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VARIATIONAL INEQUALITIES OF BINGHAM TYPE IN THREE DIMENSIONS

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Introduction

The flow of Bingham type through a domain Ω in the d-th dimensional space $\mathbf{R}^d$ $(d \geq 2)$ during the time $(0, T)$ is a flow of an incompressible visco-plastic fluid governed by the equations for a velocity vector $u = (u^1, \ldots, u^d)$ and a stress tensor $\sigma = (\sigma_{ij})_{i,j=1}^d$:

$$(0.1) \qquad \begin{aligned} &\frac{\partial u}{\partial t} + u\cdot\nabla u = f + \nabla\sigma \\ &\nabla\cdot u = 0 \end{aligned} \qquad \text{in } \Omega \times (0, T)$$

and by the constituent law:

$$(0.2) \qquad \begin{aligned} &\sigma^D = \left\{\eta(|D|) + \frac{g}{|D|}\right\} D && \text{when } D \neq 0 \\ &|\sigma^D| \leq g && \text{when } D = 0 \end{aligned}$$

which is equivalent to

$$\eta(|D|)D = \begin{cases} (1 - g/|\sigma^D|)\sigma^D & \text{when } |\sigma^D| > g \\ 0 & \text{when } |\sigma^D| \leq g \end{cases}$$

where $\sigma^D = \sigma + \pi I_d$ is the deviation of σ (i.e., $\pi = -\operatorname{tr}(\sigma)/d$ is the pressure), g the yield limit, $D = D(u)$ a tensor of strain velocity with components:

$$D_{ij}(u) = \frac{1}{2}(\nabla_i u^j + \nabla_j u^i) \quad \text{with} \quad \nabla_i = \partial/\partial x_i,$$

$|\sigma|$ the length defined by

$$|\sigma| = (\sigma\cdot\sigma)^{1/2}, \quad \sigma\cdot\tau = \sigma_{ij}\tau_{ij},$$

$u\cdot\nabla = u^i\nabla_i$, $(\nabla\cdot\sigma)_i = \nabla_j\sigma_{ij}$ and $\nabla\cdot u = \nabla_i u^i = \operatorname{div} u$, the summation convention concerning repeated indices being used.

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In the present paper we consider a fluid with viscosity $\eta\,(|D|)$ such that $\lambda\eta(\lambda)$ is a nondecreasing function in $\lambda \geq 0$ satisfying

$$c_1\lambda^{p-1} \leq \lambda\eta(\lambda) \leq c_2\lambda^{p-1}, \quad \lambda \geq 0$$

for some positive constants c_1, c_2 and $p > 1$. The various interesting examples of $\eta(\lambda)$ may be found in Astarita-Marrucci [1]. Introducing a convex functional of u:

$$\varphi(u) = \int_\Omega dx \int_0^{|D(u)|} (\lambda\eta(\lambda) + g)\, d\lambda, \tag{0.3}$$

we can deduce after Duvaut-Lions [5] the equations (0.1)-(0.2) subject to the boundary condition $u = 0$ to the evolution inequality

$$\int_\Omega (u'(t) + B(u(t))\cdot(v - u(t))\, dx + \varphi(v) - \varphi(u(t)) \geq \int_\Omega f(t)\cdot(v - u(t))\,dx \tag{0.4}$$

for all $t \in (0, T)$ and all v such that $\nabla\cdot v = 0$ in Ω and $v = 0$ on the boundary $\partial\Omega$ of Ω, where $u' = du/dt$ and $B(u) = u\cdot\nabla u$. The inequality (0.4) is called to be of Bingham type if $g > 0$.

The problem we consider here is to find a solution $u(t) = u(x, t)$ of inequality (0.4) of Bingham type satisfying the boundary condition

$$u(x, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \tag{0.5}$$

and the initial condition

$$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega. \tag{0.6}$$

The fluid which is obeyed by (0.2) with constant viscosity η is called a Bingham fluid, whose flow was first studied by Duvaut-Lions [5,6] introducing a variational inequality such as (0.4). They obtained, among other things, a weak solution (for the definition see Theorem 1). In Naumann-Wulst [13,14] strong solutions (for the definition see Corollary 1) were looked for in the case $\eta(\lambda) = \lambda^{p-2}$, $(\sqrt{97} - 1)/4 \leq p < 3$, under the condition that Ω is a smooth and bounded domain in $\mathbf{R}^3$. The existence of a strong solution for a Bingham fluid was investigated by Kim [7,8] in the plane as well as in the third dimensional bounded domain.

The main result of this paper consists of three theorems. Theorem 1 is concerned with the existence of weak solutions to the initial-boundary value problem (0.4)~(0.6) with $p > 6/5$ where φ is allowed to depend explicitly on t. As a corollary we obtain strong solutions for $p \geq 11/5$ (see Corollary 1). This result is a slight improvement of a result of Naumann-Wulst [14, Theorem 1.1 (i)]. In Theorem 2 we derive the energy inequality of strong form, provided that Ω is an exterior domain and $\eta(\lambda) = \mu\lambda^{p-2}$ with positive constant μ and $p \geq 9/5$. The regularity of velocity field u of Bingham fluid with variable viscosity and yield limit will be investigated in Theorem 3. This is nothing but a simple extension of the result of Kim [8].

The distinctive feature of the present paper is to construct Yosida's approximation $\mathscr{L}_n = n\left\{1 - \left(1 + \frac{1}{n} L_n\right)^{-1}\right\}$ of a multivalued operator $L_n(v) = e_n(v) + B(v) + \partial\varphi(v)$ which is regularized by adding the term $e_n(v) = -\xi_n \exp(\lambda_n \|\nabla v\|^c) \Delta v$ where $c > 4$ and $\xi_n, \lambda_n \to 0$ as $n \to \infty$. In fact, it is proved in Section 3 that the inverse of an operator $\left(1 + \frac{1}{n} L_n\right)$ exists. The evolution equation $u_n'(t) + \mathscr{L}_n(t, u_n(t)) = f_n(t)$ which approximates (0.4) will be solved by the method of successive approximation. A weak solution which is seeked for in Theorem 1 will be found in Section 4 as a limit of a subsequence of $\{u_n\}$.

The proof of Theorem 2 is achieved in Section 5 by taking a test function of the form $\mathrm{rot}\{\zeta_\lambda(F_\lambda * (\zeta_\lambda \,\mathrm{rot}\, u_n))\}$ $(\lambda \to 0)$ where F_λ denotes a fundamental solution of operator $\lambda - \Delta$ and ζ_λ a cut-off function such that $\zeta_\lambda(x) = 1$ for $|x| > 2/\lambda$ and $= 0$ for $|x| > 1/\lambda$. This device for the proof comes into action thanks to the plastic term $g|D(u)|$. For the Navier-Stokes equation where $p = 2$ and $g = 0$ we refer to Miyakawa-Sohr [11].

Theorem 3 is able to be applied to problems of heat transfer in a Bingham fluid with viscosity and yield limit depending on the temperature, which will be investigated elsewhere.

We devote Section 1 to preparations for the present study. Theorems 1 ~ 3 are stated in Section 2, along with three corollaries and four remarks where Theorems 1 ~ 3 are examined in the case that $d = 2$. Sections 4 ~ 6 are devoted to the proof of Theorems 1-3, respectively.

§1. Preliminaries

By $\mathscr{V}$ we denote the set of $v = (v^1, \ldots, v^d) \in C_0^\infty(\mathbf{R}^d)^d$ such that $\nabla \cdot v = 0$ everywhere and by L^p $(1 \leq p \leq \infty)$ the set of all L^p-function from $\mathbf{R}^d$ $(d \geq 2)$ into $\mathbf{R}$ equipped with the usual L^p-norm $\|\cdot\|_p$. Especially, we simply write $\|\cdot\|_2 = \|\cdot\|$. Further, the following abbreviations are used: $\|v\|_p = \||v|\|_p$,

$\| \nabla v \|_p = \| |\nabla v| \|_p$ and $\| D(v) \|_p = \| |D(v)| \|_p$ for vector field v, where ∇v and $D(v)$ denote tensors with components $\nabla_i v^j$ and $D_{ij}(v) = \nabla_i v^j + \nabla_j v^i$, and $|\cdot|$ respective length with respect to the euclidian metric.

We start with stating the two fundamental inequalities.

Korn's inequality. For any $p \in (1. \infty)$ there exists a positive constant K_p such that

$$\| \nabla v \|_p \leq K_p \| D(v) \|_p, \quad v \in C_0^\infty(\mathbf{R}^d)^d. \tag{1.1}$$

Sobolev's inequality. For any $p \in [1, d)$ there exists a positive constant S_p such that

$$\| v \|_{p*} \leq S_p \| D(v) \|_p, \quad v \in C_0^\infty(\mathbf{R}^d)^d, \tag{1.2}$$

where $p^* = dp/(d-p)$.

For the proof of (1.1) we refer to Mosolov-Mjasnikov [12] and its bibliography. Combining (1.1) and the usual Sobolev inequality (see Berger [2]), we immediately obtain (1.2) for p, $1 < p < d$. The inequality (1.2) with $p = 1$ has been proved by Strauss [16].

The following proposition is nothing but a straightforward extension of the result of Renardy [15].

PROPOSITION 1.1. *There exists a sequence of operators* $T_{\varepsilon,\lambda,\mu}$ $(\varepsilon, \lambda, \mu > 0)$; $u \to u_{\varepsilon,\lambda,\mu} = T_{\varepsilon,\lambda,\mu} u$ *of* L^q_σ $(1 \leq q < \infty)$ *into* $\mathscr{V}$ *such that*

(i) $u_{\varepsilon,\lambda,\mu} \to u$ *in* L^q,

(ii) $\nabla u_{\varepsilon,\lambda,\mu} \to \nabla u$ *in* L^p, *if* $\nabla_i u^j \in L^p$ $(1 \leq i, j \leq d)$ *and* $p > 1$,

and

(iii) $D(u_{\varepsilon,\lambda,\mu}) \to D(u)$ *in* L^r, *if* $D_{ij}(u) \in L^r$ $(1 \leq i, j \leq d)$ *for* $r \geq 1$ *such that* $1/r - 1/q \leq 2/d$,

as $\mu \to 0$, $\lambda \to 0$ *and* $\varepsilon \to 0$, *one after another, where*

$$L^q_\sigma = \{u \in (L^q)^d; \nabla\cdot u = 0\} \text{ for } q > 1,$$
$$L^1_\sigma = \left\{u \in (L^1)^d; \nabla\cdot u = 0 \text{ and } \int u dx = 0\right\}.$$

Proof. For a C^∞-function $\xi(t)$ on $[0, \infty)$ such that $\xi(t) = 1$ for $t < 1$, $= 0$ for $t > 2$ and $0 \leq \xi(t) \leq 1$ we introduce two functions on $\mathbf{R}^d$:

(1.3) $$\eta(x) = \xi(|x|) \quad \text{and} \quad \rho(x) = \eta(x) / \int \eta(x)\, dx,$$

and a cut-off function:

$$\phi(x) = 1/\mathrm{vol}(B_1) \text{ on } B_1 \quad \text{and} \quad = 0 \text{ outside } B_1,$$

where and in what follows B_R denotes an open ball of radius R with center the origin. For positive numbers λ, μ, ε we set

$$\eta_\mu(x) = \eta(\mu x),\ \rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon) \text{ and } \phi_\lambda(x) = \lambda^d \phi(\lambda x).$$

Denoting by G the fundamental solution of the laplacian, we define

$$G_{\varepsilon,\lambda} = G * (\delta - \phi_\lambda) * \rho_\varepsilon,$$

where $f * g$ denotes the convolution of f and g, and δ the Dirac function. The use of Fourier transformation asserts that $G_{\varepsilon,\lambda}$ is rapidly decreasing along with its all derivatives. In the course of the proof we also use the well-known inequality in the literature:

(1.4) $$\|f * g\|_r \leq \|f\|_p \|g\|_q \quad (1 \leq p,\ q,\ r \leq \infty \text{ and } 1/p + 1/q = 1 + 1/r)$$

and the lemma due to Renardy [15]: *Suppose that* $f \in L^r$ $(1 \leq r < \infty)$ *and further assume that* $\int f(x)\,dx = 0$ *in the case* $r = 1$. *Then, we have*

(1.5) $$\phi_\lambda * f \to 0 \quad in \quad L^r \quad as \quad \lambda \to 0.$$

We now define an operator $T_{\varepsilon,\lambda,\mu}$ of L^q_σ $(q \geq 1)$ into $\mathcal{V}$:

(1.6) $$u^j_{\varepsilon,\lambda,\mu} = (T_{\varepsilon,\lambda,\mu}u)^j = -\nabla_k\{\eta_\mu(G_{\varepsilon,\lambda} * \mathrm{rot}_{kj}u)\},$$

where $\mathrm{rot}_{kj}u = \nabla_k u^j - \nabla_j u^k$. A simple calculation leads to

$$u^j_{\varepsilon,\lambda,\mu} = \eta_\mu\{(\delta - \phi_\lambda) * \rho_\varepsilon * u^j\} - (\nabla_k \eta_\mu)(G_{\varepsilon,\lambda} * \mathrm{rot}_{kj}u)$$

and

(1.7) $$\begin{aligned} \nabla_i u^j_{\varepsilon,\lambda,\mu} &= \eta_\mu\{(\delta - \phi_\lambda) * \rho_\varepsilon * \nabla_i u^j\} \\ &\quad - \{\nabla_i \eta_\mu(\nabla_k G_{\varepsilon,\lambda} * \nabla_k u^j) + \nabla_k \eta_\mu(\nabla_i G_{\varepsilon,\lambda} * \mathrm{rot}_{kj}u)\} \\ &\quad - (\nabla_i \nabla_k \eta_\mu)(G_{\varepsilon,\lambda} * \mathrm{rot}_{kj}u) \equiv a_{ij} + b_{ij} + c_{ij}. \end{aligned}$$

The assertions (i) and (ii) immediately follow from the above two equalities. To prove (iii) we derive from (1.7)

$$\begin{aligned} D_{ij}(u_{\varepsilon,\lambda,\mu}) &= \eta_\mu\{(\delta - \phi_\lambda) * \rho_\varepsilon * D_{ij}(u)\} + (b_{ij} + b_{ji})/2 + (c_{ij} + c_{ji})/2 \\ &= A_{ij} + B_{ij} + C_{ij}. \end{aligned}$$

It is easy to see by (1.5) that $A_{ij} \to D_{ij}(u)$ in L^r. The use of (1.4) and the identity

$$\nabla_i \nabla_j u^k = \nabla_j D_{ki}(u) + \nabla_i D_{jk}(u) - \nabla_k D_{ij}(u) \tag{1.8}$$

guarantee us that $B_{ij} \to 0$ in L^r as $\mu \to 0$.

