ON A LOCAL HÖLDER CONTINUITY FOR A MINIMIZER OF THE EXPONENTIAL ENERGY FUNCTIONAL

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0. Introduction

Let $\Omega \subset \mathbf{R}^m$ be a bounded domain with smooth boundary, where $m \geq 2$. We consider the exponential energy functional

$$(0.1) \qquad E(u) := \int_\Omega e^{|\nabla u|^2} \, dx$$

for $u : \Omega \to \mathbf{R}^n$, where $n \geq 2$.

For minima of certain functionals with suitable growth conditions, many authors have established regularity results [1,5,7,12]. For example, in [7], Hardt and Lin proved that mappings minimizing the L^p-norm of the gradient between compact Riemannian manifolds are smooth except singular sets with finite $(\dim M - [p] - 1)$-dimensional Hausdorff measure.

If a functional has sufficiently rapid growth, we can expect full regularity of minima. In fact, quite recently, Duc and Eells [2], Eells and Lemaire [3] show that, in the case of $n = 1$, a minimizer u of E satisfies $u \in C^\infty(\Omega)$ for any smooth boundary data provided that $\Omega \subset \mathbf{R}^m$ is a strictly convex bounded domain.

In this paper, we consider a local regularity of minima of E for the case of $n \geq 2$. Our main theorem is stated as follows:

(0.2) THEOREM. *For a given boundary data* $g \in \cap_{1 < p < \infty} W^{1-1/p, p}(\partial\Omega, \mathbf{R}^n)$, *there exists a unique minimizer of E on the space* $\{u \in \cap_{1 < p < \infty} W^{1,p}(\Omega, \mathbf{R}^n) : u|_{\partial\Omega} = g\}$. *If $B_R(a) \subset \Omega$, then there exists a number μ satisfying $0 < \mu < 1$ such that the minimizer belongs to* $C^{1,\mu}(B_{R/4}(a))$.

From now on, we abbreviate $W^{1-1/p,p}(\partial\Omega, \mathbf{R}^n)$, $W^p(\Omega, \mathbf{R}^n)$ to $W^{1-1/p,p}(\partial\Omega)$, $W^p(\Omega)$, respectively.

For the sake of simplicity, we also use the following notations of function spaces. Let W be the Banach space defined by

$$W := \bigcap_{1<p<\infty} W^{1,p}(\Omega),$$

and the space of traces of elements of W:

$$B := \bigcap_{1<p<\infty} W^{1-1/p,p}(\partial\Omega).$$

For a given $g \in B$, the function space on which we consider the functional E is denoted by

$$W_g := \{u \in W : u|_{\partial\Omega} = g\}.$$

Moreover, to prove Theorem, we consider a Hilbert space

$$X_g := \{u \in H^1(\Omega) : u|_{\partial\Omega} = g\}$$

for a given $g \in H^{1/2}(\Omega)$.

In this paper, $B_r(a)$ expresses the ball centered at $a \in \Omega$ with radius r, and we abbreviate $B_r := B_r(a)$ when there is no confusion. Moreover d stands for the distance from the center a to $\partial\Omega$, i.e, $d := \text{dist}(a, \partial\Omega)$.

1. The existence of a minimizer

The aim of this section is to prove the following theorem.

(1.1) THEOREM. *For a given boundary data $g \in B$, there exists a unique minimizer of the functional E on W_g.*

The proof of Theorem 1.1 is divided into the following propositions.

First, we prove an existence and uniqueness theorem of minima on the space X_g.

(1.2) PROPOSITION. *For a given boundary data $g \in B$, there exists a unique minimizer of the functional E on X_g.*

Proof. We note that the set X_g is a closed convex subspace in $H^1(\Omega)$ and the functional E is strongly convex and strongly lower semi-continuous on X_g. Therefore, E is weakly lower semi-continuous on X_g.

Moreover, we show that the functional E is coercive on X_g. Namely, any sequence $\{u_n\}$ in X_g which satisfies $\|u_n\| \to \infty$ contains a subsequence $\{u_n'\}$ such that $E(u_n') \to \infty$. Let $\{u_n\} \subset X_g$ be a sequence which satisfies $\|u_n\| \to \infty$. Since Ω is bounded, using Poincaré's inequality, we see $\|\nabla u_n\|_{L^2(\Omega)} \to \infty$. On the other hand, for any $u \in X_g$, we have

$$(1.3) \qquad E(u) = \int_\Omega e^{|\nabla u|^2}\, dx \geq \int_\Omega dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx.$$

Therefore we get $E(u_n) \to \infty$.

Since E is weakly lower semi-continuous, strongly convex and coercive, there exists a unique minimizer on X_g, (cf. [8]).   □

Hereafter we denote the minimizer of E in X_g by u_X.

