show that $\delta K_{\hat{S}/\mathbf{P}^1} + 2L_1 + \hat{Z}$ is nef as in [18, Corollary 1]. Therefore,

$$\begin{aligned} 0 &\leq (\delta K_{\hat{S}/\mathbf{P}^1} + 2L_1 + \hat{Z})(\hat{K} - 2n_2\hat{D}) \\ &= (\delta + 1)\hat{K}^2 - 4(g-1)(n_1 + (\delta + 1)n_2 - \delta). \end{aligned}$$

From this, we get

$$\hat{K}^2 \geq \frac{4(g-1)}{2g-7}(n_1 + (2g-7)n_2 - 2g + 8). \tag{5.3}$$

On the other hand, since L_1 is nef, we have

$$
(5.4)\qquad
\begin{aligned}
\hat{K}^2 &= \hat{K}(2L_1 + 2n_1\hat{D} + \hat{Z})\\
&= 2(2L_2 + \hat{Z} + 2n_2\hat{D})(L_1 + n_1\hat{D}) + \hat{K}\hat{Z}\\
&= 2(2L_2 + \hat{Z})\, L_1 + 4(g-1)n_1 + 12n_2 + \hat{K}\hat{Z}\\
&\geq 4(g-1)n_1 + 12n_2 + \hat{K}\hat{Z}.
\end{aligned}
$$

Since $\delta K_{\hat{S}/\mathbf{P}^1} + 2L_1 + \hat{Z}$ is nef, we have

$$\hat{K}\hat{Z} \geq \frac{2\delta}{\delta+1}(n_1 + 1) - 2\delta$$

as in [18, Corollary 2]. Therefore, we get

$$(5.5)\qquad \hat{K}^2 \geq \frac{4}{2g-7}\left((2g^2 - 8g + 3)n_1 + 3(2g-7)n_2 - 2(g-4)^2\right).$$

Since $n_1 + n_2 = n - 1$, (5.2) is derived from (5.3) when $(2g-1)n_1 - (2g-7)n_2 + 6 \leq 0$, and otherwise, it is derived from (5.5). Therefore, the inequality (5.2) has been established. Q.E.D.

We continue the proof of Theorem 5.1. Assume that $L^2 \leq 4n - 8$. Since $L^2 \geq 3(n-1)$, we have $n \geq 5$ and, by (5.1), we get $4 \leq g \leq 6$. We show that $|D|$ is free from base points. Recall that we have $\delta = \hat{Z}\hat{D} = (2Z + E)\hat{D}$ and $\delta = 2g - 8$. Assume that $|D|$ has a base point. Then $E\hat{D}$ is positive. Since δ is even, $E\hat{D}$ is an even integer. Note that we have $LD = (\hat{M} + Z)\hat{D} = 3 + Z\hat{D}$. Therefore, we have $(g, LD, E\hat{D}) = (5, 3, 2)$, $(6, 4, 2)$ or $(6, 3, 4)$. On the other hand. Hodge's index theorem shows $(LD)^2 \geq L^2D^2$. Since $D^2 \geq E\hat{D}$, we get $(LD)^2 \geq L^2(E\hat{D})$. Therefore, we in particular have $L^2 \leq 8$. This contradicts $L^2 \geq 3n - 3$, since $n \geq 5$. Thus $|D|$ has no base points.

As for $L^2 = 4n - 6$, we need the following:

LEMMA 5.3. *Let S be a surface of type* (IIa) *with $L^2 = 4n - 6$, and assume that $n = 3$ or 4. Then S has a pencil of non-hyperelliptic curves of genus 4 which is free from base points.*

Proof. When $n = 3$, we have $p_g = 10$ and $K^2 = 24$. Then the assertion follows from [12, §4] or [11]. We assume that $n = 4$. Then $L^2 = 10$ and $\deg V = 3$. Therefore, $|L|$ has one base point P. Let $\sigma: \tilde{S} \to S$ be the blowing-up at P, and put $E = \sigma^{-1}(P)$. We let $|M|$ denote the variable part of $|\sigma^*L|$. Then $M^2 = 9$ and $ME = 1$. M induces a holomorphic map $f: \tilde{S} \to V$. As is well-known, V is either Σ_1 embedded by $|\Delta_0 + 2\Gamma|$ or a cone over rational cubic curve. In the latter case, we can lift f to a holomorphic map of $\tilde{S}$ onto Σ_3. This can be seen as follows. Let G denote the fixed part of the linear system coming from that of hyperplanes through the vertex of V. Then we can find an irreducible pencil $|\tilde{D}|$ such that $M = [3\tilde{D} + G]$ ([9, Lemma 1.5]). We have $MG = 0$ and it follows $9 = M^2 = 3M\tilde{D} = 9\tilde{D}^2 + 3\tilde{D}G$. Then $\tilde{D}^2 = 0$ or 1. If $\tilde{D}^2 = 1$, then $\tilde{D}G = 0$ and $G^2 = 0$. By Hodge's index theorem, we have $G = 0$. Then $1 = ME = 3\tilde{D}E$, which is absurd. Therefore, the inverse image of the vertex is G. Since Σ_3 is obtained from V by blowing up the vertex, we can lift f to a holomorphic map onto Σ_3. Therefore, in either case, we have a holomorphic map $h : \tilde{S} \rightarrow \Sigma_d$, where $d = 1$, 3, such that $M = h^*(\Delta_0 + ((d+3)/2)\Gamma)$. We put $D = \sigma_* h^* \Gamma$. Since E is an irreducible curve with $ME = 1$, $h_*(E)$ is either a fiber Γ, or Δ_0, $d = 1$. If $h_*(E)$ is a fiber, then $Eh^*\Gamma = 0$. It follows that $|D|$ is a pencil of curves of genus 4 without base points. If $h_*(E) = \Delta_0$, then $Eh^*\Gamma = 1$. It would follow $D^2 = 1$, which is absurd since S is an even surface. Q.E.D.