Our final goal is to show that $C_{ij} \to 0$ in L^r as $\mu \to 0$. To do so let us first remark that C_{ij} is represented as a linear combination of terms of the form $U = (\nabla_j \nabla_k \eta_\mu)(G_{\varepsilon,\lambda} * \nabla u)$. Let us assume $r \geq q$. The inequality (1.4) then leads to

$$\| U \|_r \leq C\mu^2 \| \nabla G_{\varepsilon,\lambda} \|_p \| u \|_q, \quad p \geq 1.$$

Thus, $\| U \|_r \to 0$ as $\mu \to 0$. If $r < q$, we use Hölder's inequality:

$$\| U \|_r \leq C\mu^{2-d/p} \left(\int_{|x| \geq 2/\mu} | \nabla G_{\varepsilon,\lambda} * u |^q dx \right)^{1/q},$$

where $1/p + 1/q = 1/r$. Application of (1.4) with $p = 1$ implies $\nabla G_{\varepsilon,\lambda} * u \in L^q$ and our assumption on q and r yields $2 - d/p \geq 0$. Consequently, $\| U \|_r \to 0$ as $\mu \to 0$. Q. E. D.

In this section we always assume

Ω an arbitrary domain in $\mathbf{R}^d$ $(d \geq 2)$,

H the closure of $\mathcal{V}(\Omega) = \{v \in \mathcal{V};\ \mathrm{supp}\, v \subset \Omega\}$ by norm $\| v \|$.

and

$Y_p(\mathbf{R}^d)$ the closure of $\mathcal{V}$ by norm $\| D(v) \|_p$ $(p \geq 1)$.

It is easy to see that $Y_p(\mathbf{R}^d)$ is imbedded in $L^p_{\mathrm{loc}}(\mathbf{R}^d)^d$. Therefore, we may introduce the Banach spaces which play important parts in the paper:

$V_p = Y_p(\mathbf{R}^d) \cap H$ equipped with norm $\| v \|_{V_p} = \| D(v) \|_p + \| v \|$

and, setting $V = V_2$,

$W_p = V_p \cap V$ equipped with norm $\| v \|_{W_p} = \| v \|_{V_p} + \| v \|_V$.

It is evident that every function in V_p vanishes outside of the closure $\bar{\Omega}$ of Ω. According to Lions [9, p.6], we can assert that V_p is separable for any $p \geq 1$ and further reflexive if $p > 1$ and that $V_p \subset H \subset V_p'$, where H is identified with its dual H', each space is dense in the following and the injections are one to one and continuous. These assertions hold true for W_p as well.

There are given two separable Banach spaces X and Y such that $X \subset Y \subset H$, where each space is dense in the following and the injections are one to one and continuous. Denoting by $\langle\ ,\ \rangle_X$ the duality between X' and X, it is easily verified that $\langle f, u\rangle_X = \langle f, u\rangle_Y$ for $u \in X$ and $f \in Y'$. So it will be allowed to write it as $\langle f, u\rangle$ without any confusion. In particular, $\langle f, u\rangle$ means the inner product in H if $u, f \in H$.

LEMMA 1.1. *Suppose that* $2 \leq d \leq 4$.

(i) *For all* $r \geq 1$ *we have* $V_r = \{u \in H\,;\, D_{ij}(u) \in L^r\ (1 \leq i, j \leq d)\}$.

(ii) *For all* $q, r \in [1, p]$ *such that* $q < d$ *we have* $V_p \cap V_1 \subset L^{q*} \cap V_r$ $(q^* = dq/(d-q))$.

More precisely, there exists a positive constant $C_{q,r}$ *such that*

$$\| v \|_{q^*}^q + \| \nabla v \|_r^r \leq C_{q,r}(\| D(v) \|_p^p + \| D(v) \|_1), \quad v \in V_p \cap V_1. \tag{1.9}$$

(iii) *If* Ω *is smooth and* $p \geq d/(d-1)$, *then* $v|_{\Omega} \in W_0^{1,p}(\Omega)^d$ *for all* $v \in V_p \cap V_1$ *where* $W_0^{1,p}(\Omega)$ *denotes the set of functions belonging to the usual Sobolev space* $W^{1,p}(\Omega)$ *such that* $\cdot|_{\partial\Omega} = 0$.

Proof. The assertion (i) is an easy consequence of Proposition 1.1. The use of interpolation inequality;

$$\| f \|_\nu \leq \| f \|_\lambda^\alpha \| f \|_\mu^\beta \quad (1 \leq \lambda \leq \nu \leq \mu < \infty) \tag{1.10}$$
$$\text{with} \quad \beta = \frac{1 - \lambda/\nu}{1 - \lambda/\mu} \quad \text{and} \quad \alpha + \beta = 1$$

and the Young inequality:

$$A^\alpha B^\beta \leq \alpha A + \beta B \quad \text{for } A, B \geq 0 \tag{1.11}$$

lead to

$$\| D(v) \|_r^r \leq \frac{r-1}{p-1} \| D(v) \|_p^p + \frac{p-r}{p-1} \| D(v) \|_1, \quad v \in C_0^\infty(\mathbf{R}^d)$$

for $1 \leq r \leq p$. Making use of (1.1) and (1.2), and keeping in mind (i) we obtain (1.9).

To prove (iii) we assume $v \in V_p \cap V_1$ and $p \geq d/(d-1)$. Then, (1.9) implies $v \in W^{1,p}(\mathbf{R}^d)^d$. Observing that $v = 0$ outside of $\bar{\Omega}$ and that Ω is smooth, we obtain $v|_{\partial\Omega} = 0$. Q. E. D.

LEMMA 1.2. (i) *Suppose* $p \in [2, d+2)$ *and let us set* $q = dp/(d+2)$. *Then, we have*

$$\| \phi \|_p^p \leq \| \phi \|^{p-q} \| \phi \|_{q*}^q, \quad \phi \in C_0^\infty(\mathbf{R}^d).$$

(ii) *Suppose* $p \in (2d/(d+2), 2) \cup [d+2, \infty)$. *Then, there exist positive constants* K, Λ *and* $\theta \in (0,1)$ *such that*

$$\| \phi \|_{p,B_{1/\lambda}} \le K\lambda^{-\theta}(\| \nabla\phi \|_p + \| \phi \|), \quad \phi \in C_0^\infty(\mathbf{R}^d) \tag{1.12}$$

for all $\lambda \in (0, \Lambda)$, *where* $\| \phi \|_{p,M} = \left(\int_M | \phi |^p \, dx \right)^{1/p}$.

Proof. Observing $q^* \ge p$ and applying (1.10) to $f = \phi$, we readily get (i). To prove (ii) we first assume $p \ge d+2$. Choose r so that $r^* > p > d > r > 1$ and set

$$\eta_n(x) = \eta(2^{1-n}\lambda x), \quad n = 1,2,\ldots.$$

Then, by virtue of (1.10) we have

$$\| \eta_n\phi \|_p \le \| \eta_n\phi \|^\alpha \| \eta_n\phi \|_{r^*}^\beta, \quad \beta = (p-2)r^*/p(r^*-2).$$

Hence, Hölder's inequality yields

$$\| \eta_n\phi \|_p \le C\left(\frac{2^n}{\lambda}\right)^{d\beta(1/r-1/p)} \| \nabla(\eta_n\phi) \|_p^\beta \tag{1.13}$$

for all $\phi \in C_0^\infty(\mathbf{R}^d)$ with $\| \phi \| = 1$. Choosing again r so close to d that

$$0 < \theta = d\beta(1/r - 1/p) < 1,$$

we obtain from (1.13) that

$$\begin{aligned} \| \eta_n\phi \|_p &\le C\left(\frac{2^n}{\lambda}\right)^\theta (\| \nabla(\eta_n\phi) \|_p + 1) \\ &\le C_1\lambda^{1-\theta} \| \phi \|_{p,B_n} + C_2\left(\frac{2^n}{\lambda}\right)^\theta (\| \nabla\phi \|_p + 1), \end{aligned} \tag{1.14}$$

where $B_n = \{x \,; |x| < 2^n/\lambda\}$ and $C_i (i = 1, 2)$ are positive constants not depending on λ and n.

Set

$$a_n = \| \phi \|_{p,B_n}, \quad \delta = C_1\lambda^{1-\theta} \text{ and } M = C_2\lambda^{-\theta}(\| \nabla\phi \|_p + 1).$$

Then, (1.14) becomes $a_{n-1} \le \delta a_n + 2^{n\theta}M$, and hence

$$a_0 \le \delta^n a_n + 2^\theta M(1 - 2^\theta\delta)^{-1} \le \delta^n a_n + 4M$$

for $\lambda < (4C_1)^{1/(\theta-1)} = \Lambda$. By passage to limit we get $a_0 \le 4M$. This concludes (1.12), provided $K = 4C_2$.

We now suppose $2d/(d+2) < p < 2$. By virtue of Hölder's inequality we have

$$\| \phi \|_{p,B_{1/\lambda}} \leq \lambda^{-\theta} \| \phi \|, \quad \theta = d(1/p - 1/2).$$

Our hypothesis implies $0 < \theta < 1$. Q. E. D.

Given $T > 0$ and a separable Banach space X equipped with norm $\| \cdot \|_X$, let us denote by $L^r(0, T; X)$ $(1 \leq r < \infty)$ the set of all functions $u(t)$ of the interval $(0, T)$ into X such that $\| u(t) \|_X^r$ is integrable over $(0, T)$. It then follows from theorem due to Pettis and Bochner (see Yosida [18]) that there exists a sequence of finitely valued functions $u_n(t)$ such that $u_n(t) \to u(t)$ for a.e. $t \in (0, T)$ in X and $u_n \to u$ in $L^r(0, T; X)$. By $L^\infty(0, T; X)$ we denote the set of all functions $u(t)$ such that $\| u(t) \|_X$ is essentially bounded in $(0, T)$. We use the abbreviation:

$$L^r_{\mathrm{loc}}(0, \infty; X) = \bigcup_{T>0} L^r(0, T; X) \quad (1 \leq r \leq \infty),$$

which is a Fréchet space. By $C(I; X)$ (resp. $C_w(I; X)$) we denote the set of continuous functions (resp. weakly continuous functions) of I into X.

It is not difficult to show that the space $L^p(0, T; V_q)$ $(p, q \geq 1)$ is separable and its dual is equal to $L^{p'}(0, T; V_q')$ $(1' = \infty)$, and hence it is reflexive if $p, q > 1$.

For a, b such that $0 \leq a < b$ we set

$$\mathscr{B}^p_{a,b} = L^p(a, b; V_p) \cap L^1(a, b; V_1). \quad p > 1, \tag{1.15}$$

which is Banach space equipped with norm

$$\| v \|_{a,b} = \left(\int_a^b \| v \|^p_{V_p} dt \right)^{1/p} + \int_a^b \| v \|_{V_1} dt. \tag{1.16}$$

Here $L^r(a, b; X)$ is defined with $(0, T)$ replaced by (a, b). By $\langle \, , \rangle_{a,b}$ we denote the duality between $\mathscr{B}^p_{a,b}$ and its dual $(\mathscr{B}^p_{a,b})'$. Then, we can prove

LEMMA 1.3. *The space* $C_0^\infty(0, T; V_p \cap V_1)$ *is dense in* $\mathscr{B}^p_{0,T}$.