(1.4) PROPOSITION.   *The minimizer u_X of E in X_g belongs to W_g, provided $g \in B$.*

Proof.   Since $E(u_X) < \infty$, we obtain

$$\infty > E(u_X) = \int_\Omega e^{|\nabla u_X|^2}\, dx \geq \sum_{j=0}^{k} \frac{1}{(2j)!} \int_\Omega |\nabla u_X|^{2j}\, dx.$$

Therefore we have $u_X \in W^{1,2j}(\Omega)$ for $j = 2,3,4,\cdots$. Since $u_X|_{\partial\Omega} = g \in B$, we get $u_X \in W_g$.   □

(1.5) PROPOSITION.   *The minimizer u_X of E in X_g is unique in W_g.*

Proof.   First we note that $B \subset H^{1/2}(\partial\Omega)$. If $g \in B$, then it is easy to see that $X_g \supset W_g$. Therefore we have

$$\inf_{W_g} E(u) \geq \inf_{X_g} E(u).$$

Moreover since $u_X \in W_g$, we have

$$E(u_X) = \inf_{W_g} E(u).$$

This equality implies that u_X is a minimizer of E in W_g. Since the functional E is strongly convex and W_g is convex, the proof of Proposition 1.4 shows the uniqueness of the minimizer.   □

2. Gradient bound

In the following sections, we prove the regularity of a minimizer of E by Moser's iteration method. To do so, we first have to prove that a minimizer of E satisfies the Euler-Lagrange equation of the functional E, since the tangent space of the space W does not coincide with itself. Proposition 2.1 is found in [3], but we describe the proof of it for the readers' convenience.

(2.1) PROPOSITION.   *The minimizer of the functional E in the class W_g satisfies the Euler-Lagrange equation of E weakly.*

Proof.   Let $e(\cdot)$ be the energy density of the functional E, i.e, $e(u) = e^{|\nabla u|^2}$. For $\phi \in W$ with $E(\phi) < \infty$, set

$$f(t,x) := \frac{e(u + t(\phi - u)) - e(u)}{t} \quad \text{for} \quad 0 \leq t \leq 1.$$

Since $e(\phi) \geq 1$ for any ϕ, we have

$$(2.2)\ 0 \leq \int_\Omega f(t,\,x)\,dx \leq t \int_\Omega e^{|\nabla \phi|^2}\,dx + (1-t) \int_\Omega e^{|\nabla u|^2}\,dx \quad \text{for } 0 \leq t \leq 1.$$

Moreover, since $e(\cdot)$ is convex, we have

$$(2.3) \quad \begin{aligned} 0 &\leq \frac{\partial f(t,x)}{\partial t}, \\ 0 &\leq f(t,x) \quad \text{for} \quad 0 \leq t \leq 1. \end{aligned}$$

Therefore, by (2.2) and (2.3), $\lim_{t \downarrow 0} f(t,\,x)$ exists and we have

$$(2.4) \quad \begin{aligned} \lim_{t \downarrow 0} f(t,\,x) &= \nabla_i u^\alpha e^{|\nabla u|^2} (\nabla_i \phi^\alpha - \nabla_i u^\alpha) \in L^1, \\ \int_\Omega \nabla_i u^\alpha e^{|\nabla u|^2} (\nabla_i \phi^\alpha - \nabla_i u^\alpha)\,dx &\geq 0. \end{aligned}$$

Now we can replace ϕ to $\phi + s\varphi$ for any $s \in \mathbf{R}$ and any $\varphi \in C_0^\infty(\Omega, \mathbf{R}^n)$ in (2.4). Letting $s \to \infty$ or $s \to -\infty$, we have

$$\int_\Omega \nabla_i u^\alpha \nabla_i \varphi^\alpha e^{|\nabla u|^2}\,dx = 0$$

for any $\varphi \in C_0^\infty(\Omega, \mathbf{R}^n)$. $\square$

In the followings, we shall prove a bound of the gradient of minimizers. Since it will be proved by Moser's iteration method, we shall first describe the proof in the case of $m \geq 3$, for the sake of simplicity. After the completion of the proof in that case, we shall make a remark about the proof in the case of $m = 2$.

In order to justify the calculation in the proof, it is sufficient to have

$$\nabla_i u^{\alpha} e^{\frac{\gamma}{2}|\nabla u|^2} \in W_{\text{loc}}^{1,2}(\Omega) \quad \text{for} \quad \gamma > -1.$$

(2.5) PROPOSITION.   *Assume $m \geq 3$. Let u be a minimizer of E. If $B_r(a) \subset \Omega$, then we have*

$$\nabla_i u^{\alpha} e^{\frac{\gamma}{2}|\nabla u|^2} \in W^{1,2}(B_r(a)) \quad \text{for} \quad \gamma > -1.$$

Proof. Let $\Delta_h \varphi$ be the difference quotient of a function φ in the direction e_k. Since u is a minimizer, u satisfies the Euler-Lagrange equation of E weakly:

(2.6)
$$\int_{\Omega} \nabla_i u^{\alpha} \nabla_i \varphi^{\alpha} e^{|\nabla u|^2} dx = 0 \quad \text{for all} \quad \varphi \in C_0^{\infty}(\Omega).$$

In (2.6), replace φ to $\Delta_{-h}\varphi$, and substitute $\varphi^{\alpha} := \Delta_h u^{\alpha} \eta^2$. Then we have

$$\int_{\Omega} \Delta_h(\nabla_i u^{\alpha} e^{|\nabla u|^2}) \, \nabla_i(\Delta_h u^{\alpha} \eta^2) \, dx = 0.$$

Using $|\nabla |\Delta_h u|^2|^2 = |\Delta_h |\nabla u|^2|^2$ and Hölder's inequality, we have

$$\int_{\Omega} \left(|\Delta_h \nabla u|^2 + \frac{1}{2}|\Delta_h |\nabla u|^2|^2\right) e^{|\nabla u|^2} \eta^2 \, dx$$

$$\leq C \int_{\Omega} (|\Delta_h \nabla u||\Delta_h u| + |\nabla u||\Delta_h u||\Delta h |\nabla u|^2|) |\nabla \eta| \eta e^{|\nabla u|^2} \, dx.$$