We complete the proof of Theorem 5.1 for $L^2 = 4n - 6$. For this purpose, we freely use the notation in the proof of Lemma 5.2. We can assume that $n \geq 5$ by virtue of Lemma 5.3. Since $L^2 = 4n - 6$, it follows from (5.1) that $4 \leq g \leq 8$. We remark that $g = 8$ occurs only when $n = 5$, $K^2 = \hat{K}^2$ and the equality holds in (5.2). If $|D|$ has a base point, then $K^2 > \hat{K}^2$ and, therefore, we have $g \leq 7$. But then, similarly as in the case $L^2 \leq 4n - 8$, one can show that $|D|$ has no base points.

We can exclude $g = 7$, 8 as follows: By what we have shown above, we have $\hat{S} = S$ in the proof of Lemma 5.2. Therefore we have $K\hat{Z} \geq 0$, and it follows from (5.4) that $4(4n - 6) = K^2 \geq 4(g - 1)n_1 + 12n_2$. Consider the case $g = 7$. Then this gives $3n_2 \geq 2n$. On the other hand, it follows from (5.3) that $11n \geq 18n_2$. Hence $g = 7$ cannot happen. Quite similarly, we can exclude the case $g = 8$. Therefore, we have $4 \leq g \leq 6$.

Q.E.D. of Theorem 5.1.

§6. Surfaces of type (IIb)

By Lemma 4.5, we have $L^2 = 4n - 6$ or $4n - 8$ if V is not ruled by lines. In the course of the study, we sometimes need to lift a map into a cone to its nonsingular model by applying a method due to Horikawa [9].

LEMMA 6.1 *Let S be a surface of type* (IIb) *with $L^2 = 4n - 8$. Then S has a pencil of trigonal curves of genus 7, which is free from base points.*

Proof. It follows from Lemma 4.5 that V is isomorphic to either Σ_d embedded into $\mathbf{P}^n$ by $|2\Delta_0+\beta\Gamma|$, where $\beta>2d$ and $n=3(\beta-d)+2$, or the Veronese transform of a cone over a rational normal curve. In any cases, we get $\deg V=(4/3)(n-2)$. Since $L^2=3\deg V$, $|L|$ is free from base points.

Assume that V is isomorphic to Σ_d. Then L comes from $[2\Delta_0+\beta\Gamma]$. It follows that the pencil $|D|$ induced from $|\Gamma|$ is a pencil of trigonal curves of genus 7. Note that d must be even by Lemma 4.1.

Assume that V is the Veronese transform of a cone. Then there exists a line bundle L_0 such that $L=2L_0$ and L_0 induces a holomorphic map of degree three onto the cone over a rational normal curve of degree $m-1$ in $\mathbf{P}^{m-1}$, where $n=3m-1$. Consider the pull-back by Φ_{L_0} of the linear system of hyperplanes through the vertex of the cone, and let G be the fixed part. We have an irreducible pencil $|D|$ such that $L_0=[(m-1)D+G]$ ([9, Lemma 1.5]). Since $|L_0|$ is free from base points, we have $L_0G=0$. Then $3m-3=(L_0)^2=(m-1)L_0D$, i.e., $L_0D=3$. Since $(L_0)^2$ must be a positive even integer, m is an odd integer not less than 3. We have $3=L_0D=(m-1)D^2+DG\geq(m-1)D^2$. It follows $D^2=0$, since S is an even surface. Therefore, D is of genus 7. Note that, since $G\neq 0$ and $D^2=0$, the holomorphic map of S onto the cone induced by L_0 can be lifted to a holomorphic map of S onto Σ_{m-1}. Q.E.D.

We assume that $L^2=4n-6$ in the following. Let $\sigma:\tilde{S}\to S$ be, as before, a composite of blowing-ups such that the variable part $|M|$ of $|\sigma^*L|$ is free from base points. We denote by Z the fixed part of $|\sigma^*L|$. Since S is of type (IIb), we have $M^2\geq 4n-8$ by Lemma 4.5.

LEMMA 6.2. *Let S be a surface of type* (IIb) *with* $L^2=4n-6$. *Assume that* $\deg V=(4n-8)/3$. *Then one of the following occurs.*

(1) *$|L|$ has no fixed component and σ is a composite of two blowing-ups.*

(2) *σ is the identity map, $MZ=LZ=1$ and $Z^2=0$.*

(3) *σ is the identity map, $MZ=2$, $LZ=0$ and $Z^2=-2$.*

Proof. Since $4n-6=(\sigma^*L)^2=(\sigma^*L)M+(\sigma^*L)Z=M^2+MZ+(\sigma^*L)Z$, we have $MZ+(\sigma^*L)Z=2$. Let Z_0 be the fixed part of $|L|$. Then we can find an exceptional divisor E such that $Z=\sigma^*Z_0+E$, and we have $ME=\sum m_i^2$, where the m_i are the multiplicity of the base points appearing in σ. From $MZ+(\sigma^*L)Z=2$, it follows $M\sigma^*Z_0+\sum m_i^2+LZ_0=2$.

Assume first that $|Z|$ has no fixed components. Then Z consists of exceptional curves, and we have $(\sigma^*L)\, Z = 0$. We have $2 = MZ = \sum m_i^2$. Therefore, σ is a composite of two blowing-ups and we have (1).