Proof. Let $u \in \mathscr{B}^p_{0,T}$. Since V_p and V_1 are separable, we can find a sequence of finitely valued functions $u_n(t)$ such that $u_n(t) \to u(t)$ for a.e. $t \in (0, T)$ in $V_p \cap V_1$ and $u_n \to u$ in $\mathscr{B}^p_{0,T}$. Based on this fact, we may define the Bochner integral

$$(1.17)\qquad u_\varepsilon(t)=\rho_\varepsilon * u(t)=\int_0^T \rho_\varepsilon(s)u(t-s)\,ds,\quad t\in(\varepsilon,\,T-\varepsilon),$$

and prove that u_ε belongs to $C^\infty(\varepsilon,\,T-\varepsilon\,;\,V_p\cap V_1)$ and converges to u in $\mathscr{B}^p_{\delta,T-\delta}$ as $\varepsilon\to 0$ for all $\delta\in(0,\,T/2)$, where $\rho_\varepsilon(t)=\varepsilon^{-d}\rho(t/\varepsilon)$ (for $\rho(t)$ see (1.3)).

Let $\zeta_\delta\in C_0^\infty(0,\,T)$ be a function such that $0\le\zeta_\delta(t)\le 1$ for all t and $\zeta_\delta(t)=1$ for $t\in(\delta,\,T-\delta)$. It then easily follows that $\zeta_\delta u_\varepsilon\to\zeta_\delta u$ as $\varepsilon\to 0$ and $\zeta_\delta u\to u$ as $\delta\to 0$ in $\mathscr{B}^p_{0,T}$. This concludes the lemma. Q. E. D.

LEMMA 1.4. *Let $u\in\mathscr{B}^p_{0,T}$ with $u'=du/dt\in(\mathscr{B}^p_{0,T})'$, which always means that*

$$(1.18)\qquad \langle u',\,\phi\rangle_{0,T}=-\int_0^T\langle u,\,\phi'\rangle\,dt,\ \phi\in C_0^\infty(0,\,T\,;\,V_p\cap V_1).$$

If $p\ge 2$, we then have, after a possible modification of the value $u(t)$ on a set of measure zero,

$$(1.19)\qquad \|u(t)\|^2-\|u(s)\|^2=2\langle u',\,u\rangle_{s,t}\quad\textit{for all}\quad 0\le s<t\le T.$$

If we further suppose $u\in C_w([0,\,T]\,;\,H)$, then $u\in C([0,\,T]\,;\,H)$.

Proof. The space $L^\infty(0,\,T\,;\,V_p\cap V_1)$ is dense in $L^2(0,\,T\,;\,H)$ and hence so is $\mathscr{B}^p_{0,T}$ if $p\ge 2$. Observing the injection $\mathscr{B}^p_{0,T}\to L^2(0,\,T\,;\,H)$ is one to one and continuous, we have

$$\mathscr{B}^p_{0,T}\subset L^2(0,\,T\,;\,H)\subset(\mathscr{B}^p_{0,T})',$$

if $p\ge 2$, where the injection $L^2(0,\,T\,;\,H)\to(\mathscr{B}^p_{0,T})'$ is also one to one and continuous. The proof of the lemma will be thus achieved by a similar argument as in Temam [17, p. 260]. Defining u_ε by (1.17), we have

$$\int_0^T\langle u_\varepsilon',\,\phi\rangle\,dt=\langle u',\,\rho_\varepsilon*\phi\rangle_{0,T}\le C\|\rho_\varepsilon*\phi\|_{0,T}\le C\|\phi\|_{0,T}$$

and on the other hand

$$\int_0^T\langle u_\varepsilon',\,\phi\rangle\,dt=-\int_0^T\langle u_\varepsilon,\,\phi'\rangle\,dt\to\langle u',\,\phi\rangle_{0,T}\quad\text{as }\varepsilon\to 0$$

for all $\phi\in C^\infty(0,\,T\,;\,V_p\cap V_1)$ with supp $\phi\subset(\varepsilon,\,T-\varepsilon)$. By virtue of Lemma 1.3, we can conclude that $\{u_\varepsilon'\}$ is bounded in $(\mathscr{B}^p_{0,T})'$ and that

$$(1.20)\qquad u_\varepsilon\to u\quad\text{in}\quad\mathscr{B}^p_{\delta,T-\delta},$$

(1.21) $$u'_\varepsilon \to u' \quad \text{weakly}^* \text{ in } \quad (\mathscr{B}^p_{\delta,T-\delta})'$$

as $\varepsilon \to 0$, for all $\delta \in [0, T/2)$.

According to (1.20), we have

$$\| u_\varepsilon(t) \| \to \| u(t) \| \quad \text{in } L^1_{\text{loc}}(0, T).$$

Hence, we can extract a subsequence, again denoted by $\{u_\varepsilon\}$, of $\{u_\varepsilon\}$ so that

(1.22) $$\| u_\varepsilon(t) \| \to \| u(t) \| \quad \text{as} \quad \varepsilon \to 0 \quad \text{for all} \quad t \in (0, T) \setminus E,$$

where E is a subset of $(0, T)$ of measure zero.

Let $s, t \in (0, T) \setminus E$ and $s < t$. Integration of the equality

$$\frac{d}{d\tau} \| u_\varepsilon(\tau) \|^2 = 2 \langle u'_\varepsilon(\tau), u_\varepsilon(\tau) \rangle$$

over (s, t) leads to

$$\| u_\varepsilon(t) \|^2 - \| u_\varepsilon(s) \|^2 = 2 \langle u'_\varepsilon, u_\varepsilon \rangle_{s,t}.$$

Letting $\varepsilon \to 0$ here, we easily see (1.19), keeping in mind (1.20)~(1.22). Since the right-hand side of (1.19) is continuous in s and t, we get (1.19) for all $0 \leq s < t \leq T$, modifying, if necessary, the value of $u(t)$ on E. The latter half of the lemma easily follows from the continuity of $\| u(t) \|$. Q. E. D.

Finally, we describe a few statements about functional φ and operator B. Regarding the properties which are maintained by the functional (0.3), we are going to introduce a class of functionals on V_p. For each $t \geq 0$ we consider a functional $\varphi_t(u) = \varphi(t, u)$ on V_p, $p \geq 1$, possessing the properties (A.1)~(A.3):

(A.1) For each $t \geq 0$ φ_t is a proper, convex and lower-semicontinuous function on V_p such that $\varphi_t(0) = 0$.

(A.2) There exist positive constants μ_i and g_i $(i = 1,2)$ such that for all $t \geq 0$ and all $v \in W_p$

(1.23) $$\begin{aligned} &\varphi_t(u) \geq \mu_1 \| D(u) \|_p^p + g_1 \| D(u) \|_1, \quad u \in V_p \cap V_1, \\ &| \langle \partial\varphi_t(u), v \rangle | \leq \mu_2 \int_\Omega | D(u) |^{p-1} | D(v) | \, dx + g_2 \| D(v) \|_1, \; u \in \mathscr{D}(\partial\varphi_t), \end{aligned}$$

where $\partial\varphi_t(u)$ denotes the set of subgradients of φ at u:

$$\partial\varphi_t(u) = \{ w \in W'_p ;\, \varphi_t(v) - \varphi_t(u) \geq \langle w, v - u \rangle, \quad v \in W_p \},$$

$\mathscr{D}(\partial\varphi_t)$ the effective domain of $\partial\varphi_t$:

$$\mathscr{D}(\partial\varphi_t) = \{u \in W_p\,;\, \partial\varphi_t(u) \neq \phi\},$$

and hence $\partial\varphi_t$ may be regarded as a mapping of $\mathscr{D}(\partial\varphi_t)$ into the set of subsets of W_p'.

(A.3) There exists a positive constant $\varepsilon(h)$ depending on $h \geq 0$ such that $\varepsilon(h) \to 0$ as $h \to 0$, and for all $s,\, t \geq 0$ and all $v \in V_p \cap V_1$

$$|\varphi(s, v) - \varphi(t, v)| \leq \varepsilon(|s-t|)(\|D(v)\|_p^p + \|D(v)\|_1).$$

It may be easily shown that $0 \in \mathscr{D}(\partial\varphi_t) \subset W_p \cap V_1$ and

$$\varphi_t(u) \leq \mu_2 \|D(u)\|_p^p + +g_2\|D(u)\|_1, \quad u \in \mathscr{D}(\partial\varphi_t).$$

For a future convenience we set

(1.24) $\Phi_p =$ the set of φ_t, $t \geq 0$, satisfying (A.1)~(A.3).

It is well-known (see Brezis [3]) that $\varphi(t, v(t))$ is measurable function of $t \geq 0$ if $v \in L^p(0, T\,;\, V_p)$ and a mapping $v \to \int_0^T \varphi(t, v(t))dt$ is convex and lower-semicontinuous.

Finally, we describe two lemmas concerning operator $B(u) = u \cdot \nabla u$.

LEMMA 1.5. *Suppose $d = 3$. For each $p > 6/5$ there exists a positive constant γ_p such that*

$$|\langle u_1 \cdot \nabla u_2, v\rangle| \leq \gamma_p (\|u_1\|\,\|u_2\|)^{a/2} (\|\nabla u_1\|_l \|\nabla u_2\|_l)^{b/2} \|\nabla v\|_q \tag{1.25}$$

for all u_1, u_2, v in $\mathscr{V}$, where $a + b = 2$ and

$$b = p-1, \quad l = p, \quad q = \frac{6p}{(5p-6)(p-1)} \quad \textit{when } 6/5 < p < 11/5,$$
$$b = \frac{6}{5p-6}, \quad l = p, \quad q = p \quad \textit{when } 9/5 \leq p < 3,$$
$$b = 1, \quad l = \frac{6p}{5p-6}, \quad q = p \quad \textit{when } 12/5 \leq p < \infty.$$

When $d = 2$, the inequality (1.25) is valid for all $p > 1$, provided that

$$b = p-1, \quad l = p, \quad q = \frac{p}{(p-1)^2} \quad \textit{when } 1 < p < 2,$$
$$b = 1, \quad l = p', \quad q = p \quad \textit{when } 2 \leq p < \infty,$$

where $p' = p/(p-1)$.

Proof. We start with case $d = 3$.

(i) Let $p \in (6/5, 11/5)$. By integration by part we have, using Hölder's inequality,

$$|\langle u_1 \cdot \nabla u_2, v\rangle| \leq C \| u_1 \|_{2q'} \| u_2 \|_{2q'} \| \nabla v \|_q, \; q' = q/(q-1). \tag{1.26}$$

Applying (1.10) with $\lambda = 2$, $\mu = p^* = 3p/(3-p)$ and $\nu = 2q'$, we get, using (1.2),

$$\| u_i \|_{2q'} \leq C \| u_i \|^{a/2} \| \nabla u_i \|_p^{b/2}, \quad i = 1, 2.$$

Substituting these into (1.26) leads to (1.25).

(ii) Let $p \in [9/5, 3)$. Take $q = p$ in (1.26). Keeping in mind that $2 < 2p' \leq p^*$, we obtain analogously as in (i)

$$\| u_i \|_{2p'} \leq C \| u_i \|^{\alpha} \| \nabla u_i \|_p^{\beta},$$

where $\alpha + \beta = 1$ and $\beta = 3/(5p-6)$. Combining this with (1.26) $(q = p)$, we arrive at (1.25).

(iii) Let $p \in [12/5, \infty)$. Since $2 < 2p' < r = 2p/(p-2)$ and $1/r = 1/l - 1/3$, we have

$$\| u_i \|_{2p'} \leq C \| u_i \|^{1/2} \| \nabla u_i \|_l^{1/2}.$$

Inserting this into (1.26) with $q = p$ leads to (1.25).

Exactly as above we can show (1.25) for the case $d = 2$. Q. E. D.

The following lemma is an immediate consequence of Proposition 1.1 and the previous lemma.