Therefore we have

(2.7)
$$\int_{\Omega} \left(|\Delta_h \nabla u|^2 + \frac{1}{2}|\Delta_h |\nabla u|^2|^2\right) e^{|\nabla u|^2} \eta^2 \, dx$$

$$\leq C \int_{\Omega} (|\Delta_h \nabla u|^2 + |\nabla u|^2 |\Delta_h u|^2) |\nabla \eta|^2 e^{|\nabla u|^2} \, dx.$$

On the other hand, since $u^{\alpha} |\Delta_h u|^2 \eta^2 \in W_0^{1,2}(\Omega)$, and u is a minimizer of E, we obtain

$$\int_{\Omega} \nabla_i u^{\alpha} \nabla_i(u^{\alpha} |\Delta_h u|^2 \eta^2) \, e^{|\nabla u|^2} \, dx = 0.$$

Therefore we have

$$\int_\Omega |\nabla u|^2 |\Delta_h u|^2 e^{|\nabla u|^2} \eta^2 \, dx$$

$$\leq \int_\Omega |\nabla u| |\nabla|\Delta_h u|^2| |u| e^{|\nabla u|^2} \eta^2 \, dx$$

$$+ 2 \int_\Omega |\nabla u| |\Delta_h u|^2 |u| e^{|\nabla u|^2} |\nabla \eta| \eta \, dx$$

$$\leq \varepsilon \int_\Omega |\nabla|\Delta_h u|^2|^2 e^{|\nabla u|^2} \eta^2 \, dx + C \sup_\Omega |u| \int_\Omega |\nabla u|^2 e^{|\nabla u|^2} \eta^2 \, dx$$

$$+ \varepsilon \int_\Omega |\nabla u|^2 |\Delta_h u|^2 e^{|\nabla u|^2} \eta^2 \, dx + C \sup_\Omega |u| \int_\Omega |\Delta_h u|^2 e^{|\nabla u|^2} \eta^2 \, dx.$$

Hence we obtain

$$(2.8) \quad \begin{aligned} &\int_\Omega |\nabla u|^2 |\Delta_h u|^2 e^{|\nabla u|^2} \eta^2 \, dx \\ &\leq \varepsilon \int_\Omega |\nabla|\Delta_h u|^2|^2 e^{|\nabla u|^2} \eta^2 \, dx \\ &\qquad\qquad + C \sup_\Omega |u| \int_\Omega (|\nabla u|^2 + |\Delta_h u|^2) e^{|\nabla u|^2} \eta^2 \, dx. \end{aligned}$$

Combining (2.7) and (2.8), we obtain

$$\int_\Omega (|\nabla \Delta_h u|^2 + |\Delta_h|\nabla u|^2|^2) e^{|\nabla u|^2} \eta^2 \, dx$$

$$\leq C r^{-4} \int_\Omega (|\Delta_h u|^2 + |\nabla u|^2) e^{|\nabla u|^2} \, dx.$$

On the other hand, we have

$$\int_\Omega |\nabla u|^2 e^{|\nabla u|^2} \eta^2 \, dx \leq C r^{-2} E_0 \sup_\Omega |u|,$$

by putting $u^\alpha \eta \in W_0^{1,2}(\Omega)$ as a test function. Therefore we conclude that

$$(2.9) \quad \int_\Omega (|\nabla \Delta_h u|^2 + |\Delta_h|\nabla u|^2|^2) e^{|\nabla u|^2} \eta^2 \, dx \leq C E_0.$$

To show $\nabla u e^{\frac{\gamma}{2}|\nabla u|^2} \eta^2 \in W_0^{1,2}(\Omega)$ for $-1 < \gamma < 0$, it is sufficient to show

$$\int_\Omega (|\nabla^2 u|^2 + |\nabla u|^2 |\nabla|\nabla u|^2|^2) \, dx < C.$$

Therefore, by (2.9), the claim follows.   □

(2.10) PROPOSITION. *Assume $m \geq 3$. Let u be a minimizer of E. If $B_r(a) \subset \Omega$, then there exists a positive number ε_0 depending only on m such that*

$$\left(r^{-m} \int_{B_{r/2}(a)} e^{(1+\varepsilon)|\nabla u|^2} dx \right)^{\frac{m-2}{m}} \leq C r^{-m} E_0$$

for all $0 < \varepsilon \leq \varepsilon_0$, where $C = C(m, \varepsilon)$ and $E_0 = \int_{\Omega} e^{|\nabla u|^2} dx$.