Assume that $|L|$ has a fixed component. We claim that σ is the identity map. If $ME = 2$, then $M\sigma^* Z_0 = LZ_0 = 0$. Since $LZ_0 = (\sigma^*L)\sigma^*Z_0 = M\sigma^*Z_0 + Z_0^2$, we have $Z_0^2 = 0$. By Hodge's index theorem, we get $Z_0 = 0$. This contradicts the assumption that $|L|$ has a fixed component. If $ME = 1$, then $M\sigma^*Z_0 + LZ_0 = 1$ which is equivalent to $2M\sigma^*Z_0 + Z_0^2 = 1$. This implies that Z_0^2 is odd. Since S is an even surface, this is absurd. Hence, we have $ME = 0$. This shows that σ is the identity map, and we have $Z_0 = Z$.

We have $MZ + LZ = 2$. If $MZ = 0$ and $LZ = 2$, then we have $Z^2 = 2$, which contradicts Hodge's index theorem. If $MZ = LZ = 1$, then $Z^2 = 0$ and we are in (2). If $MZ = 2$ and $LZ = 0$, then $Z^2 = -2$ and we are in (3). Q.E.D.

LEMMA 6.3. (1) *and* (2) *of Lemma* 6.2 *cannot occur.*

Proof. We first show that (1) of Lemma 6.2 cannot happen. Assume contrarily that (1) is the case. Let P_1 and P_2 be two base points of $|L|$ which may be infinitely near. Put $E_i = \sigma^{-1}(P_i)$. Then $Z = E_1 + E_2$. We have $MZ = 2$ and $Z^2 = -2$.

Assume that V is isomorphic to $\sum_d$. Then $M = f^*[2\Delta_0 + \beta\Gamma]$. Since $MZ = 2$, we have either (a) $f_*Z \sim \Gamma$, or (b) $f_*Z \sim \Delta_0$, $\beta = 2d + 2$, or (c) $f_*Z = 2\Delta_0$, $\beta = 2d + 1$. Put $\tilde{D} = f^*\Gamma$ and $D = \sigma_*\tilde{D}$. If (a) is the case, then $\tilde{D}Z = 0$. Put $\beta_0 = [\beta/2]$ and $M_0 = f^*[\Delta_0 + \beta_0\Gamma]$. Then M_0 is nef and $M_0 Z = 1$. Hence we can assume that $M_0E_1 = 1$ and $M_0E_2 = 0$. Since $M = 2M_0$ or $2M_0 + \tilde{D}$, we have $ME_2 = 0$. This is a contradiction, since E_2 must contain a (-1)-curve. If (b) is the case, then we have $\tilde{D}Z = 1$. It follows that $D^2 = 1$, which contradicts that S is an even surface. If (c) is the case, then we have $D^2 \geq 2$ since $\tilde{D}Z = 2$. We have $LD = (\sigma^*L)\tilde{D} = 6 + 2 = 8$. Since $(LD)^2 \geq L^2D^2$ by Hodge's index theorem, we get $n \leq 9$. On the other hand, we have $n = 3(\beta - d) + 2 = 3d + 5$ and $d > 0$. Therefore, we have $n = 8$, $d = 1$. This implies that V is isomorphic to $\sum_1$ embedded by $|2\Delta_0 + 3\Gamma|$. We have an irreducible curve C on V which is linearly equivalent to $\Delta_0 + \Gamma$. Then $C^2 = 1$ and $Zf^*C = 0$. Therefore, we get $(\sigma_*f^*C)^2 = 3$, which contradicts that S is an even surface.

When V is the Veronese transform of a cone over a rational normal curve of degree d, we can find a line bundle M_0 such that $M = 2M_0$ and $|M_0|$ induces a holomorphic map onto the cone. We can write $M_0 = [d\tilde{D} + G]$ with an irreducible pencil $|\tilde{D}|$ and an effective divisor G which is the divisorial part of the inverse image of the vertex. Note that we have $M_0G = 0$ since $|M_0|$ is free from base points. Since $3 = M_0\tilde{D} = d\tilde{D}^2 + G\tilde{D} \geq d\tilde{D}^2$, we have either (i) $\tilde{D}^2 = 0$, $G\tilde{D} = 3$, or (ii) $\tilde{D}^2 = 1$ ($d = 2$ or 3). If (i) is the case, we can lift $\tilde{S} \to V$ to a holomorphic map of $\tilde{S}$ onto Σ_d, and we can show that this leads us to a contradiction as in the previous case $V = \Sigma_d$. Assume that (ii) is the case. Since $M_0G = 0$ and $(\sigma^*L)G \geq 0$, we have $ZG \geq 0$. We have $1 = M_0Z = d\tilde{D}Z + ZG$. Therefore, we get $\tilde{D}Z = 0$. We put $D = \sigma_*\tilde{D}$. Since $\tilde{D}^2 = 1$, it follows that $D^2 = 1$. This contradicts that S is an even surface. In sum, (1) of Lemma 6.2 has been excluded.

We next show that (2) of Lemma 6.2 is also inadequate. Assume contrarily that (2) is the case. We can assume that $n \geq 6$ by Lemma 4.5. When V is the Veronese transform of a cone, we can lift $f: S \to V$ to a holomorphic map onto Σ_d as in the proof of Lemma 6.1, which we also denote by f. Note that d must be even by Lemma 4.1. Since $M = f^*[2\Delta_0 + \beta\Gamma]$ and $MZ = 1$, we have $\beta = 2d + 1$ and $f^*Z = \Delta_0$. It follows $ZD = 1$, where we put $D = f^*\Gamma$. Therefore, $(M - D)Z = 0$. Since we have $(M - D)^2 = 4(n - 5) > 0$, it follows from Hodge's index theorem that $Z = 0$. This contradicts $ZD = 1$. Q.E.D.