LEMMA 1.6. *Suppose that $d = 3$ and $u \in \mathscr{B}^p_{0,T} \cap L^\infty(0, T; H)$. Then, $B(u) = u \cdot \nabla u$ is contained in $L^{r'}(0, T; V_q')$, where*

$$r = p, \quad q = q(p) = \begin{cases} 6p/\{(5p-6)(p-1)\}, & p \in (6/5, 11/5) \\ p, & p \in [11/5, \infty) \end{cases} \tag{1.27}$$

$$(\text{or } r' = p(5p-6)/6, \quad q = p, \quad p \in [9/5, 11/5)).$$

§2. Results and remarks

THEOREM 1 (Existence of weak solutions). *Suppose that Ω is a domain in $\mathbf{R}^3$,* *that* φ_t *is contained in the set* Φ_p, $p > 6/5$, *which appears in* (1.24), *and that the prescribed data* u_0 *and* f *satisfy*

$$u_0 \in H \quad \text{and} \quad f \in L^2_{\mathrm{loc}}(0, \infty; H). \tag{2.1}$$

There then exists a weak solution, i.e., *a vector field* u *satisfying*

$$u \in \bigcup_{T>0} \mathscr{B}^p_{0,T} \cap C_w([0, T]; H) \quad (\mathscr{B}^p_{0,T} = L^p(0, T; V_p) \cap L^1(0, T; V_1)) \tag{2.2}$$

with a derivative $u'(t) = du(t)/dt$:

$$u' \in \{\bigcup_{T>0} \mathscr{B}^p_{0,T} \cap L^p(0, T; V_q)\}' \quad \text{in the sense (1.18)}, \tag{2.3}$$

the initial condition

$$u(0) = u_0, \tag{2.4}$$

the evolutional inequality

$$\int_0^T \langle v', v - u\rangle\, dt - \frac{1}{2}(\| v(T) - u(T)\|^2 - \| v(0) - u_0\|^2) + \int_0^T \langle B(u), v\rangle\, dt + \int_0^T \{\varphi(t, v) - \varphi(t, u)\}\, dt \geq \int_0^T \langle f, v - u\rangle\, dt \tag{2.5}$$

for all $T > 0$ *and all* $v \in W^p_{0,T}$:

$$W^p_{0,T} = \{v \in \mathscr{B}^p_{0,T} \cap L^p(0, T; V_q) \cap C_w([0, T]; H);\ v' \in (\mathscr{B}^p_{0,T})'\} \tag{2.6}$$

and the energy inequality

$$\frac{1}{2}\| u(t)\|^2 + \int_0^t \varphi(\tau, u)\, d\tau \leq \frac{1}{2}\| u_0\|^2 + \int_0^t \langle f, u\rangle\, d\tau \quad \text{for all} \quad t > 0, \tag{2.7}$$

where $q = q(p)$ *is the same as in* (1.27). *In particular,*

$$u \in L^p(\Omega \times (0, T)) \quad \text{for any } T > 0 \text{ when } 2 \leq p < 5. \tag{2.8}$$

COROLLARY 1 (Existence of strong solutions). *Suppose* $p \geq 2$ *in Theorem* 1 *and let* u *be a weak solution satisfying*

$$u \in L^q_{\mathrm{loc}}(0, \infty; V_p) \quad \text{with } q = q(p) \text{ from (1.27)}. \tag{2.9}$$

Then, it is a strong solution, i.e., a weak solution possessing the further properties:

$$\text{(i) } u \in C([0, T]; H), \quad \text{(ii) } u' \in (\bigcup_{T>0} \mathscr{B}^p_{0,T})', \tag{2.10}$$

$$(2.11)\quad \langle u', v-u\rangle_{0,T} + \int_0^T \langle B(u), v-u\rangle\,dt + \int_0^T \{\varphi(t,v)-\varphi(t,u)\}\,dt$$
$$\geq \int_0^T \langle f, v-u\rangle\,dt \quad \textit{for all } T>0 \textit{ and all } v \in \mathscr{B}^p_{0,T}$$

and the energy inequality of strong form

$$(2.12)\qquad \frac{1}{2}\|u(t)\|^2 + \int_s^t \varphi(\tau,u)\,d\tau \leq \frac{1}{2}\|u(s)\|^2 + \int_s^t \langle f,u\rangle\,d\tau$$

for all $0 \leq s < t$, *where* $\langle\ ,\ \rangle_{0,T}$ *denotes the duality between* $\mathscr{B}^p_{0,T}$ *and its dual. Particularly, if* $p \geq 11/5$, *there then exists a strong solution.*

Proof. If $p \geq 11/5$, then (2.3) implies (ii) of (2.10). Suppose $p < 11/5$. Application of (1.25) yields

$$\int_0^T \|B(u)\|^{p'}_{V_p'}\,dt \leq \gamma_p \sup_{0\leq t\leq T}\|u(t)\|^{ap'}\int_0^T \|\nabla u\|_p^{bp'}\,dt,$$

from which (ii) of (2.10) follows (see (4.3)). Here, $b = 6/(5p-6)$ and $p' = p/(p-1)$. Then, (i) of (2.10) is an easy consequence of Lemma 1.4.

For any $v \in C^1([0,T]; V_p \cap V_1)$ it follows from Lemma 1.4 that

$$(2.13)\quad \int_0^T \langle v', v-u\rangle\,dt \leq \langle u', v-u\rangle_{0,T}$$
$$+\frac{1}{2}(\|u(T)-v(T)\|^2 - \|u_0 - v(0)\|^2),$$

and hence we have (2.11) for such v. Let $v \in \mathscr{B}^p_{0,T}$. We make an extension of $v(t)$ so that $v(t)=0$ for $t<0$ and for $t>T$, and define a mollifier

$$(2.14)\qquad v_\varepsilon(t) = \int_{-\infty}^{\infty} \rho_\varepsilon(s) v(t-s)\,ds,$$

which belongs to $C^1([0,T]; V_p \cap V_1)$ and converges to v in $\mathscr{B}^p_{0,T}$ as $\varepsilon \to 0$. Inserting $v = v_\varepsilon$ in (2.11) and letting $\varepsilon \to 0$, we obtain (2.11) for all $v \in \mathscr{B}^p_{0,T}$ and all $T>0$. In fact, since φ_t is convex, we have

$$(2.15)\qquad \varphi(t, v_\varepsilon(t)) \leq \int_{-\infty}^{\infty} \rho_\varepsilon(s)\varphi(t-s, v(t-s))\,ds$$
$$+\int_{-\infty}^{\infty} \rho_\varepsilon(s)\{\varphi(t, v(t-s)) - \varphi\,(t-s), v(t-s))\}\,ds = \mathrm{I}_\varepsilon(t) + \mathrm{II}_\varepsilon(t).$$

Keeping in mind that $\varphi(t, v(t))$ is integrable on $(0, T)$, we get $I_\varepsilon(t) \to \phi(t, v(t))$ in $L^1(0, T)$. An elementary calculation gives us

$$\int_0^T |\, \mathrm{II}_\varepsilon(t) \,|\, dt \le \int_{-\infty}^{\infty} \rho_\varepsilon(s)\, ds \int_{-s}^{T} |\, \varphi(\tau + s, v(\tau)) - \varphi(\tau, v(\tau)) \,|\, d\tau.$$

Employing the Lebesgue theorem, we can derive from (A.3) that

$$\lim_{s \to 0} \int_{-s}^{T} |\, \varphi(\tau + s, v(\tau)) - \varphi(\tau, v(\tau)) \,|\, d\tau = 0,$$

which proves $\mathrm{II}_\varepsilon(t) \to 0$ in $L^1(0, T)$ and hence (2.15) yields

$$\limsup_{\varepsilon \to 0} \int_0^T \varphi(t, v_\varepsilon(t))\, d\tau \le \int_0^T \varphi(t, v(t))\, dt.$$

The inequality (2.12) is an easy consequence of (2.11) and Lemma 1.4. Q. E. D.

COROLLARY 2 (Uniqueness of strong solutions). *Suppose in Theorem* 1 *that* φ_t *is written in the form*

$$\varphi_t(v) = \hat{\varphi}_t(v) + \int_\Omega \mu(t) \,|\, D(v) \,|^2\, dx \tag{2.16}$$

where $\hat{\varphi}_t \in \Phi_r$, $r \le 1$, *and* $\mu \in C([0, \infty), L^\infty(\Omega))$ *satisfying* $\mu \ge \mu_0$ *for a positive constant* $\mu_0 > 0$. *Then, we have*:

(i) $\varphi_t \in \Phi_p$ *with* $p = \max(2, r)$.

(ii) *Let* u_* *be a weak solution and* u *be a strong solution satisfying* (2.10) *and* (2.11), *and further assume that* $u \in L^{2q/(2q-3)}(0, T; V_q)$ *for* $q = q(p)$ *from* (1.27) *and for all* $T > 0$. *Then,* $u = u_*$.

Proof. (i) If $p \ge 2$, then $|\, D(u) \,|\,|\, D(v) \,| \le (|\, D(u) \,|^{p-1} + 1) \,|\, D(v) \,|$. If $p < 2$, we have, using (1.11),

$$\begin{aligned} |\, D(u) \,|^{p-1} \,|\, D(v) \,| &= (|\, D(u) \,|\,|\, D(v) \,|)^{p-1} \,|\, D(v) \,|^{2-p} \\ &\le (p-1) \,|\, D(u) \,|\,|\, D(v) \,| + (2-p) \,|\, D(v) \,|. \end{aligned}$$

Consequently, (i) follows from (1.23).

(ii) It is evident that $p \ge 2$ leads to $2q/(2q-3) \ge p$. Therefore, we have $u \in L^p(0, T; V_q)$ and hence it follows from (ii) of (2.10) that u is in $W^p_{0,T}$ for $T > 0$. We choose $v = u$ as a test function in the variational inequality (2.5) with u and T replaced by u_* and t, and get

$$(2.17)\qquad \int_0^t \{\langle u', u - u_*\rangle + \langle B(u_*), u\rangle + \hat{\varphi}(\tau, u) - \hat{\varphi}(\tau, u_*)\}\, d\tau$$

$$\geq \frac{1}{2}\| u(t) - u_*(t) \|^2 + \int_0^t \{\langle 2\mu D(u_*), D(u - u_*)\rangle + \langle f, u - u_*\rangle\}\, d\tau.$$

Inserting $v = u_*$ into (2.11) and adding this to (2.17), we obtain

$$(2.18)\qquad \| w(t) \|^2 + 2\mu_0 \int_0^t \| \nabla w \|^2\, d\tau \leq 2\int_0^t \langle B(w), u\rangle\, d\tau,\ w = u - u^*,$$

from which we are going to derive $w(t) = u(t) - u_*(t) = 0$ for every t. To do so, we use (1.2), (1.10) $(2 < 2q' < 6)$ and (1.11) to get the following:

$$(2.19)\qquad \text{LHS of (2.18)} \leq 2\int_0^t \| \nabla u \|_q \| w \|_{2q'}^2\, d\tau$$

$$\leq 2\int_0^t \| \nabla u \|_q \| w \|^{2\alpha} \| w \|_6^{2\beta}\, d\tau \leq 2\Big(\eta \int_0^t \| w \|_6^2\, d\tau\Big)^{\beta} \Big(\eta^{-\beta/\alpha}\int_0^t \| \nabla u \|_q^{1/\alpha} \| w \|^2\Big)\, d\tau\Big)^{\alpha}$$

$$\leq 2\beta\eta \int_0^t \| w \|_6^2\, d\tau + 2\alpha\eta^{-\beta/\alpha}\int_0^t \| \nabla u \|_q^{1/\alpha} \| w \|^2\, d\tau$$

$$\leq 2\mu_0 \int_0^t \| \nabla w \|^2\, d\tau + 2\alpha\eta^{-\beta/\alpha}\int_0^t \| \nabla u \|_q^{1/\alpha} \| w \|^2\, d\tau,$$

where $d = 1 - 3/2q$, $\beta = 1 - \alpha$ and $\eta = \mu_0/\beta S_2^2$. From this it follows that

$$\| w(t) \|^2 \leq C\int_0^t \| \nabla u \|_q^{1/\alpha} \| w \|^2\, d\tau.$$

Keeping in mind that $\| \nabla u \|_q^{1/\alpha} \in L^1(0, T)$, we conclude that $u(t) = u_*(t)$ for all t. Q. E. D.

COROLLARY 3 (Energy decay). *Let u be a weak solution which is obtained in Theorem 1. Then, the following statements hold.*

(i) *If $\in L^1(0, \infty; H)$ and if u satisfies (2.12), then $\| u(t) \| \to 0$ as $t \to \infty$.*

(ii) *If f satisfies $\| f(t) \|_3 \leq g_1/S_1$ for all $t \geq 0$, then $\| u(t) \| \leq \| u_0 \|$ for all $t \geq 0$, where S_1 and g_1 are constants appearing in (1.2) and (1.23), respectively.*

(iii) *Assume that u is a strong solution satisfying (2.9) and $u' \in L^r(0, \infty; V_p' \cap L^3(\Omega))$ for some $r \geq p'$. If f satisfies $\| f(t) \|_3 < g_1/S_1$ for all $t \geq T_0$, then there exists $T_1 \geq T_0$ such that $u(t) = 0$ for all $t \geq T_1$.*

Proof. (i) From (2.12) with $s = 0$ it follows by using Gronwall's lemma that

$$\| u(t) \|^2 + 2 \int_0^t \varphi(\tau, u)\, d\tau \leq \text{const.} \quad \text{for all } t > 0, \tag{2.20}$$

which implies $u \in \mathscr{B}^p_{0,\infty} \cap L^\infty(0, \infty; H)$. Hence, $u(t) \in V_p \cap V_1$ for a.e. $t > 0$. Applying (1.9) with $v = u(t)$ and $q = 6/5$, we obtain $u \in L^{6/5}(0, \infty; H)$ since $q^* = 2$. Therefore, the proof of (i) will be achieved by carrying out the same device as in Miyakawa-Sohr [11].