Proof. First we take a negative number γ as $\gamma > \max\{-1, -4/m\}$. In (2.6), replace φ to $\nabla_k \varphi$, set $w := e^{|\nabla u|^2}$ and substitute $\varphi := \nabla_k u^\alpha w^{\gamma/2} \eta^2$. Then we have

$$
\begin{aligned}
0 &= \int_{B_r} \nabla_k (\nabla_i u^\alpha e^{|\nabla u|^2}) \, \nabla_i \, (\nabla_k u^\alpha w^{\gamma/2} \eta^2) \, dx \\
&= \int_{B_r} (\nabla_k \nabla_i u^\alpha + \nabla_i u^\alpha \nabla_k |\nabla u|^2) \, \nabla_k \nabla_i u^\alpha w^{\gamma/2+1} \eta^2 \, dx \\
&\quad + \frac{\gamma}{2} \int_{B_r} (\nabla_k \nabla_i u^\alpha + \nabla_i u^\alpha \nabla_k |\nabla u|^2) \, \nabla_k u^\alpha w^{\gamma/2+1} \, \nabla_i |\nabla u|^2 \eta^2 \, dx \\
&\quad + 2 \int_{B_r} (\nabla_k \nabla_i u^\alpha + \nabla_i u^\alpha \nabla_k |\nabla u|^2) \, \nabla_k u^\alpha w^{\gamma/2+1} \eta \, \nabla_i \eta \, dx \\
&= \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}
$$

(2.11)

By elementary calculations, the first term of the right hand side of (2.11) is equal to

$$(2.12) \qquad \text{(I)} = \int_{B_r} |\nabla \nabla u|^2 w^{\gamma/2+1} \eta^2 \, dx + \frac{1}{2} \int_{B_r} |\nabla |\nabla u|^2|^2 w^{\gamma/2+1} \eta^2 \, dx$$

and the second term is equal to

$$(2.13) \qquad \text{(II)} = \frac{\gamma}{4} \int_{B_r} |\nabla |\nabla u|^2|^2 \, w^{\gamma/2+1} \, dx + \frac{\gamma}{2} \int_{B_r} |\langle \nabla u, \nabla |\nabla u|^2 \rangle|^2 w^{\gamma/2+1} \, dx,$$

where $\langle \cdot, \cdot \rangle$ is the inner product $\mathbf{R}^n$. Since $-1 < \gamma < 0$, the sum of (2.12) and (2.13) is estimated by

$$
\begin{aligned}
(2.14) \qquad (\text{I} + \text{II}) &= \int_{B_r} (|\nabla \nabla u|^2 + \frac{\gamma}{2} |\langle \nabla u, \nabla |\nabla u|^2 \rangle|^2) w^{\gamma/2+1} \eta^2 \, dx \\
&\quad + \frac{1}{2} \left(1 + \frac{\gamma}{2} \right) \int_{B_r} |\nabla |\nabla u|^2|^2 w^{\gamma/2+1} \eta^2 \, dx
\end{aligned}
$$

$$\geq \frac{1}{2}\left(1+\frac{\gamma}{2}\right)\int_{B_r}|\nabla|\nabla u|^2|^2 w^{\gamma/2+1}\eta^2\,dx.$$

On the other hand, the third term is estimated from above by

$$(2.15) \qquad \begin{aligned} \text{(III)} &\leq C\int_{B_r}(1+|\nabla u|^2)\,|\nabla|\nabla u|^2|\,|\nabla\eta|\,\eta\,w^{\gamma/2+1}\,dx \\ &\leq \frac{C}{\delta}\int_{B_r}|\nabla|\nabla u|^2|\,|\nabla\eta|\,\eta\,w^{\gamma/2+1+\delta}\,dx, \end{aligned}$$

for any $\delta>0$, since $xe^x\leq\frac{1}{\delta}\,e^{(1+\delta)x}$ for any $\delta>0$ and $x\geq0$.

Inequalities (2.14) and (2.15) then give that

$$(2.16) \qquad \begin{aligned} &\frac{1}{2}\left(1+\frac{\gamma}{2}\right)\int_{B_r}|\nabla|\nabla u|^2|^2 w^{\gamma/2+1}\eta^2\,dx \\ &\leq \frac{C}{\delta}\int_{B_r}|\nabla|\nabla u|^2|\,|\nabla\eta|\,\eta\,w^{\gamma/2+1+\delta}\,dx. \end{aligned}$$

Since the left hand side and the right hand side of (2.16) are equal to

$$\text{(LHS)} = \frac{2}{(1+\gamma/2)}\int_{B_r}|\nabla(w^{\frac{1}{2}(\gamma/2+1)})|^2\eta^2\,dx,$$

$$\text{(RHS)} = \frac{C}{\delta}\,\frac{2}{(1+\gamma/2)}\int_{B_r}|\nabla(w^{\frac{1}{2}(\gamma/2+1)})|\,\eta\,w^{\frac{1}{2}(\gamma/2+1)+\delta}\,|\nabla\eta|\,dx,$$

and $w\geq1$ and $-1<\gamma<0$, by taking $\delta=-\gamma/4>0$ and the Hölder's inequality, we obtain

$$\begin{aligned} \int_{B_r}|\nabla(w^{\frac{1}{2}(\gamma/2+1)})|^2\,\eta^2\,dx &\leq C\,\frac{2\delta'}{-\gamma}\int_{B_r}|\nabla(w^{\frac{1}{2}(\gamma/2+1)})|^2\,\eta^2\,dx \\ &\quad + C\,\frac{2}{-\gamma\delta'}\int_{B_r}w^{\gamma/4+1}\,|\nabla\eta|^2\,dx, \end{aligned}$$

for any $\delta'>0$. Therefore, by taking $\delta'=-\frac{\gamma}{4C}$, there exists a positive constant $C=C(m,\gamma)$ such that

$$(2.17) \qquad \int_{B_r}|\nabla(w^{\frac{1}{2}(\gamma/2+1)})|^2\,\eta^2\,dx \leq C\int_{B_r}w^{\gamma/4+1}\,|\nabla\eta|^2\,dx.$$