LEMMA 6.4. *Let S be a surface of type* (IIb) *with $L^2 = 4n - 6$. Assume that* $\deg V = (4n - 8)/3$. *Then the ruling of V induces on S a pencil $|D|$ of trigonal curves of genus g, $7 \leq g \leq 8$, without base points. Furthermore, $n = 14, 20, 26$ if* $g(D) = 8$.

Proof. By Lemmas 6.2 and 6.3, we have $MZ = 2$, $LZ = 0$ and $Z^2 = -2$. Since $KZ = 2LZ = 0$, we see that Z consists of (-2)-curves.

When V is the Veronese transform of a cone, we can lift $f: S \to V$ to a holomorphic map of S onto Σ_d as in the proof of Lemma 6.1. Therefore, we assume that $|M|$ induces a holomorphic map $f: S \to \Sigma_d$ of degree 3 such that $M = f^*[2\Delta_0 + \beta\Gamma]$. Note that d must be even by Lemma 4.1. We put $\beta_0 = [\beta/2]$ and $M_0 = f^*[\Delta_0 + \beta_0\Gamma]$. Then $M = 2M_0$ or $2M_0 + D$, where $D = f^*\Gamma$.

Since M_0 is nef and $MZ = 2$, we have either (i) $M_0Z = 1$, or (ii) $M_0Z = 0$ and $DZ = 2$. When (i) is the case, Z is a (-2)-curve. To see this, let Z_1 be an irreducible component of Z with $M_0Z_1 > 0$. Since $M_0Z = 1$, we have $M_0Z_1 = 1$ and $M_0Z_2 = 0$, where $Z_2 = Z - Z_1$. We have $0 = LZ_1 = MZ_1 + Z_1^2 + Z_1Z_2$. It follows $Z_1Z_2 = 0$. Since $Z_1^2 = Z^2 = -2$, we get $Z_2^2 = 0$. Therefore, Hodge's index theorem shows $Z_2 = 0$. Hence $Z = Z_1$. Since $M_0Z = 1$, we have either $f_*Z \sim \Gamma$ or $f_*Z = \Delta_0$ ($\beta = 2d + 2$). If $f_*Z \sim \Gamma$, then we have $g(D) = 7$, whereas we have $g(D) = 8$ when $f_*Z \sim \Delta_0$. We show that $n \leq 26$ when $f_*Z \sim \Delta_0$. Assume contrarily that $n > 26$. Since $n = 3(\beta - d) + 2 = 3d + 8$ and since d is even, we have $d \geq 8$. Then $L^2 \geq 122$ and it follows that $(L - 7D)^2 > 0$. We have $(L - 7D)(D + Z) = 0$. Since $(D + Z)^2 = 0$, Hodge's index theorem shows that $D + Z = 0$, which is absurd. Hence, $n \leq 26$ and we have $d = 0, 2, 4$ or 6. Note that, when $d = 0$, we can assume $g(D) = 7$ by considering another ruling of Σ_0.

We show that (ii) cannot happen. Assume contrarily that (ii) is the case. Then, we have $f_*Z = 2\Delta_0$ and $\beta = 2d + 1$. Since $\beta \geq d + 2$, d is positive. Recall that we have $n = 3(\beta - d) + 2 = 3d + 5$ and $L^2 = 4n - 6$. Therefore, we get $L^2 \geq 38$ and it follows that $(L - 2D)^2 > 0$. Since $DZ = 2$ and $Z^2 = -2$, we have $(D + 2Z)^2 = 0$. Since $(L - 2D)(D + 2Z) = 0$, it follows from Hodge's index theorem that $D + 2Z = 0$, which is absurd. Q.E.D.

LEMMA 6.5. *Let S be a surface of type* (IIb) *with $L^2 = 4n - 6$. Assume that $\deg V = (4n - 7)/3$ or $(4n - 6)/3$. Then the ruling of V induces on S a pencil $|D|$ of trigonal curves of genus 7 without base points, except when $n = 6, 7$ and $\deg V = n$. In the exceptional cases, $D^2 = 2$ and D is of genus 8 $(n = 6)$ or 9 $(n = 7)$.*

Proof. We separately consider the cases $3 \deg V = 4n - 7$ and $3 \deg V = 4n - 6$.

(a) $3 \deg V = 4n - 7$.

Since $L^2 - 3 \deg V = 1$, $|L|$ has one base point P. Let $\sigma : \tilde{S} \to S$ be the blowing-up at P, and put $E = \sigma^{-1}(P)$. We let $|\tilde{D}|$ denote the irreducible pencil on $\tilde{S}$ which is mapped onto the pencil of conics on V, and put $D = \sigma_*\tilde{D}$. If we put $m = \tilde{D}E$, then we have $LD = 6 + m$ and $D^2 = \tilde{D}^2 + m^2$. When $D^2 = 0$, $|D|$ is a pencil of trigonal curves of genus 7. We investigate the case $D^2 > 0$. Suppose first that $m > 0$. By Hodge's index theorem, we have $(LD)^2 \geq L^2D^2$. Therefore.

$$(6 + m)^2 \geq (4n - 6)D^2 \geq (4n - 6)m^2.$$

It follows that $4n - 6 \leq (6/m + 1)^2$. Recall that we have $n \geq 6$ by Lemma 4.5. Since $4n - 7 = 3 \deg V$, we have $n \geq 7$. Therefore, we get $m = 1$, $n = 7, 10$ or 13. We have $D^2 = \tilde{D}^2 + 1$. Since D^2 is an even integer, $\tilde{D}^2$ is a positive odd integer. Hence $D^2 \geq 2$, and we have $49 \geq (4n - 6) \times 2$. Therefore, we get $n = 7$, $LD = 7$ and $D^2 = 2$. It follows that any general member of $|D|$ is a nonsingular curve of genus 9.