(ii) Using (1.2) and (1.23), we can derive from (2.7)

$$\frac{1}{2} \| u(t) \|^2 + \int_0^t \{\mu_1 \| \nabla u \|_p^p + (g_1 - S_1 \| f \|_3) \| D(u) \|_1\}\, d\tau \leq \frac{1}{2} \| u_0 \|^2,$$

which implies (ii).

(iii) After a simple calculation we obtain from (2.11) that

$$\varphi(t, u(t)) \leq \langle f(t) - u'(t), u(t) \rangle \text{ for a.e. } \quad t \geq 0. \tag{2.21}$$

On the other hand it easily follows from the assumption that there exists $T_1 \geq T_0$ such that $\| u'(T_1) \|_3 + \| f(T_1) \|_3 \leq g_1 / S_1$ and (2.21) is valid for $t = T_1$. Inserting $t = T_1$ into (2.21), we readily obtain $\varphi(T_1, u(T_1)) \leq g_1 \| D(u(T_1)) \|_1$, and hence $u(T_1) = 0$. It is easy to see that u is a weak solution for $t \geq T_1$ with initial data $u(T_1) = 0$. Thus, part (ii) guarantees that $u(t) = 0$ for all $t \geq T_1$. Q. E. D.

THEOREM 2 (Case of exterior domain). *Suppose that the complement of Ω is compact and that $\varphi(u) = \mu \| D(u) \|_p^p + g \| D(u) \|_1$ with $p \geq 9/5$ and positive constants μ, g. Then, for any data* (2.1) *there exists a weak solution u satisfying the energy inequality of strong form*

$$\frac{1}{2} \| u(t) \|^2 + \int_s^t \{p\mu \| D(u) \|_p^p + g \| D(u) \|_1\}\, d\tau \leq \frac{1}{2} \| u(t) \|^2 + \int_s^t \langle f, u \rangle\, d\tau \tag{2.22}$$

for $s = 0$, a.e. $s > 0$ and all $t \geq s$.

In the last theorem we consider a Bingham fluid with variable viscosity μ and yield limit g, which is occupied in a bounded and smooth domain Ω in $\mathbf{R}^3$. We recall that V_p ($p \geq 3/2$) is identified with the closure of $\mathscr{V}(\Omega)$ by norm $\| \nabla v \|_p$ (see Lemma 1.1 (iii)). Set

$$\varphi(t, u) = \int_\Omega \{\mu(t) \mid D(u) \mid^2 + g(t) \mid D(u) \mid\}\, dx \quad \text{for} \quad u \in V. \tag{2.23}$$

For prescribed data u_0 and f:

$$u_0 \in V \quad \text{and} \quad f \in W^{1,1}_{\text{loc}}(0, \infty ; H) \tag{2.24}$$

we consider the problem: *To find a strong solution satisfying the evolutional inequality*

$$\langle u'(t) + B(u(t)), v - u(t)\rangle + \varphi(t, v) - \varphi(t, u(t)) \geq \langle f(t), v - u(t)\rangle, \tag{2.25}$$

for $v \in V$ *and for* a.e. $t > 0$, *and the initial condition*

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.26}$$

Before stating the theorem we introduce two function spaces $\mathcal{M}$ and $\mathcal{G}$ in which μ and g are contained, respectively. To do so, for $b > 6$ we define a and α as follows:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2} \quad \text{and} \quad \frac{1}{a} + \frac{1}{3} = \frac{1}{\alpha} + \frac{1}{2}. \tag{2.27}$$

It is obvious that $2 < a < 3$, $1/\alpha + 1/b = 1/3$ and hence $3 < \alpha < 6$. Then, we define

$$\mathcal{M} = \{\mu \in C([0, \infty) ; W^{1,\alpha}(\Omega)) ; \mu' \in L^2_{\text{loc}}(0, \infty ; L^b(\Omega))\},$$
$$\mathcal{G} = W^{1,2}_{\text{loc}}(0, \infty ; L^2(\Omega)).$$

Denoting by γ_0, γ_1 and c_0 positive constants such that

$$|\langle B(u), v\rangle| \leq \frac{\gamma_0}{8} \| \nabla u \|^2 \| v \|_3, \quad \| v \|_3^4 \leq c_0 \| v \|^2 \| \nabla v \|^2 \tag{2.28}$$

and

$$|\langle B(u), v\rangle| \leq \frac{1}{8} (\eta \| \nabla u \|^2 + 4\gamma_1 \eta^{-3} \| u \|^2) \| \nabla v \|, \quad \eta > 0 \tag{2.29}$$

for all u, $v \in V$, and setting for all $T > 0$

$$A_T = \Big(\| u_0 \|^2 + \int_0^T \| f \| \, dt\Big) \exp\Big(\int_0^T \| f \| \, dt\Big),$$

$$M_T = C\mu_1\mu_0^{-2}\big(\sup_{0 \leq t \leq T} \| \nu(t) \nabla\mu(t) \|_\alpha^2 + 1\big) \int_0^T \| \nu\mu' \|_b^2 \, dt,$$

$$G_T = \int_0^T \| \sqrt{\nu} g' \|^2 \, dt,$$

$$I_T = \left\{\| f(0) - \chi \|^2 + \int_0^T \| f' \| \, dt + (\max_{0 \le t \le T} \| f(t) \|^2 + g_1^2) \, M_T + G_T\right\}$$
$$\times \exp\left(\int_0^T \| f' \| \, dt + \gamma_1\mu_0 + M_T\right),$$
$$J_T = M_T \exp\left(\int_0^T \| f' \| \, dt + \gamma_1\mu_0 + M_T\right)$$

and

$$E_T = (18\mu_0^{\lambda-2} A_T^{1+\lambda} J_T)^{1/\lambda} + 18\mu_0 A_T J_T + \{18A_T(\max_{0 \le t \le T} \| f(t) \|^2 + I_T)\}^{1/2}$$

with $\nu = 1/\mu$, $\lambda = 3/\alpha - 1/2$, positive constants $\mu_i (i = 0, 1)$ and some positive constant C depending only on α and Ω, we can state the last theorem.

THEOREM 3. *Let Ω be a bounded and smooth domain in $\mathbf{R}^3$ and let μ_i, g_i $(i = 0, 1)$ be positive constants. Suppose that $\mu \in \mathcal{M}$, $g \in \mathcal{G}$, $\mu_0 \le \mu \le \mu_1$ and $g_0 \le g \le g_1$, and that u_0 and f satisfy* (2.24) *and*

(2.30) $$\chi - B(u_0) \in \partial\varphi(0, u_0) \text{ for some } \chi \in H.$$

If one of the following conditions

(2.31) $$\text{(i) } \mu_0^5/\gamma_0^4 > c_0 A_T E_T \quad \text{with} \quad \gamma_1 = 0 \quad \text{and} \quad \text{(ii) } \mu_0^3 > T^{1/2} E_T$$

is fulfilled, then we can find a strong solution u satisfying (2.25), (2.26) *and*

(2.32) $$\mu_0 \| \nabla u(t) \|^2 \le E_T,$$
$$\| u'(t) \|^2 + \frac{\mu_0}{4} \int_0^T \| \nabla u' \|^2 \, dt \le I_T + J_T(\mu_0 E_T + \mu_0^{\lambda-2} A_T^{\lambda} E_T^{2-\lambda})$$

for all $t \le T$. Moreover, the u is unique in the sense that every weak solution is equal to u. In particular, if f is in $L^\infty_{\mathrm{loc}}(0, \infty\,; L^3(\Omega)^3)$, the following

(2.33) $$\sup_{0 \le t \le T} \| \nabla u(t) \|_q \; (2 \le q \le 6) \text{ and}$$
$$\int_0^T \| \nabla u \|_q^p \, dt \left(q > 6, \; \frac{1}{p} = \frac{1}{4}\left(1 - \frac{6}{q}\right)\right)$$

are bounded from above by positive continuous functions of the arguments

$$\| \chi \|, \; \mu_0, \; \mu_1, \; g_1, \; \int_0^T (\| f \| + \| f' \|) dt,$$
$$\sup_{0 \le t \le T} \| \nu \nabla u(t) \|_\alpha, \; \int_0^T \| \nu\mu' \|_b^2 \, dt, \; \int_0^T \| \sqrt{\nu} \, g' \|^2 \, dt.$$

Remark 1. Suppose $d=2$. Reviewing Lemma 1.5 and the procedure carried out in Section 3, we obtain a new version of Theorem 1: *Let* $p>1$. *For any data* (2.1) *there exists a weak solution* $u(t)$ *satisfying* (2.2)~(2.7) *for all* $T>0$ *and all* $v\in W_{0,T}^{p}$, *where* $q=p/(p-1)^2$ *if* $1<p<2$ *and* $q=p$ *if* $p\geq 2$. Accordingly, it follows from Corollaries 1 and 2, by taking $q=p$ and applying the inequality $\| w\|_{2p'}\leq$ const. $\| w\|^{1/p'}\|\nabla w\|^{1/p}$ in the place of (2.19), *that there exists exactly one strong solution if* $p\geq 2$ *and* φ_t *is written in the form* (2.16).

Remark 2. The conclusion of Theorem 2 remains valid even if $\varphi(u)$ is replaced by

$$\sum_{j=1}^{N}\mu_j\| D(u)\|_{p_j}^{p_j}\quad\text{with}\quad \max(p_j)\geq 9/5\quad\text{and}\quad \min(p_j)=1.$$

Remark 3. Let φ be a functional not depending on t and satisfying (A.1)~(A.2) for $p>6/5$, provided W_p is replaced by $V_p\cap V_{9/5}$. Then, it is easily shown that for any $f\in H$ there exists a solution $u\in V_p\cap V_1$ to the stationary problem:

$$\langle B(u),v\rangle+\varphi(v)-\varphi(u)\geq\langle f,v-u\rangle,\quad v\in V_q\cap V_1,\tag{2.34}$$

where $q=3p/(5p-6)$ for $p\in(6/5,9/5)$ and $q=p$ for $p\geq 9/5$. In fact, observing (1.26) with $2q'=p^*(6/5<p<9/5)$ and (1. 9) ($q=6/5$), we can find $u_\xi\in\mathcal{D}(\partial\varphi)\subset V_p\cap V_{9/5}$ satisfying $f\in B(u_\xi)+e_\xi(u_\xi)+\partial\varphi(u_\xi)$ as in Proposition 3.1, where $e_\xi(v)=-\xi\nabla(|\nabla v|^{-1/5}\nabla v)$ and ξ is a positive constant. A desired solution u is given as a limit of u_ξ (cf. Lemma 1.5).

Remark 4. Suppose $d=2$. For any $b>2$ we define a and α by $1/a+1/b=1/2$ and $\alpha=a>2$. Then, Theorem 3 remains valid without condition (2.31). More precisely, under the same hypotheses as in Theorem 3 we can prove that if u_0 and f satisfy (2.30), then there exists one and only one solution of (2.25)-(2.26) in $t\leq T$ satisfying

$$u\in L^\infty(0,T;V_q)\text{ for any }q\geq 2,\text{ and }u'\in L^2(0,T;V)\cap L^\infty(0,T;H).$$

§3. Regularized problem

For positive numbers λ and ξ we define an operator $e_{\lambda,\xi}$ of $V=V_2$ into its dual V' by

$$\langle e_{\lambda,\xi}(u),v\rangle=\xi\langle\exp(\lambda\|\nabla u\|^c)\nabla u,\nabla v\rangle\quad\text{for all }v\in V\text{ with }c>4.$$

It is easy to see that $e_{\lambda,\xi}$ is monotone and $B(u_n) = u_n \cdot \nabla u_n \to u \cdot \nabla u$ weakly in V' if $u_n \to u$ weakly in V. Accordingly, $A = e_{\lambda,\xi} + B : u \to e_{\lambda,\xi}(u) + B(u)$ is a pseudo-monotone operator of V into V', i.e., if $\| u \|_V \leq 1$, then $\| A(u) \|_{V'}$ is bounded, and if $u_j \to u$ weakly in V as $j \to \infty$ and $\limsup_{j\to\infty} \langle A(u_j), u_j - u\rangle \leq 0$, then $\liminf_{j\to\infty} \langle A(u_j), u_j - v\rangle \geq \langle A(u), u - v\rangle$ for all $v \in V$. It is readily seen that the A may be regarded as a pseudo-monotone operator of $W_p = V_p \cap V$ into W_p'.

PROPOSITION 3.1. *Let $\varphi \in \Phi_p$, $p > 6/5$, which does not depend on t, let $L_{\lambda,\xi}$ be a mapping from $\mathfrak{D}(\partial\varphi) = \{v \in W_p ; \partial\varphi(v) \neq \phi\} \subset W_p \cap V_1$ into the set of subsets of W_p':*

$$L_{\lambda,\xi}(v) = e_{\lambda,\xi}(v) + B(v) + \partial\varphi(v)$$

and let

$$Y_{\xi,n} = (\gamma^{-4} n\xi^3)^{1/4} \quad \textit{with } \chi = \gamma_2 \textit{ from } (1.25).$$

Then, the following statements hold.