By the Sobolev embedding theorem, (2.17) yields

$$\left(r^{-m}\int_{B_{r/2}}w^{(\gamma/2+1)\frac{m}{m-2}}\,dx\right)^{\frac{m-2}{m}} \leq Cr^{-m}\int_{B_r}w^{\gamma/4+1}\,dx.$$

Put $1 + \varepsilon := \left(1 + \dfrac{\gamma}{2}\right) \dfrac{m}{m-2}$, then we obtain the inequality

$$\left(r^{-m} \int_{B_{r/2}} e^{(1+\varepsilon)|\nabla u|^2} dx\right)^{\frac{m-2}{m}} Cr^{-m} \int_{B_r} e^{|\nabla u|^2} dx \le Cr^{-m} E_0,$$

and $1 + \gamma/4 = \dfrac{(1+\varepsilon)(m-2)+1}{2m}$. Finally we remark that we can not take $\gamma = 0$. $\qquad\square$

(2.18) PROPOSITION. *Assume $m \ge 3$. Let u be a minimizer of E. For a finite number q satisfying $q > \dfrac{m}{2}$, if $B_r(a) \subset \Omega$, then the following inequality holds:*

$$\left(r^{-m} \int_{B_{r/2}(a)} e^{q|\nabla u|^2} dx\right)^{\frac{1}{q}} \le C,$$

where $C = C(m, E_0, d, q)$.

Proof. In (2.6), if we take $\gamma \ge 0$, then the first and second terms are estimated by

$$(\mathrm{I} + \mathrm{II}) \ge \frac{1}{2}\left(1 + \frac{\gamma}{2}\right) \int_{B_r} |\nabla|\nabla u|^2|^2 w^{\gamma/2+1} \eta^2 \, dx.$$

Hence we have

$$(2.19) \quad \int_{B_r} |\nabla(w^{\frac{1}{2}(\gamma/2+1)})|^2 \eta^2 \, dx \le \frac{C}{\delta} \int_{B_r} |\nabla(w^{\frac{1}{2}(\gamma/2+1)})| \, w^{\frac{1}{2}(\gamma/2+1)+\delta} |\nabla\eta| \eta \, dx,$$

for any $\delta > 0$, where the constant $C = C(\gamma)$.

Take $\alpha = \gamma/2 + 1$. Then using the Young's inequality and the Sobolev embedding theorem, (2.19) yields that

$$(2.20) \qquad \left(r^{-m} \int_{B_{\sigma r}} w^{\alpha \frac{m}{m-2}} dx\right)^{\frac{m-2}{m}} \le \frac{Cr^{-m}}{(1-\sigma)^2} \int_{B_r} w^{\alpha+\delta} \, dx,$$

where $C = C(\delta, \gamma)$.

Taking $\alpha = 1$ and $\delta = \varepsilon$, by Proposition 2.10 and (2.20), we have

$$\left(r^{-m} \int_{B_{\sigma^2 r}} e^{\frac{m}{m-2}|\nabla u|^2} dx\right)^{\frac{m-2}{m}} \le \frac{C}{(1-\sigma)^2} r^{-m} \int_{B_{\sigma r}} e^{(1+\varepsilon)|\nabla u|^2} dx$$

$$\le \left(\frac{C}{(1-\sigma)^2}\right)^2 (r^{-m} E_0)^{\frac{m}{m-2}}.$$

In the following, take $\alpha = \left(\dfrac{m}{m-2}\right)^{(k-1)/2}$ and $\delta = \left(\dfrac{m}{m-2}\right)^{k/2} - \left(\dfrac{m}{m-2}\right)^{(k-1)/2}$

Then finite times iterations of this procedure implies that

$$(2.21) \qquad \left(r^{-m} \int_{B_{\sigma^k r}} e^{\left(\frac{m}{m-2}\right)^{k/2}|\nabla u|^2} dx\right)^{\left(\frac{m-2}{m}\right)^{k/2}} \leq C (r^{-m} E_0)^{\left(\frac{m}{m-2}\right)^{k/2}},$$

where the constant $C = (m, k, \sigma, \gamma)$. Therefore for any finite number q satisfying $q > \dfrac{m}{2}$ the following inequality is true:

$$\left(r^{-m} \int_{B_{r/2}} e^{q|\nabla u|^2} dx\right)^{1/q} \leq C.$$

$\square$

(2.22) PROPOSITION.   *Assume $m \geq 3$. Let u be a minimizer of E. If $B_{2r}(a) \subset \Omega$, then there exists a positive number p satisfying $1 < p < \dfrac{m}{m-2}$ such that for $\alpha \geq 2$ and $0 < \sigma < 1$,*

$$\left(r^{-m} \int_{B_{\sigma r}(a)} (1 + |\nabla u|^2)^{\alpha \frac{m}{m-2}} dx\right)^{\frac{m-2}{m}} \leq C \left(r^{-m} \int_{B_r(a)} (1 + |\nabla u|^2)^{\alpha p} dx\right)^{1/p},$$

where the constant $C = C(m, d, E_0, \sigma, \alpha, p)$.