Suppose that $m = 0$ and $\tilde{D}^2 > 0$. Then D^2 is positive. Since D^2 must be even, we have $36 \geq (4n - 6)D^2 \geq 2(4n - 6)$. Since $n \geq 7$, this is absurd.

(b) $3 \deg V = 4n - 6$.

Since $L^2 = 3 \deg V$, $|L|$ is free from base points. Let $|D|$ be the pencil induced by a ruling of V. Then $LD = 6$. Since D^2 is a nonnegative even integer, we can put $D^2 = 2m$. If $m = 0$, then $|D|$ is a pencil of trigonal curves of genus 7. Assume that $m > 0$. By Lemma 4.5, we have $n \geq 6$. Since $36 = (LD)^2 \geq L^2D^2 = 2m(4n - 6)$, we get $(n, m) = (6, 1)$. In this case, any general member of $|D|$ is a nonsingular curve of genus 8, since $D^2 = 2$ and $LD = 6$. Q.E.D.

§7. Surfaces of type (IIc)

LEMMA 7.1. *Let S be a surface of type* (IIc). *Then the semi-canonical image is the image of a quadric surface in $\mathbf{P}^3$ by the holomorphic map associated with $\mathcal{O}(3)$.*

Proof. A description of $\tilde{V}$ can be found in Lemma 4.5. We show that the cases (2b) and (3c) are inadequate by using the notation there.

Assume that (2b) is the case. Since $L^2 = 3 \deg V + 1$, we see that $|L|$ has one base point P. Let $\sigma : \tilde{S} \to S$ be the blowing-up at P, and put $E = \sigma^{-1}(P)$. Let $|M|$ denote the variable part of $|\sigma^* L|$. Then $M^2 = 45$ and $ME = 1$. We let $f : \tilde{S} \to V = \Sigma_1$ denote the natural holomorphic map. Then $M = f^*(3\Delta_0 + 4\Gamma)$. Since $ME = 1$, we have $f_* E = \Delta_0$. Therefore, we have $Ef^*\Gamma = 1$. Put $D = \sigma_* f^* \Gamma$. Then we have $D^2 = 1$, which is absurd since S is an even surface.

Assume that (3c) is the case. In this case, $|L|$ is free from base points, because $L^2 = 3 \deg V$. Let $\tilde{\Gamma}$ denote the proper transform of the fiber of Σ_1 through the point x. Then it is a (-1)-curve, and it gives a Cartier divisor l on V with $l^2 = -1$. This is because $|\tilde{H}|$ induces a biholomorphic map in a neighbourhood of $\tilde{\Gamma}$. Hence Lemma 4.1 implies that (3c) is inadequate. Q.E.D.

THEOREM 7.2. *Let S be a surface of type* (IIc). *Then S has a pencil of trigonal curves of genus* 10 *without base points, and the canonical model is defined in the weighted projective space* $\mathbf{P}(1, 1, 1, 1, 4)$ *by*

$$A_2 = 0, \quad u^3 + B_8 u + B_{12} = 0, \tag{7.1}$$

where (x_0, x_1, x_2, x_3, u) is a system of coordinates with $\deg x_i = 1$, $\deg u = 4$, and the A_k are homogeneous forms of degree k in the x_i.

Proof. V is as in (3b) of Lemma 4.5. Then $|L|$ is free from base points, since $L^2 = 3 \deg V$. In this case, there exists a line bundle L_0 on S which satisfies $L = 3L_0$ and induces a holomorphic map of degree 3 onto a quadric surface in $\mathbf{P}^3$. Let

(x_0, x_1, x_2, x_3) be a system of homogeneous coordinates on $\mathbf{P}^3$, which is identified with a basis for $H^0(L_0)$. We assume that the preimage V_0 of V is defined by a quadric equation $A_2 = 0$ in the x_i.

We claim that $h^0(4L_0) = 26$. By the Riemann-Roch theorem, we have $\chi(4L_0) = 35$. Furthermore, we have $h^2(4L_0) = h^0(2L_0) = h^0(V_0, \mathcal{O}(2)) = 9$. Therefore, it is sufficient to show that $h^1(2L_0) = 0$. For this purpose, we choose a general member C of $|L_0|$. Then it is of genus 22. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}(2L_0) \to \mathcal{O}(3L_0) \to \mathcal{O}_C(3L_0|_C) \to 0.$$

Recall that, since $3L_0 = L$, we have $h^0(3L_0) = 16$ and $h^1(3L_0) = 0$ by the proof of Lemma 1.2. Hence it suffices for our purpose to prove the following:

LEMMA 7.3. $h^0(C, \mathcal{O}(3L_0|_C)) = 7$.

Proof. Since $H^1(\mathcal{O}_S) = 0$ and $h^0(L_0) = 4$, $h^0(L_0|_C) = 3$ and we see that $|L_0|_C|$ induces a holomorphic map of degree 3 onto an irreducible conic curve. Hence, we can find a line bundle M_C which satisfies $L_0|_C = 2M_C$ and which induces a holomorphic map of degree 3 onto $\mathbf{P}^1$. Note that we have $K_C = 14M_C$ and $h^0(iM_C) \geq i + 1$ for any $i \geq 0$.