(i) *For any $u \in W_p'$ there exists $v \in \mathfrak{D}(\partial\varphi)$ such that*

$$u \in \Big(1 + \frac{1}{n} L_{\lambda,\xi}\Big)(v) \quad (n = 1,2,\ldots). \tag{3.1}$$

(ii) *Let v_i $(i = 1,2)$ be solutions of* (3.1) *with $u = u_i \in H$. Then, we have*

$$\| \nabla v_i \| \leq Y_{\xi,n} \quad \textit{and} \quad \| \delta v \|^2 + \frac{\xi}{n} \| \nabla\delta v \|^2 \leq 2 \| \delta u \|^2, \tag{3.2}$$

if $u_i \in H_{\lambda,\xi,n} = \{u ; \| u \| \leq M_{\lambda,\xi,n}\}$, where $\delta v = v_2 - v_1$, $\delta u = u_2 - u_1$ and

$$M_{\lambda,\xi,n} = \Big(\frac{2\xi}{n}\Big)^{1/2} Y_{\xi,n} \exp\Big(\frac{\lambda}{2} Y_{\xi,n}^c\Big).$$

Proof. (i) The existence of v follows from Theorem 8.5 of Lions [9, Ch. 2]. In fact, (1.23) implies $c_1 \| \nabla v \|_p^p \leq \varphi(v)$ and by definition we have $\langle e_{\lambda,\xi}(v), v\rangle \geq \xi \| \nabla u \|^2$, and hence, it follows that the operator $\Big(1 + \frac{1}{n} L_{\lambda,\xi}\Big)$ is coercive over W_p:

$$\frac{\langle v + n^{-1}A(v), v\rangle + n^{-1} \varphi(v)\rangle}{\| v \|_{W_p}} \to \infty \quad \text{if } \| v \|_{W_p} \to \infty.$$

(ii) The relation (3.1) yields

$$\|v\|^2 + \frac{2}{n}\{\langle e_{\lambda,\xi}(v), v\rangle + \varphi(v)\} \leq \|u\|^2, \tag{3.3}$$

and hence $\langle e_{\lambda,\xi}(v), v\rangle \leq n\|u\|^2/2$. If $u \in H_{\lambda,\xi,n}$, then

$$\|\nabla v\|^2 \exp(\lambda\|\nabla v\|^c) \leq \frac{n}{2\xi}\|u\|^2 \leq Y_{\xi,n}^2 \exp(\lambda Y_{\xi,n}^c).$$

So that

$$\|\nabla v\| \leq Y_{\xi,n} = (\gamma^{-4}n\xi^3)^{1/4}. \tag{3.4}$$

Keeping in mind the following three inequalities:

$$\begin{aligned} &\langle e_{\lambda,\xi}(v_1) - e_{\lambda,\xi}(v_2), v_1 - v_2\rangle \geq \xi\|\nabla(v_1 - v_2)\|^2, \\ &\langle B(v_1) - B(v_2), v_1 - v_2\rangle = -\langle B(v_1 - v_2), v_1\rangle \\ &\qquad\qquad \leq \gamma\|v_1 - v_2\|^{1/2}\|\nabla(v_1 - v_2)\|^{3/2}\|\nabla v_1\|, \\ &\langle\partial\varphi(v_1) - \partial\varphi(v_2), v_1 - v_2\rangle \geq 0, \end{aligned} \tag{3.5}$$

we can deduce from the relation $u_i \in \left(1 + \frac{1}{n}L_{\lambda,\xi}\right)(v_i)$ that

$$\|\delta v\|^2 + \frac{1}{n}\{\xi\|\nabla\delta v\|^2 - \gamma\|\delta v\|^{1/2}\|\nabla v_1\|\|\nabla\delta v\|^{3/2}\} \leq \langle\delta u, \delta v\rangle.$$

Applying (1.11) and then (3.4) with $v = v_1$, we obtain after a simple calculation

$$\frac{3}{4}\|\delta v\|^2 + \frac{\xi}{4n}\|\nabla\delta v\|^2 \leq \langle\delta u, \delta v\rangle, \tag{3.6}$$

from which (3.2) follows by using Schwarz' inequality. Q. E. D.

There are given $u_0 \in H$ and $f \in L^2_{\text{loc}}(0, \infty; H)$. Let $a_n \in H$ and $f_n \in C([0, \infty); H)$, and assume that

$$a_n \to u_0 \text{ in } H \quad\text{and}\quad f_n \to f \text{ in } L^2_{\text{loc}}(0, \infty; H). \tag{3.7}$$

We then choose λ so that $M_{\lambda,\xi,n} = A_n\exp(2nT)$, that is,

$$\lambda = 2(\gamma^{-4}n\xi^3)^{-c/4}\{2nT + \log(2^{-1/2}\gamma n^{1/4}\xi^{-5/4}A_n)\}, \tag{3.8}$$

where

$$A_n = \frac{1}{2n}\{\max_{0\leq t\leq T}\|f_n(t)\| + 2n\|a_n\|\}.$$

It is evident that $\|a_n\| \leq M_{\lambda,\xi,n}$. Substitution of $\xi = \xi_n = n^{-\alpha}$ and $T = T_n = n^\beta$

into (3.8) yields λ_n. If we set $M_n = M_{\lambda_n,\xi_n,n}$ and $Y_n = Y_{\xi_n,n}$, and choose α and β as

$$0 < \alpha < \frac{1}{3}\left(1 - \frac{4}{c}\right) \quad \text{and} \quad 0 < \beta < \frac{c}{4}(1 - 3\alpha),$$

it then easily follows that

$$\text{(3.9)} \qquad \begin{aligned} &\xi_n \to 0,\ T_n \to \infty \text{ and } \lambda_n \to 0 \quad \text{as } n \to \infty \\ &Y_n = (\gamma^{-4} n \xi_n^3)^{1/4}, \quad M_n = A_n \exp(2nT_n). \end{aligned}$$

PROPOSITION 3.2. *Let $\varphi_t \in \Phi_p$, $p > 6/5$, $u_0 \in H$ and $f \in L^2_{\mathrm{loc}}(0, \infty; H)$, and assume that $a_n \in H$ and $f_n \in C([0, \infty); H)$ satisfy* (3.7). *Then, there exist sequences $\xi_n > 0$, $T_n > 0$, $\lambda_n > 0$, $Y_n > 0$, and $M_n > 0$, satisfying* (3.9), *such that the following statements hold:*

(i) *For any u belonging to*

$$H_n = \{u \in H ; \| u \| \leq M_n\}$$

there corresponds exactly one $v \in \mathscr{D}(\partial\varphi_t)$ such that $u \in \left(1 + \frac{1}{n} L_n(t, \cdot)\right)(v)$ and $\| \nabla v \| \leq Y_n$, where

$$\text{(3.10)} \qquad L_n(t,v) = e_n(v) + B(v) + \partial\varphi(t, v) \quad \textit{with } e_n = e_{\lambda_n,\xi_n}.$$

(ii) *Let $\mathscr{L}_n(t, \cdot)$ be Yosida's approximation of L_n:*

$$\mathscr{L}_n(t, \cdot) = n\left\{1 - \left(1 + \frac{1}{n} L_n(t, \cdot)\right)^{-1}\right\} : H_n \to H.$$

Then, there exists exactly one function $u_n(t)$ in $C^1([0, T_n]; H_n)$ satisfying

$$\text{(3.11)} \qquad \begin{aligned} &u_n' + \mathscr{L}_n(t, u_n(t)) = f_n(t) \quad \textit{in } (0, T_n), \\ &u_n(0) = a_n. \end{aligned}$$

Proof. Choose ξ_n, T_n, λ_n, Y_n and M_n as above. The proof of (i) is an immediate consequence of Proposition 3.1. So we devote our attention to part (ii). Setting $v = \left(1 + \frac{1}{n} L_n(t, \cdot)\right)^{-1}(u) \in \mathscr{D}(\partial\varphi_t)$, we immediately obtain

$$\text{(3.12)} \qquad \begin{aligned} &n(u - v) = \mathscr{L}_n(t, u) \in L_n(t, v) \\ &\| v \|^2 + \frac{2}{n}\{ \langle e_n(v), v \rangle + \varphi(t, v)\} \leq \| u \|^2. \end{aligned}$$

Let $b_n - M_n - \| a_n \|$. We set $U_n = \{u \in H ; \| u - a_n \| \leq b_n\}$, which is a subset of H_n, and define

$$\mathscr{F}_n(t, u) = f_n(t) - \mathscr{L}_n(t, u) \quad \text{for } (t, u) \in [0, T_n] \times U_n.$$

We are going to prove that $\mathscr{F}_n$ is a continuous function of $[0, T_n] \times U_n$ into H. With each $t_i \in [0, T_n]$ and $u_i \in U_n\ (i = 1, 2)$ we associate $v_i \in W_p \cap V_1$ in a manner that $u_i \in v_i + \frac{1}{n} L_n(t_i, v_i)$. Then, we have $\| \nabla v_i \| \leq Y_n$ and (3.12) with $u = u_i$ and $v = v_i$. Therefore, we have

(3.13) $$\| \mathscr{F}_n(t_2, u_2) - \mathscr{F}_n(t_1, u_1) \| \leq \| f_n(t_2) - f_n(t_1) \| + n(\| \delta u \| + \| \delta v \|),$$

where $\delta v = v_2 - v_1$ and $\delta u = u_2 - u_1$.

From (3.10) and (3.12) it follows that

(3.14) $$\langle n(u_i - v_i) - e_n(v_i) - B(v_i), v_j - v_i \rangle \leq \varphi(t_i, v_j) - \varphi(t_i, v_i)$$

for $(i, j) = (1, 2)$ and $= (2, 1)$. Adding these, we obtain

$$\begin{aligned} &\langle n\delta(v - u) + \delta e_n(v) + \delta B(v), \delta v \rangle \\ &\leq \varphi(t_2, v_1) - \varphi(t_1, v_1) - \varphi(t_2, v_2) + \varphi(t_1, v_2) \end{aligned}$$

and hence, writing the RHS of the above inequality as $\Phi(t_1, t_2)$,

$$n \| \delta v \|^2 + \xi_n \| \nabla \delta v \|^2 + \langle \delta v \cdot \nabla v_1, \delta v \rangle \leq n \langle \delta u, \delta v \rangle + \Phi(t_1, t_2).$$

Employing Hölder's inequality and the inequality $\| \nabla v_1 \| \leq Y_n$ in the term $\langle \delta v \cdot \nabla v_1, \delta v \rangle$, we get analogously as in (3.6)

(3.15) $$3 \| \delta v \|^2 + \frac{\xi_n}{n} \| \nabla \delta v \|^2 \leq 4 \langle \delta u, \delta v \rangle + 4\Phi(t_1, t_2).$$

So that $\| \delta v \|^2 \leq 2 \| \delta u \|^2 + 4\Phi$. Hence, combining this with (3.13) concludes the continuity of $\mathscr{F}_n$. In fact, (A.2) and (A.3) implies $\Phi(t_1, t_2) \to 0$ as $t_2 \to t_1$, since $\varphi(t_i, v_i) \leq \| u_i \|^2 \leq (b_n + \| a_n \|)^2$.

It is not difficult to see that

$$\begin{aligned} &\| \mathscr{F}_n(t, u) \| \leq \alpha_n + \beta_n \| u - a_n \| \quad \text{with } a_n = 2nA_n \text{ and } \beta_n = 2n, \\ &\| \mathscr{F}_n(t, u_1) - \mathscr{F}_n(t, u_2) \| \leq 3n \| u_1 - u_2 \|,\ u_i \in U_n \quad (i = 1, 2). \end{aligned}$$

These permit us to apply the method of successive approximation to obtain one and only one $u_n \in C^1([0, T_n]; H_n)$ satisfying (3.11), because $M_n = A_n \exp(2nT_n)$ implies

$$\alpha_n \beta_n^{-1} \{\exp(\beta_n T_n) - 1\} \leq b_n.$$

This completes the proof of part (ii). Q. E. D.

Remembering that $u_n(t) \in H_n$, we define $v_n(t) \in \mathscr{D}(\partial\varphi_t)$ by

$$v_n(t) = \left(1 + \frac{1}{n} L_n(t, \cdot)\right)^{-1}(u_n(t)). \tag{3.16}$$

It then follows from (3.15) that $v_n \in C([0, \infty) ; V)$. Furthermore, we have

LEMMA 3.1. *For each n it follows that*

(P.1) $n(u_n(t) - v_n(t)) = \mathscr{L}_n(t, u_n(t)) \in L_n(t, v_n(t)), \quad 0 \leq t \leq T_n,$

(P.2) $\| v_n(t) \|^2 + \frac{2}{n} \{\langle e_n(v_n(t)), v_n(t)\rangle + \varphi(t, v_n(t))\} \leq \| u_n(t) \|^2, \ 0 \leq t \leq T_n,$

(P.3) $$\frac{1}{2} \| u_n(t) \|^2 + \int_s^t \{\langle e_n(v_n), v_n\rangle + \varphi(\tau, v_n)\} \, d\tau + \frac{1}{n} \int_s^t \| \mathscr{L}_n(\tau, u_n) \|^2 \, d\tau$$
$$\leq \frac{1}{2} \| u_n(s) \|^2 + \int_s^t \langle f_n, u_n\rangle \, d\tau, \quad 0 \leq s < t \leq T_n$$

and

(P.4) $$\| u_n(t) \|^2 + \int_0^T \{\langle e_n(v_n), \mathrm{v}_n\rangle + \varphi(t, v_n)\} \, dt + \frac{1}{n} \int_0^T \| \mathscr{L}_n(t, u_n) \|^2 \, dt \leq K_T^2,$$

for t, $0 \leq t < T \leq T_n$, where K_T is a positive constant independent of t.