Proof. In (2.6), replace φ to $\nabla_k \varphi$, set $w := 1 + |\nabla u|^2$ and substitute $\varphi := \nabla_k u^\alpha w^{\gamma/2} \eta^2$ for $\gamma \geq 0$, then we have

$$\int_{B_r} (\nabla_k \nabla_i u^\alpha + \nabla_i u^\alpha \nabla_k |\nabla u|^2) \nabla_k \nabla_i u^\alpha w^{\gamma/2} \eta^2 e^{|\nabla u|^2} dx$$

$$(2.22) \qquad + \frac{\gamma}{2} \int_{B_r} (\nabla_k \nabla_i u^\alpha + \nabla_i u^\alpha \nabla_k |\nabla u|^2) \nabla_k u^\alpha \nabla_i w w^{\gamma/2-1} \eta^2 e^{|\nabla u|^2} dx$$

$$= -2 \int_{B_r} (\nabla_k \nabla_i u^\alpha + \nabla_i u^\alpha \nabla_k |\nabla u|^2) \nabla_k u^\alpha w^{\gamma/2} \nabla_i \eta \, \eta \, e^{|\nabla u|^2} dx.$$

By a similar calculation to the proof of Proposition 2.10, we have

$$\text{the first term of (2.22)} = \int_{B_r} |\nabla^2 u|^2 w^{\gamma/2} \eta^2 e^{|\nabla u|^2} dx$$

$$+ \int_{B_r} |\nabla|\nabla u|^2|^2 w^{\gamma/2} \eta^2 e^{|\nabla u|^2} dx,$$

$$\text{the second term of (2.22)} = \int_{B_r} |\langle \nabla u, \nabla|\nabla u|^2\rangle|^2 w^{\gamma/2-1} \eta^2 e^{|\nabla u|^2} dx$$

$$+ \int_{B_r} |\nabla |\nabla u|^2 |^2 w^{\tau/2-1} \eta^2 e^{|\nabla u|^2} \, dx,$$

$$| \text{the third term of (2.22)} | \leq 2 \int_{B_r} |\nabla w| |\nabla \eta| w^{\tau/2+1} \eta e^{|\nabla u|^2} \, dx$$

$$\leq C \int_{B_r} |\nabla w^{\frac{\tau+4}{4}}|^2 \eta^2 e^{|\nabla u|^2} \, dx$$

$$+ \frac{1}{2} \int_{B_r} w^{\frac{\tau+4}{2}} |\nabla \eta|^2 e^{|\nabla u|^2} \, dx.$$

Therefore we have

$$\int_{B_r} |\nabla (\eta w^{\frac{\tau+4}{4}})|^2 \, dx \leq C \int_{B_r} w^{\frac{\tau+4}{2}} |\nabla \eta|^2 e^{|\nabla u|^2} \, dx.$$

By the Sobolev embedding theorem, we obtain

$$(2.23) \qquad \left(r^{-m} \int_{B_{\sigma r}} w^{\frac{\tau+4}{2} \frac{m}{m-2}} \, dx \right)^{\frac{m-2}{m}} \leq C r^{-m} \int_{B_r} w^{\frac{\tau+4}{2}} e^{|\nabla u|^2} \, dx.$$

Applying the Hölder's inequality to (2.23), we obtain the inequality

$$\left(r^{-m} \int_{B_{\sigma r}} w^{\frac{\tau+4}{2} \frac{m}{m-2}} \, dx \right)^{\frac{m-2}{m}} \leq C \left(r^{-m} \int_{B_r} w^{p \frac{\tau+4}{2}} \, dx \right)^{1/p} \left(r^{-m} \int_{B_r} e^{q|\nabla u|^2} \, dx \right)^{1/q}.$$

On the other hand, choose $q > \dfrac{m}{2}$ then $1 < p < \dfrac{m}{m-2}$ and the integral $r^{-m} \int_{B_r} e^{q|\nabla u|^2} \, dx$ is bounded from above. Therefore, setting $\alpha = \dfrac{\tau+4}{2} \geq 2$, we complete the proof. $\qquad \square$

(2.24) PROPOSITION. *Assume $m \geq 3$. Let u be a minimizer of E. If $B_{2r}(a) \subset \Omega$, then there exists a constant $C = C(m,d,E_0)$ such that*

$$\sup_{B_{r/2}(a)} |\nabla u|^2 \leq C.$$

Proof. By Proposition 2.22, there exists a positive number p satisfying $p < \dfrac{m}{m-2}$ such that for any $\alpha \geq 2$ and any $0 < \sigma < 1$

$$\left(r^{-m} \int_{B_{\sigma r}} v^{\alpha \frac{m}{m-2}} \, dx \right)^{\frac{m-2}{m}} \leq C \left(r^{-m} \int_{B_r} v^{\alpha p} \, dx \right)^{1/p},$$

where $v := 1 + |\nabla u|^2$. Take

$$r_k := \sigma^k r, \quad B_k := B_{r_k},$$
$$\alpha_k := 2 \left(\frac{1}{p} \frac{m}{m-2} \right)^k.$$

Then we obtain a recursion formula

$$\left(r^{-m} \int_{B_k} v^{\alpha_k \frac{m}{m-2}} dx \right)^{\alpha_k^{-1} \frac{m-2}{m}} \leq C \left(r^{-m} \int_{B_{k-1}} v^{\alpha_{k-1}} dx \right)^{\alpha_{k-1}^{-1}} \quad \text{for} \quad k \geq 1.$$