We have $h^0(2M_C) = 3$ and, by the Riemann-Roch theorem, $h^0(12M_C) = 18$. We have $h^0(3M_C) - h^0(11M_C) = -12$ and $h^0(3M_C) \geq 4$. Since C is nonhyperelliptic, it follows from Clifford's theorem that $h^0(4M_C) \leq 6$. Hence we have $(h^0(4M_C), h^0(10M_C)) = (5, 14)$ or $(6, 15)$. Similarly, we get $6 \leq h^0(5M_C) \leq 8$ and $7 \leq h^0(6M_C) \leq 9$. By the base-point-free pencil trick, the following sequence is exact for any $m > 0$:

(7.2) $$0 \to H^0((m-1)M_C) \to H^0(mM_C) \otimes H^0(M_C) \to H^0((m+1)M_C).$$

From (7.2) with $m = 11$, we see $h^0(11M_C) \leq 16$. Hence $h^0(3M_C) = 4$ and $h^0(11M_C) = 16$. Assume that $h^0(4M_C) = 6$ and consider (7.2) for $m = 4$. It shows $h^0(5M_C) \geq 8$ and, therefore, $h^0(5M_C) = 8$. But then, we would have $h^0(6M_C) \geq 10$ from (7.2) with $m = 5$, which contradicts $h^0(6M_C) \leq 9$. Hence we get $h^0(4M_C) = 5$. Then, since $h^0(6M_C) \leq 9$, (7.2) with $m = 5$ gives $h^0(5M_C) \leq 7$. Assume that $h^0(5M_C) = 7$. Then $h^0(6M_C) = 9$, and (7.2) for $m = 6$ gives $h^0(7M_C) \geq 11$. By (7.2) with $m = 7$, we get $h^0(8M_C) \geq 13$, which contradicts Clifford's theorem since C is nonhyperelliptic. Therefore, $h^0(5M_C) = 6$. Quite similarly, we get a contradiction if we assume that $h^0(6M_C) > 7$, by considering (7.2) for $m = 6, 7$. Hence $h^0(6M_C) = 7$. Q.E.D.

In $H^0(4L_0) \simeq \mathbf{C}^{26}$, we have quartic forms in the x_i. Modulo A_2, these present 25 linearly independent elements. Therefore, we have a new element $\eta \in H^0(4L_0)$. We consider $H^0(12L_0) = H^0(2K)$ which is 275 dimensional. Here, we have the following elements

$$\eta^3,\ B_4\eta^2,\ B_8\eta,\ B_{12},$$

where the B_k are homogeneous forms of degree k in the x_i. Modulo A_2, these give 276 section. Therefore, we have a relation of the form

$$B_0\eta^3 + B_4\eta^2 + B_8\eta + B_{12} = 0.$$

It is clear that B_0 is a nonzero constant. Therefore, we can assume that $B_0 = 1$ and $B_4 = 0$ by a suitable linear change of η if necessary. We have shown that we can lift $S \to V_0 \subset \mathbf{P}^3$ to a holomorphic map into $\mathbf{P}(1, 1, 1, 1, 4)$ by putting $u = \eta$. The image is defined by Eq. (7.1). It can be checked that it is nothing but the canonical model of S. Q.E.D. of Theorem 7.2.

In view of a conjecture in [17, (0.0)], it may be interesting to summarize the obtained results in the following form:

THEOREM 7.4. *Let S be a surface of type* (II) *or* (III). *Then it has a pencil $|D|$ of trigonal curves of genus g, $3 \leq g \leq 10$. Furthermore, the numerical characters satisfy*

$$K^2 \geq \frac{24(g-1)}{5g+1}p_g - \frac{8(g-1)(2g+1)}{5g+1} \tag{7.3}$$

except when $g = 5$, or $g = 8$ with $(p_g, K^2) = (78, 296), (102, 392)$. When $g = 5$, they satisfy

$$K^2 \geq \frac{40}{11}p_g - \frac{152}{11}.$$

Proof. We only have to show the inequalities in the statement. When $g = 3$, (7.3) follows from Castelnuovo's inequality $K^2 \geq 3p_g - 7$ or [13, Theorem 1.2]. We assume that $g \geq 4$. Put $h^0(L) = n + 1$ as usual. Then it follows from (1.1) that

$$L^2 = 2p_g - 4n - 2 + 2h^1(L) \geq 2p_g - 4n - 2.$$

If we have an inequality of the form $L^2 \geq an - b$ with $a > 0$, then, since $K = 2L$, we get an inequality of the form

$$K^2 \geq \frac{8a}{a+4}\left(p_g - 1 - \frac{2b}{a}\right)$$

by eliminating n. Note that, when $g = 4$, we have $L^2 \geq 3(n-1)$ by the proof of Lemma 1.4 and (5.1). Hence, for $g < 7$, the desired inequality follows from (5.1). When $g \geq 7$, we get (7.3) from Lemmas 6.1, 6.4, 6.5 and Theorem 7.2. Q.E.D.

§8. Quadrics through the canonical image

Firstly, we work in a more general situation: Let S be a canonical surface, X its canonical image and $Q(X)$ the quadric hull of X. The following is implicitly stated in [17, (0.2) Remark].

LEMMA 8.1. *Let S be a canonical surface with $K^2 \leq 4p_g - 12 + q$, and let W be the irreducible component of $Q(X)$ containing X. Then* $\dim W \leq 3$.