Proof. Properties (P.1) and (P.2) easily follow from (3.12). Keeping in mind

$$w_n(t) = \mathscr{L}_n(t, u_n) - B(v_n) - e_n(v_n) \in \partial\varphi(t, v_n), \quad u_n(0) = a_n, \tag{3.17}$$

we can derive

$$\begin{aligned} &\varphi(t, v_n(t)) - \varphi(s, v_n(s)) \\ &\quad \leq \langle w_n(t), v_n(t) - v_n(s)\rangle + \varphi(t, v_n(s)) - \varphi(s, v_n(s)). \end{aligned}$$

Therefore, (A.3) implies the continuity of $\varphi(t, v_n(t))$ in $t \geq 0$, because $v_n \in C([0, \infty) ; V)$ and $\varphi(0, v_n(t))$ is bounded in $0 \leq t \leq T_n$. On the other hand, from (3.11) and (P.1) it immediately follows that for all $t \geq 0$

$$\langle u_n', u_n\rangle + \langle \mathscr{L}_n(t, u_n), v_n\rangle + \frac{1}{n} \| \mathscr{L}_n(t, u_n) \|^2 = \langle f_n, u_n\rangle.$$

Hence, we have by virtue of (3.17)

$$\langle u_n', u_n \rangle + \langle e_n(v_n), v_n \rangle + \varphi(t, v_n) + \frac{1}{n} \| \mathscr{L}_n(t, u_n) \|^2 \leq \langle f_n, u_n \rangle.$$

Integration over $\Omega \times (s, t)$ of the above gives (P.3). Application of Gronwall's lemma to (P.3) yields (P.4). Q. E. D.

§4. Proof of Theorem 1

For $p > 6/5$ we define $q = q(p)$ by (1.27). Recalling the fact that $V_q \cap V_1 \subset W_p$ (see Lemma 1.1 (ii)), we deduce from (3.11) and (3.17)

$$(4.1) \quad \int_0^T \langle u_n', v - v_n \rangle \, dt + \int_0^T \langle e_n(v_n), v - v_n \rangle \, dt + \int_0^T \langle B(v_n), v \rangle \, dt$$
$$+ \int_0^T \{\varphi(t, v) - \varphi(t, v_n)\} \, dt \geq \int_0^T \langle f_n, v - v_n \rangle \, dt, \quad v \in C^1([0, T]; V_q \cap V_1)$$

for all n such that $T_n \geq T$. The proof of Theorem 1 will be accomplished by passage to limit $n \to \infty$ in (4.1) after a suitable choice of a subsequence of $\{u_n\}$. To do so, using Lemma 3.1, we are going to examine the convergence properties (C.1)~(C.7) of the sequences $\{u_n\}$ and $\{v_n\}$.

LEMMA 4.1. *Suppose $p > 6/5$. Then, for any $T > 0$ we have*

$$(\text{C.1}) \qquad \lim_{n \to \infty} \int_0^T \| u_n - v_n \|^2 \, dt = 0,$$

$$(\text{C.2}) \qquad \lim_{n \to \infty} \int_0^T \langle e_n(v_n), v \rangle \, dt = 0, \quad v \in C([0, T]; V_q \cap V_1).$$

Moreover there exists a subsequence, still denoted by $\{n\}$, of $\{n\}$ such that

$$(\text{C.3}) \qquad \begin{aligned} &u_n \to u \quad \textit{weakly}^* \textit{ in } L^\infty(0, T; H) \\ &v_n \to u \quad \textit{weakly}^* \textit{ in } L^\infty(0, T; H) \quad \textit{as } n \to \infty \\ &v_n \to u \quad \textit{weakly} \textit{ in } L^p(0, T; V_p) \end{aligned}$$

and

$$(\text{C.4}) \qquad \liminf_{n \to \infty} \int_0^T \varphi(t, v_n) \, dt \geq \int_0^T \varphi(t, u) \, dt.$$

Proof. Property (C.1) immediately follows from (P.1), (P.2) and (P.4). The boundedness of $\{u_n\}$ and $\{v_n\}$ in Banach spaces $L^\infty(0, T; H)$ and $L^p(0, T; V_p) \cap L^\infty(0, T; H)$, respectively, yields (C.3). Keeping in mind (P.4), we can compute as follows:

$$\int_0^T \langle e_n(v_n),\, v\rangle\, dt \le C\int_0^T \xi_n \| \nabla v_n \| \exp(\lambda_n \| \nabla v_n \|^c)\, dt$$
$$\le C\xi_n \left\{ \int_{E_{n,N}} N^{-1} \| \nabla v_n \|^2 \exp(\lambda_n \| \nabla v_n \|^c)\, dt + \int_{(0,T)\setminus E_{n,N}} N \exp(\lambda_n N^c)\, dt \right\}$$
$$\le C\{K_T^2/N + \xi_n NT \exp(\lambda_n N^c)\},$$

which leads to (C.2), where

$$E_{n,N} = \{t \in (0,\, T)\,;\, \| \nabla v_n(t) \| > N\} \quad \text{and} \quad C = \sup_{t\in(0,T)} \| \nabla v(t) \|.$$

The property (C.4) immediately follows from lower-semicontinuity of the mapping $v \to \int_0^T \varphi(t,\, v)\, dt.$ Q. E. D.

Relying on the technique developed by Masuda [10] we can prove

LEMMA 4.2. *Suppose $p > 6/5$. Then, there exists a subsequence $\{n'\}$ of $\{n\}$ such that*

(C.5) $\lim_{n'\to\infty} \langle u_{n'}(t),\, \phi\rangle = \langle u(t),\, \phi\rangle$ *uniformly in* $[0,\, T]$ *for all* $\phi \in H$,

(C.6) $\lim_{n'\to\infty} \int_0^T \| v_{n'} - u \|_{\Omega_R}^r\, dt = 0$ *for any positive numbers* r *and* R,

and

(C.7) $\lim_{n'\to\infty} \int_0^T \langle B(v_{n'}) - B(u),\, v\rangle\, dt = 0$ *for all* $v \in C([0,\, T]\,;\, V_q)$,

where $q = q(p)$, u *is the same as in* (C.3) *and* $\Omega_R = \Omega \cap B_R$.

Proof of (C.5). For $\phi \in \mathcal{V}(\Omega)$ let us set $x_n(t) = \langle u_n(t),\, \phi\rangle$. It is easy to see that $| x_n(t) | \le K_T \| \phi \|$ and

$$| x_n(t) - x_n(s) | \le C_p\{| t - s |^\theta + \int_s^t | \langle e_n(v_n),\, \phi\rangle | \, d\tau\}$$

for all $0 \le s < t \le T_n$, where $0 < \theta \le 1$ and C_p is a positive constant. This, together with (C.3), allows us to apply the Ascoli-Arzelà theorem, which implies (C.5).

Proof of (C.6). For the proof we have only to substitute $U =$ "the restriction of $v_n - u$ onto Ω_R" into the Friedrichs type inequality: *For any* $\varepsilon > 0$ *there exists a positive integer* N *such that*

$$\| U \|_{\Omega_R} \le \varepsilon \| \nabla U \|_{p,\Omega_R} + N \sum_{k=1}^{N} | \langle \phi_k, U \rangle_{\Omega_R} |, \quad U \in W_\sigma^{1,p}(\Omega_R), \tag{4.2}$$

where $\{\phi_k\}$ *is total in* $L_\sigma^2(\Omega_R)$. The proof of (4.2) will be achieved, based on the fact that the injection mapping $W^{1,p}(\Omega_R) \to L^2(\Omega_R)$ is compact if $p > 6/5$.

Proof of (C.7). From the definition of B we have

$$\int_0^T \langle B(v_{n'}) - B(u), v \rangle \, dt = - \int_0^T \langle (v_n - u) \otimes v_n + u \otimes (v_n - u), \nabla v \rangle \, dt,$$

which is denoted by $I_n(\nabla v)$. Here, $u \otimes v$ is a tensor field such that $(u \otimes v)_{ij} = u^i v^j$. We decompose $I_n(\nabla v)$ in the form

$$I_n(\nabla v) = I_n(w_\lambda) + I_n(w_{\lambda,\mu}) + I_n(z_{\lambda,\mu}),$$

where

$$w_\lambda = (1 - \eta(\lambda x)) \nabla v, \quad w_{\lambda,\mu} = \eta(\lambda x) \{1 - \xi(\mu | \nabla v |)\} \nabla v$$

and

$$z_{\lambda,\mu} = \eta(\lambda x) \, \xi(\mu | \nabla v |) \, \nabla v$$

for small λ, $\mu > 0$. Here ξ and η are cut-off function defined by (1.3).

Using Lemma 1.5 and the Dini theorem concerning a monotone decreasing sequence of continuous functions, we can prove that for any $\varepsilon > 0$ there exist λ and μ so small that $| I_n(w_\lambda) | < \varepsilon$ and $| I_n(w_{\lambda,\mu}) | < \varepsilon$. We fix such λ, μ. Since $\operatorname{supp} z_{\lambda,\mu} \subset B_{2/\lambda}$ and $| z_{\lambda,\mu} | \le 2/\mu$, it follows that

$$| I_n(z_{\lambda,\mu}) | \le \frac{2}{\mu} \int_0^T \| v_n - u \|_{\Omega_{2/\lambda}} (\| v_n \| + \| u \|) dt.$$

Hence, (C.6) implies

$$\lim_{n' \to \infty} I_{n'}(z_{\lambda,\mu}) = 0 \quad \text{and} \quad \limsup_{n' \to \infty} | I_{n'}(\nabla v) | \le 2\varepsilon.$$

This asserts (C.7). Q. E. D.

We are now ready to prove Theorem 1. Substituting $n = n'$ into (4.1) and letting $n' \to \infty$, we can conclude (2.5) for $v \in C^1([0, T] ; V_q \cap V_1)$ with the aid of (C.1)~(C.7). In fact, the first term of the LHS of (4.1) is calculated as follows:

$$\int_0^T \langle u_n', v - v_n\rangle\, dt = \int_0^T \{\langle v', v - u_n\rangle + \langle u_n' - v', v - u_n\rangle + \langle u'_n, u_n - v_n\rangle\}\, dt$$

$$\leq \int_0^T \langle v', v - u_n\rangle\, dt - \frac{1}{2}(\|u_n(T) - v(T)\|^2 - \|a_n - v(0)\|^2) + \int_0^T \langle f_n, \frac{1}{n}\mathscr{L}_n u_n\rangle\, dt$$

and hence we have by (3.7)

$$\limsup_{n'\to\infty} \int_0^T \langle u_{n'}', v - v_{n'}\rangle\, dt \leq \int_0^T \langle v', v - u\rangle\, dt - \frac{1}{2}(\|u(T) - v(T)\|^2 - \|u_0 - v(0)\|^2).$$

The other terms of (4.1) will be handled without any difficulty by keeping in mind (C.2), (C.7) and (C.4).

To prove (2.5) for any v belonging to the space $W_{0,T}^p$ from (2.6) we extend $v(t)$ outside the interval $[0, T]$ as follows: $v(t) = v(-t)$ for $t < 0$ and $= v(2T - t)$ for $t > T$. Let $v_\varepsilon(t)$ be a mollifier defined by (2.14). It is easily seen that $v_\varepsilon \in C^1([0, T]; V_q \cap V_1)$, $v_\varepsilon \to v$ in $\mathscr{B}_{0,T}^p \cap L^p(0, T; V_q)$ and $v_\varepsilon' \to v'$ weakly* in $(\mathscr{B}_{0,T}^p)'$. Substituting $v = v_\varepsilon$ into (2.5) and tending $\varepsilon \to 0$, we have (2.5) for any $v \in W_{0,T}^p$ because Lemma 1.4 implies $v \in C([0, \infty); H)$ and hence $v_\varepsilon(t) \to v(t)$ uniformly in $C([0, T]; H)$.

Our next purpose is to prove (2.3). Taking account of (3.17), we can infer from (1.23), using (P.2) and (P.4),

$$\left|\int_0^T \langle w_n, v\rangle\, dt\right| \leq C\left\{\left(\int_0^T \|\nabla v\|_p^p\, dt\right)^{1/p} + \int_0^T \|D(v)\|_1\, dt\right\}$$

for all $v \in \mathscr{B}_{0,T}^p$. This guarantees the existence of β such that $w_n \to \beta$ weakly* in $(\mathscr{B}_{0,T}^p)'$. Thus, it easily follows from (C.7) that

$$-\int_0^T \langle u, \phi'\rangle\, dt = \int_0^T \langle f - B(u) - \beta, \phi\rangle\, dt \tag{4.3}$$

for all $\phi \in C_0^\infty(0, T; V_q \cap V_1)$. According to (1.18) and Lemma 1.3, we can conclude (2.3), observing Lemma 1.6.