Therefore it holds that

$$\left(r^{-m} \int_{B_k} v^{\alpha_k \frac{m}{m-2}} dx \right)^{\alpha_k^{-1} \frac{m-2}{m}} \leq C \left(r^{-m} \int_{B_0} v^p dx \right)^{1/p} \quad \text{for} \quad k \geq 1.$$

Letting $k \to \infty$, we have

$$\sup_{B_{r/2}} v \leq C \left(r^{-m} \int_{B_r} v^p dx \right)^{1/p}.$$

Therefore

$$\sup_{B_{r/2}} |\nabla u|^2 \leq C \left(r^{-m} \int_{B_r} (1 + |\nabla u|^2)^p dx \right)^{1/p}$$
$$\leq C \left(r^{-m} \int_{B_r} e^{|\nabla u|^2} dx \right)^{1/p} \leq C \left(r^{-m} E_0 \right)^{1/p}.$$

$\square$

(2.25) *Remark.* In the case of $m = 2$, the argument similar to the proof of Proposition 2.10 yields

$$(2.10') \qquad \left(r^{-2} \int_{B_{r/2}(a)} e^{(1+\varepsilon)|\nabla u|^2} dx \right)^{\frac{2-q}{q}} \leq C r^{-2} \int_{B_r(a)} e^{|\nabla u|^2} dx,$$

for any $1 < q < 2$ and $0 < \varepsilon < \varepsilon_0$, where the constant $C = C(\varepsilon, |\Omega|)$. The inequality (2.10') is proved by taking γ as $\frac{1}{4}(1/q - 1) < \gamma < 0$ in the proof of Proposition 2.10.

We also obtain inequalities similar to (2.18) and (2.22). Therefore Proposition 2.24 also holds in case of $m = 2$.

3. Hölder continuity of a minimizer

In the final section, we shall prove the main theorem. The proof of the

theorem is divided into several propositions. Using the bound of the gradient of minimizers, we shall obtain an estimate of the growth order and the mean value of them on a small ball.

(3.1) PROPOSITION. *For $0 < r < R_0/4$ satisfying $B_{R_0}(a) \subset \Omega$ and any constant vector $V \in \mathbf{R}^{nm}$ satisfying*

$$| V | \leq \sup_{B_{R_0}(a)} | \nabla u |,$$

there exists a constant vector $V_\sigma \in \mathbf{R}^{nm}$ such that

(3.2)
$$| V_\sigma | \leq \sup_{B_{R_0}(a)} | \nabla u |,$$

(3.3)
$$\int_{B_{\sigma r}(a)} | \nabla u - V_\sigma |^2 \, dx$$
$$\leq C\sigma^{m+2} \int_{B_r(a)} | \nabla u - V |^2 \, dx + C\sigma^m \int_{B_r(a)} | \nabla u - V |^2 \, dx,$$

for all $0 < \sigma < 1/2$.

Proof. Let $v : B_r \to \mathbf{R}^n$ be an H^1-solution of $\Delta v = 0$ with $u = v$ on B_r. By the definition of v, for any $\varphi \in C_0^\infty(B_r)$, we have

$$\int_{B_r} \nabla_i v^\alpha \nabla_i \varphi^\alpha e^{|V|^2} \, dx = 0.$$

Set $V_\sigma := (\text{vol}(B_{\sigma r}))^{-1} \int_{B_{\sigma r}} \nabla v \, dx$. By applying the maximum principle and some well known results on the Laplace equation, we obtain

$$| V_\sigma | \leq (\text{vol}(B_{\sigma r}))^{-1} \int_{B_{\sigma r}} | \nabla v | \, dx$$

$$\leq \sup_{B_{\sigma r}} | \nabla v | \leq \sup_{\partial B_r} | \nabla v | \leq \sup_{\partial B_r} | \nabla u | \leq \sup_{B_{R_0}} | \nabla u |.$$

Therefore we conclude (3.2).

On the other hand, we have

(3.4) $$\int_{B_{\sigma r}} | \nabla u - V_\sigma |^2 \, dx \leq 2 \int_{B_{\sigma r}} | \nabla u - \nabla v |^2 \, dx + 2 \int_{B_{\sigma r}} | \nabla v - V_\sigma |^2 \, dx,$$

since $| \nabla u - V_\sigma |^2 \leq 2 | \nabla u - \nabla v |^2 + 2 | \nabla v - V_\sigma |^2$. Here we note that v satisfies a strongly elliptic system with constant coefficients: $\Delta v = 0$. Therefore v satisfies

an estimate due to Campanato [4, III (2.8) or (2.11)], namely

(3.5)
$$\int_{B_{\sigma r}} |\nabla v - V_\sigma|^2 \, dx \le C\sigma^{m+2} \int_{B_r} |\nabla v - V_1|^2 \, dx.$$

Since V_1 minimizes the integral $\displaystyle\int_{B_r} |\nabla v - c|^2 \, dx$, we have

(3.6)
$$\int_{B_r} |\nabla v - V_1|^2 \, dx \le \int_{B_r} |\nabla v - V|^2 \, dx.$$

Combining (3.5) and (3.6), we obtain

(3.7)
$$\int_{B_{\sigma r}} |\nabla v - V_\sigma|^2 \, dx \le C\sigma^{m+2} \int_{B_r} |\nabla v - V|^2 \, dx$$
$$\le C\sigma^{m+2} \int_{B_r} |\nabla u - \nabla v|^2 \, dx + C\sigma^{m+2} \int_{B_r} |\nabla u - V|^2 \, dx.$$