Proof. Put $W = r$. Since W is a non-degenerate variety in $\mathbf{P}^N$, $N = p_g - 1$, we have

$$h^0(\mathbf{P}^N, \mathscr{I}_W(2)) \leq (N+1)(N+2)/2 - \left\{(r+1)(N+1) - \frac{1}{2}r(r+1)\right\},$$

where $\mathscr{I}_W$ denotes the ideal sheaf of W in $\mathbf{P}^N$. It is clear that

$$h^0(2K) \geq \dim \operatorname{Im}\{H^0(\mathbf{P}^N, \mathcal{O}(2)) \to H^0(X, \mathcal{O}_X(2))\}.$$

Since $h^0(2K) = K^2 + \chi(\mathcal{O}_S)$ and since $X \subset W \subset Q(X)$, we have

$$K^2 \geq (r+1)p_g - \frac{1}{2}r(r+1) - (1 - q + p_g).$$

Since $K^2 \leq 4p_g - 12 + q$ by the assumption, we get $(r-4)(2N - r - 3) + 2 \leq 0$. Hence $r \leq 3$. Q.E.D.

The following is a consequence of Enriques-Babbage-Petri's theorem:

LEMMA 8.2. *Let S be a canonical surface. Suppose that S has a pencil $\{D\}$ of non-hyperelliptic curves whose general member is either a trigonal curve or a plane quintic curve. Then the canonical image X can never be an irreducible component of $Q(X)$.*

Proof. We assume that X is an irreducible component of $Q(X)$, and show that this leads us to a contradiction. We can assume that there is a holomorphic map $f: S \to C$ onto a nonsingular projective curve C, by eliminating the base points if necessary. Let $\mathscr{E}$ be the locally free subsheaf of $f_*\mathscr{O}(K)$ generated by $H^0(C, f_*\mathscr{O}(K))$. Put $r = \mathrm{rk}(\mathscr{E})$. Since S is a canonical surface, the image of the restriction map $H^0(K) \to H^0(K_D)$ has dimension at least 3. Hence $r \geq 3$. The sheaf homomorphism $f^*\mathscr{E} \subset f^*f_*\mathscr{O}(K) \to \mathscr{O}(K)$ induces a rational map $h: S \to \mathbf{P}(\mathscr{E})$ over C. Let Φ_T denote the rational map of $\mathbf{P}(\mathscr{E})$ associated with a tautological divisor $T \in |\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)|$. Then, by the construction, the canonical map Φ_K can be identified with the composite $\Phi_T \circ h$. Let Λ denote the linear subsystem of $|2T|$ which comes from hyperquadrics containing X. By the assumption, we see that $\Sigma := h(S)$ is an irreducible component of the intersection of elements in Λ. Note that an element of $|2T|$ can be considered as a relative hyperquadric, that is, it induces a hyperquadric on fibers of $\mathbf{P}(\mathscr{E}) \to C$. Since S is a canonical surface, a general fiber D' of $\Sigma \to C$ is birational to D. Furthermore, we can consider it as a projection of a canonical curve D. Let g denote the genus of D. We consider D as a canonical curve in $\mathbf{P}^{g-1}$ and $D' \subset \mathbf{P}^{r-1}$. We denote by $\Psi: \mathbf{P}^{g-1} \to \mathbf{P}^{r-1}$ the projection which induces the birational map of D onto D'.

CLAIM. *D' cannot be cut out by hyperquadrics.*

Proof. Recall that D is a trigonal curve or a plane quintic curve. If $r = g$, then $D' = D$ and the assertion follows from Enriques-Babbage-Perti's theorem. We assume that $r < g$. We consider the case where D is trigonal. Note that $Q(D)$ is a surface scroll. Since Ψ induces a birational map of D onto D', $\Psi(Q(D))$ is 2-dimensional (for, otherwise, it is a rational curve). It follows that $\Psi(Q(D))$ is also ruled by lines. Since a ruling of $Q(D)$ induces on D a g_3^1, a general line l in the corresponding ruling of $\Psi(Q(D))$ intersects with D' at three points (or more). Since a hyperquadric meets l at two points unless it contains l, we see that any hyperquadric containing D' must contain $\Psi(Q(D))$. Hence D' cannot be cut out by quadrics. The case where D is a plane quintic curve can be treated similarly, since $Q(D)$ is the Veronese surface in this case. Q.E.D. of Claim

It follows that Σ cannot be cut out by elements in Λ contradicting our initial assumption. Q.E.D.

By these lemmas, we get the following:

THEOREM 8.3. *Let S be a canonical surface with $K^2 \leq 4p_g - 12 + q$, and let X be the canonical image. The irreducible component of $Q(X)$ containing X is of dimension* 3, *provided that S has a pencil of trigonal curves or plane quintic curves.*

We now state the following:

THEOREM 8.4. *Let S be a canonical surface with $K^2 < 4\chi(\mathcal{O}_S) - 16$. Assume further that S is an even surface. Then the irreducible component of $Q(X)$ containing X is of dimension* 3. *In particular, Reid's conjecture* [15, p. 541] *is true for regular even canonical surfaces.*

Proof. Assume first that S is a regular surface. For surfaces of type (I), the assertion was shown in Lemma 2.4. As for the other types of surfaces, we can apply Theorem 8.3.

We next assume that S is an irregular surface. Since $K^2 < 4\chi(\mathcal{O}_S) - 16$, we can show that $L^2 \leq 4h^0(L) - 10$ holds as in Lemma 1.2. By Lemmas 2.1 and 2.3, Φ_L cannot be birational onto the image. Hence, as in Lemmas 1.3 and 1.4, we can show that Φ_L induces a rational map of degree 3 onto a ruled surface or it is composed of a pencil of non-hyperelliptic curves of genus not greater than 4. In either case, we see that S has a pencil of trigonal curves. Hence the assertion follows from Theorem 3.8. Q.E.D.

We give some examples of even canonical surfaces.