In order to estimate $\displaystyle\int_{B_r} |\nabla u - \nabla v|^2 \, dx$, we use the function $\varphi = u - v$ defined on B_r. From differential equations, we derive

$$\int_{B_r} \nabla_i u^\alpha \nabla_i \varphi^\alpha \left[\int_0^1 \partial_t (e^{|tV + (1-t)\nabla u|^2}) \, dt \right] dx$$
$$- \int_{B_r} \nabla_i u^\alpha \nabla_i \varphi^\alpha e^{|V|^2} \, dx + \int_{B_r} \nabla_i v^\alpha \nabla_i \varphi^\alpha e^{|V|^2} \, dx = 0.$$

Since

$$\partial_t (e^{|tV + (1-t)\nabla u|^2}) = 2(V_k^\beta - \nabla_k u^\beta)(tV_k^\beta + (1-t)\nabla_k u^\beta) e^{|tV + (1-t)\nabla u|^2},$$

we have

$$\int_{B_r} |\nabla u - \nabla v|^2 \, dx \le 2 \int_{B_r} \int_0^1 |\nabla u| \, |\nabla u - \nabla v| \, |\nabla u - V|$$
$$\times |tV + (1-t)\nabla u| \, e^{|tV + (1-t)\nabla u|^2} \, dt \, dx.$$

Since $|V| \le \sup_{B_{R_0}} |\nabla u|$ and $|tV + (1-t)\nabla u| \le |V| + |\nabla u|$, using Proposition 3.17, we have

$$\text{(the left hand side)} \le C \int_{B_r} |\nabla u - \nabla v| \, |\nabla u - V| \, dx.$$

Therefore we conclude

(3.8)
$$\int_{B_r} |\nabla u - \nabla v|^2 \, dx \leq C \int_{B_r} |\nabla u - V|^2 \, dx,$$

where the constant $C = C(m, d, E_0)$.

A calculation similar to lead (3.8) also yields that

$$\int_{B_{\sigma r}} |\nabla u - \nabla v|^2 \, dx \leq C \int_{B_{2\sigma r}} |\nabla u - V|^2 \, dx.$$

On the other hand, since $\tilde{u}^\alpha := u^\alpha + \sum_{i=1}^m V_i^\alpha x^i$ is also a weak solution of the Euler-Lagrange equation of E, if $\sigma < \frac{1}{2}$, then we have

(3.9)
$$\int_{B_{2\sigma r}} |\nabla u - V|^2 \, dx \leq C\sigma^m \int_{B_r} |\nabla u - V|^2 \, dx.$$

Combining (3.4), (3.7), (3.8) and (3.9), we derive

$$\int_{B_{\sigma r}} |\nabla u - V_\sigma|^2 \, dx \leq C\sigma^{m+2} \int_{B_r} |\nabla u - V|^2 \, dx + C\sigma^m \int_{B_r} |\nabla u - V|^2 \, dx. \qquad \square$$

(3.10) PROPOSITION. *Let* $U_r(a) = U_r \in \mathbf{R}^{nm}$ *be the average of* ∇u *in* $B_r(a)$*, i.e,*

$$U_r(a) = (\mathrm{vol}(B_r(a)))^{-1} \int_{B_r(a)} \nabla u \, dx.$$

If $B_{R_0}(a) \subset \Omega$ *and* $0 < r < R_0/4$*, then there exists a constant* μ *satisfying* $0 < \mu < 1$ *such that*

$$\int_{B_r(a)} |\nabla u - U_r|^2 \, dx \leq Cr^{m+2\mu},$$

where the constant $C = C(m, d, E_0)$.

Proof. By Proposition 3.1, we may choose a number $\sigma \in (0, 1/2)$ such that the following is true; for $r_i := \sigma^i \cdot r$ and $B_i := B_{r_i}$, there exist constant vectors $W_0, W_1, \ldots, W_k$ such that

(3.11)
$$W_0 = U_r$$

(3.12)
$$|W_{i+1}| \leq \sup_{B_{R_0}} |\nabla u|,$$

(3.13)
$$r_{i+1}^{-m} \int_{B_{i+1}} |\nabla u - W_{i+1}|^2 \, dx \leq C\sigma^2 r_i^{-m} \int_{B_i} |\nabla u - W_i|^2 \, dx + C \int_{B_i} |\nabla u - W_i|^2 \, dx$$

$$\leq C_0 \sigma^2 r_i^{-m} \int_{B_i} |\nabla u - W_i|^2 \, dx + C_1,$$

provided $|W_i| \leq \sup_{B_{R_0}} |\nabla u|^2$, where the constant C_1 depends on the bound of $|\nabla u|$ on B_{R_0}. By Proposition 3.1 again, (3.11)–(3.13) remain valid for all positive integers i. Therefore choose σ so small that σ satisfies $C_0 \sigma^2 < 1$. Then (3.13) shows that there exist constants C and μ satisfying $0 < \mu < 1$ depending on m, d, E_0 such that

$$r^{-(m+2\mu)} \int_{B_r} |\nabla u - U_r|^2 \, dx \leq C.$$

$\square$

Proposition 3.10 implies that if $B_{4r}(a) \subset \Omega$ then ∇u is Hölder continuous on $B_r(a)$ with the exponent μ.

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