EXAMPLE 1. The construction here is motivated by [11] and [14]. Let m and k be positive integers with $k \leq 2m$. Let W be the total space of the $\mathbf{P}^2$-bundle

$$\pi : \mathbf{P}(\mathcal{O}(2m-2) \oplus \mathcal{O}(4m-2-k) \oplus \mathcal{O}(10m-2-2k)) \to \mathbf{P}^1.$$

Let T and F denote a tautological divisor and a fiber, respectively. We choose sections X_0, X_1 and X_2 of $[T-(2m-2)F]$, $[T-(4m-2-k)F]$ and $[T-(10m-2-2k)F]$, respectively, such that they form a system of homogeneous coordinates on fibers of W. Let $\alpha_k \in H^0(kF)$, $P \in H^0(3X_0)$ and $Q \in H^0(2X_1)$ be general members. Then the equation

$$Q^2 - \alpha_k^2 X_2 P = 0$$

defines an irreducible hypersurface S' of W which has double curves along k conics defined by $\alpha_k = Q = 0$. We can assume that the other singular points of S' are rational double points. Let S^* be the blow up along double conics. If we introduce a new variable $w = Q/\alpha_k$, then w can be identified with a fiber coordinate on $[2T - (8m - 4 - k)F]$ and S^* is defined by

$$\alpha_k w - Q = 0,\ w^2 - X_2 P = 0$$

in the total space of $[2T - (8m - 4 - k)F]$. It is easy to see that S^* has at most rational double points, and that the dualizing sheaf is induced from $T \sim (X_2) + 2(5m - 1 - k)F$. Let S be the minimal resolution of S^*. In order to see that S is an even surface, it is sufficient to show that (X_2) induces on S a divisor of the form $2Z$, which is immediate (see, [11] or [14]).

By a standard calculation, one can show that the numerical characters of S satisfy $p_g = 16m - 3k - 3$, $q = 0$ and $K^2 = 3p_g - 7 + k$.

EXAMPLE 2. Let m and k be positive integers. Let d_0 be a nonnegative integer and assume that $m + 1 \geq (k + 1)d_0$. We consider the $\mathbf{P}^1$-bundle

$$\varpi : W = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}((k + 1)\,\Delta_0 + ((k + 1)d_0 + m + 1)\Gamma)) \to \Sigma_{2d_0}.$$

Let T denote a tautological divisor, and choose a general member $S \in |3T|$. Note that S does not meet the section T_∞ linearly equivalent to $T - \varpi^*((k + 1)\,\Delta_0 + ((k + 1)d_0 + m + 1)\Gamma)$. Since $-2T + \varpi^*((k - 1)\,\Delta_0 + ((k - 1)d_0 + m - 1)\Gamma)$ is a canonical divisor of W, we see that K is induced from $T_\infty + 2\varpi^*(k\Delta_0 + (kd_0 + m)\Gamma)$. Since $[T_\infty]$ is trivial on S, it follows $K = 2L$, where

$$L = \varpi^*[k\Delta_0 + (kd_0 + m)\Gamma]\,|_S.$$

Therefore, S is an even surface. Furthermore, $H^0(T + \varpi^*((k - 1)\Delta_0 + ((k - 1)\,d_0 + m - 1)\Gamma)$ is restricted to $H^0(K)$ isomorphically. It follows that S is a canonical surface. Let $|D|$ be a pencil on S induced by $|\Gamma|$. Then it is a pencil of trigonal curves of genus $3k + 1$.

By a standard calculation, we have $p_g = (5k + 2)\,m + 2k + 1$, $q = 0$ and $K^2 = (24k/(5k + 2))(p_g - 2k - 1)$. In other words, the slope of the fibration $S \to \mathbf{P}^1$ equals $24(g - 1)/(5g + 1)$, where $g = 3k + 1$ is the fiber genus.

EXAMPLE 3. Let (V, L_0) be a polarized $K3$ surface, and put $L_0^2 = 2k$. We choose a member $B \in |4L_0|$ which has at most simple triple points. Let $f' : S' \to V$ be the double covering with branch locus B, and let S be the minimal resolution of S'. We denote by $f : S \to V$ the natural map. Since the dualizing sheaf $\omega_{S'}$ of S' is induced by $2L_0$, we see that S is an even surface with a semi-canonical bundle $L = f^*L_0$. By a standard calculation, we have

$$h^0(\omega_{S'}) = h^0(V, \mathcal{O} \oplus \mathcal{O}(2L_0)) = 4k + 3,\ h^1(\omega_{S'}) = 0,$$
$$\omega_{S'}^2 = 2(2L_0)^2 = 16k.$$

Therefore, S is a regular surface of general type with $K^2 = 4p_g - 12$. In this way, we obtain even canonical surfaces whose canonical image is cut out by hyperquadrics: For example, let (x_0, x_1, x_2, x_3) be a system of coordinates on $\mathbf{P}^3$, and assume that V is defined by a homogeneous form A_4 of degree 4 in the x_i. We put $L_0 = \mathcal{O}_V(m)$ and follow the above construction. In this case, S' can be identified with a weighted complete intersection in $\mathbf{P}(1, 1, 1, 1, 2m)$ defined by $A_4 = u^2 + B_{4m} = 0$, where $\deg u = 2m$ and B_{4m} is a homogeneous form of degree $4m$ in the x_i. Since $L_0^2 = 4m^2$, we get $p_g(S) = 8m^2 + 3$. Furthermore, we clearly have $Q(X) = X$.

This example explains the meaning of the line $K^2 = 4p_g - 12$ in Reid's conjecture. We remark, in connection with Lemma 1.2, that a surface of degree $2n - 2$ in $\mathbf{P}^n$ is (birationally) either a ruled surface or a $K3$ surface.

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