The energy inequality (2.7) is an immediate consequence of (P.3) $(s = 0)$ and

(C.2). The inclusion (2.8) easily follows from Lemmas 1.1 and 1.2.

§5. Proof of Theorem 2

Suppose that Ω is a domain whose complement is compact. We may therefore assume that there exists a positive constant R_0 such that $E_R = \mathbf{R}^3 \setminus B_R$ is contained in Ω for all $R > R_0$. For a measurable set M we set

$$\| u \|_{r,M} = \Big(\int_M | u |^r \, dx \Big)^{1/r} \quad \text{and} \quad \| u \|_{2,M} = \| u \|_M .$$

Let $\varphi(u) = \mu \| D(u) \|_p^p + g \| D(u) \|_1$ with $p \geq 9/5$. We assume that $u_n \in H$ is the vector field constructed in Proposition 3.2, where $a_n = u_0$ and $\varphi \in \Phi_p$, $p \geq 9/5$, for all n, and that $v_n(t) \in \mathscr{D}(\partial \varphi_t)$ is defined by (3.16). The main purpose of this section is to prove

PROPOSITION 5.1. *Suppose that $p \geq 9/5$ and $T > 0$. For any $\varepsilon > 0$ there exists $R > R_0$ such that*

$$\limsup_{n \to \infty} \int_0^T \| u_n(t) \|_{E_R}^2 \, dt \leq \varepsilon. \tag{5.1}$$

Temporarily, let us assume (5.1) to hold. Since

$$\int_0^T \| u_{n'} - u \|^2 \, dt \leq 2 \int_0^T (\| u_{n'} - u \|_{B_R}^2 + \| u_{n'} \|_{E_R}^2 + \| u \|_{E_R}^2) \, dt, \tag{5.2}$$

it follows from (5.1), (C.1) and (C.5) that

$$\limsup_{n' \to \infty} \text{LHS of (5.2)} \leq 4\varepsilon,$$

which implies by using (P.4)

$$\int_0^T \| u_{n'} - u \|^r \, dt \to 0 \quad \text{as } n' \to \infty \tag{5.3}$$

for any $r > 0$. Therefore, we can extract a subsequence $\{n''\}$ of $\{n'\}$ so that $u_{n''}(s) \to u(s)$ in H for a.e. $s > 0$. Substituting $n = n''$ into (P.3) and letting $n'' \to \infty$, we obtain (2.22).

Before proving the proposition we prepare a few lemmas. For $0 < \lambda < 1$ such that $1/\lambda > R_0$ we introduce a cut-off function:

$$\zeta_\lambda(x) = \{1 - \eta(\lambda x)\}^{2p} \quad (\text{see (1.3) for } \eta(x))$$

and the fundamental solution of $\lambda - \Delta$:

$$F_\lambda = \frac{1}{4\pi|x|}\exp(-\sqrt{\lambda}\,|x|).$$

Like (1.6) we define a mapping $v \rightarrow v_\lambda$:

$$v_\lambda = \operatorname{rot}\{\zeta_\lambda(F_\lambda * (\zeta_\lambda \operatorname{rot} v))\}, \quad 1/\lambda > R_0.$$

After a simple calculation we obtain

$$v_\lambda = \zeta_\lambda\{(\delta - \lambda F_\lambda) * (\zeta_\lambda v)\} + R_\lambda v, \tag{5.4}$$

where

$$R_\lambda v = \zeta_\lambda\{F_\lambda * \operatorname{rot}(v \times \nabla\zeta_\lambda)\} + \nabla\zeta_\lambda \times \{F_\lambda * \operatorname{rot}(\zeta_\lambda v)\} + \nabla\zeta_\lambda \times \{F_\lambda * (v \times \nabla\zeta_\lambda)\}. \tag{5.5}$$

Using the inequality (1.4), the identity (1.8) and the estimations with respect to F:

$$\|\lambda F_\lambda\|_1 = 1, \quad \|\lambda^{1/2}\nabla_k F_\lambda\|_1 \le C \quad \text{and} \quad \|\nabla_i\nabla_j(F_\lambda * h)\| \le C\|h\|,\ h \in L^2, \tag{5.6}$$

we easily see that if v is in H (or V_r, $r \ge 1$), then so is v_λ, where and in what follows C denotes various positive constants not depending on λ. More precisely we can show quite easily

LEMMA 5.1. *For any $v \in C_0^\infty(\mathbf{R}^3)^3$ we have*

$$\|R_\lambda v\| \le C\lambda^{1/2}\|v\|, \quad \|\nabla R_\lambda v\| \le C\lambda\|v\|, \tag{5.7}$$

$$\|\nabla R_\lambda v\|_r \le C\lambda^{1/2}(\|\nabla v\|_r + \|v\|), \quad r > 6/5, \tag{5.8}$$

$$\|D(R_\lambda v)\|_1 \le C\lambda^{1/2}\|D(v)\|_1. \tag{5.9}$$

Proof. The proof of (5.7) is evident. Without any difficulty we can show that

$$\|D(R_\lambda v)\|_r \le C_r\lambda^{1/2}(\|D(v)\|_r + \lambda\|v\|_{r,B_{2/\lambda}})$$

for all $r \ge 1$. Consequently, the use of (1.1) and Lemma 1.2 imply (5.8). By Hölder's inequality we have

$$\|v\|_{1,B_{2/\lambda}} \le C\lambda^{-1}\|v\|_{3/2}. \tag{5.10}$$

Hence, the proof of (5.9) is achieved with the aid of (1.2). Q. E. D.

LEMMA 5.2. *Suppose that $p \geq 9/5$. Then, we have*

$$|\langle B(v), v_\lambda\rangle| \leq C\lambda^{1/2}\|v\|^a\|\nabla v\|_q^b, \quad v \in \mathcal{V}, \tag{5.11}$$

where a, b and q are positive numbers such that $a+b=3$, $b \leq q$ and $q=p$ for $p<3$ and $=2$ for $p \geq 3$.

Proof. After a simple calculation we obtain from (5.4) that

$$\langle B(v), v_\lambda\rangle = \langle \xi_\lambda v^i v^j, \lambda\nabla_i F_\lambda * (\zeta_\lambda v^j)\rangle \\ - \langle v^i v^j \nabla_i \zeta_\lambda, (\delta - \lambda F_\lambda) * (\zeta_\lambda v^j)\rangle - \langle v^i v^j, \nabla_i(R_\lambda v^j)\rangle$$

and hence, using (1.4), (5.6) and (5.7), we get

$$|\langle B(v), v_\lambda\rangle| \leq C\lambda^{1/2}\|v\|\|v\|_4^2. \tag{5.12}$$

Assume that $9/5 \leq p < 3$. Then, $2 < 4 < p^*$. Using (1.10) and (1.2), we obtain

$$\|v\|_4^2 \leq C\|v\|^{2-\beta}\|\nabla v\|_p^\beta \quad \text{with } \beta = 3p/(5p-6). \tag{5.13}$$

Evidently, $p \geq 9/5$ implies $\beta \leq p$. We now suppose $p \geq 3$. Instead of (5.13) the inequality:

$$\|v\|_4^2 \leq C\|v\|^{1/2}\|\nabla v\|^{3/2} \tag{5.14}$$

is adopted. Combining (5.12) with (5.13)-(5.14), we arrive at (5.11). Q. E. D.

Let $a \geq 1$ and $q \geq 1$. Set $z_\lambda = \zeta_\lambda^{1/p}$. Using Hölder's inequality, we have for $h \in L^q$

$$|z_\lambda^a(F_\lambda * h) - F_\lambda * (z_\lambda^a h)| \leq \frac{1}{4\pi}\int \frac{1}{|x-y|} e^{-\sqrt{\lambda}|x-y|}|z_\lambda^a(x) - z_\lambda^a(y)||h(y)|\,dy \\ \leq C\lambda \int e^{-\sqrt{\lambda}|x-y|}|h(y)|\,dy \leq C\lambda^{1-3/2q'}\left(\int e^{-\sqrt{\lambda}|x-y|}|h(y)|^q\,dy\right)^{1/q}.$$

Hence,

$$\|z_\lambda^a(F_\lambda * h) - F_\lambda * (z_\lambda^a h)\|_q \leq C\lambda^{1-3/2q'-1/2q}\|h\|_q \leq C\lambda^{-1/2}\|h\|_q. \tag{5.15}$$

With the aid of (5.15) we shall prove the last two lemmas.

LEMMA 5.3. *Let $\phi_p(v) = \|D(v)\|_p^p$, $p \geq 9/5$. Then,*

$$-\langle \partial\phi_p(v), v_\lambda\rangle \leq C\lambda^{1/2}(\|\nabla v\|_p^p + \|v\|\|\nabla v\|_p^{p-1}), \quad v \in \mathcal{D}(\partial\varphi). \tag{5.16}$$

Proof. In view of (5.4) we have

$$D(v_\lambda) = \zeta_\lambda\{(\delta - \lambda F_\lambda) * (\zeta_\lambda D(v)\}$$
$$- \{\zeta_\lambda(\varDelta F_\lambda * ([D, \zeta_\lambda]v)) + [D, \zeta_\lambda](\varDelta F_\lambda * (\zeta_\lambda v)) - D(R_\lambda v)\} = X - Y$$

and hence,

$$\text{the LHS of (5.16)} = -p\langle |D(v)|^{p-2} D(v), X - Y\rangle,$$

where $[D, \zeta]u = D(\zeta u) - \zeta D(u)$ and hence

$$([D, \zeta]u)_{ij} = \{(\nabla_i\zeta)u^j + (\nabla_j\zeta)u^i\}/2.$$

Firstly, we have in view of (5.15)

$$(5.17) \qquad -p\langle |D(v)|^{p-2}D(v), X\rangle$$
$$= -p\|z_\lambda^2 D(v)\|_p^p + p\langle |D(v)|^{p-2} D(v), z_\lambda^{2p-2}\{\lambda F_\lambda * (z_\lambda^2 D(v))\}\rangle$$
$$+ p\langle |D(v)|^{p-2} D(v), \lambda F_\lambda * (z_\lambda^{2p} D(v)) - z_\lambda^{2p-2}\{\lambda F_\lambda * (z_\lambda^2 D(v)\}$$
$$+ z_\lambda^p\{\lambda F_\lambda * z_\lambda^a D(v)\} - \lambda F_\lambda * (z_\lambda^{2p} D(v))\rangle$$
$$\leq C\|D(v)\|_p^{p-2}\lambda^{1/2}\|D(v)\|_p \leq C\lambda^{1/2}\|\nabla v\|_p^p.$$

By the same argument as is employed in the proof of (5.8) we obtain

$$p\langle |D(v)|^{p-2} D(v), Y\rangle \leq C\lambda^{1/2}\|D(v)\|_p^{p-1}(\|\nabla v\|_p + \|v\|),$$

which concludes (5.16). Q. E. D.

LEMMA 5.4. *Let $\phi_1(v) = \|D(v)\|_1$. Then,*

$$(5.18) \qquad |\langle \partial\phi_1(v), v_\lambda\rangle| \leq C\lambda^{1/2}\|D(v)\|_1, \quad v \in \mathscr{D}(\partial\varphi).$$

Proof. Let $w \in \partial\varphi_1(v)$. Then, we have

$$\langle w, v_\lambda\rangle = \langle w, \zeta_\lambda\{(\delta - \lambda F_\lambda) * \zeta_\lambda v\}\rangle + \langle w, R_\lambda v\rangle = A + B.$$

Inserting $\phi = v - t\zeta_\lambda\{(\delta - \lambda F_\lambda) * \zeta_\lambda v\}$ $(0 < t < 1)$ into the inequality $\langle w, \phi - v\rangle \leq \varphi_1(\phi) - \varphi_1(v)$, we have

$$tA \geq \varphi_1(v) - \varphi_1(\phi) = \|D(v)\|_1 - \|D(\phi)\|_1.$$

A similar calculation as in (5.17) leads to

$$D(\phi) = (1 - t\zeta_\lambda^2)D(v) + t\lambda F_\lambda * \zeta_\lambda^2 D(v)$$
$$+ t\{\zeta_\lambda(\lambda F_\lambda * \zeta_\lambda D(v)) - \lambda F_\lambda * \zeta_\lambda^2 D(v)\} + t\zeta(\zeta_\lambda)(\varDelta F_\lambda * \zeta_\lambda v)$$
$$+ t\zeta_\lambda(\varDelta F_\lambda * D(\zeta_\lambda)v).$$

Making use of (5.15), we get

$$\| D(\phi) \|_1 \leq \| D(v) \|_1 + tC\lambda^{1/2} \| \zeta_\lambda D(v) \|_1 + t \| D(\zeta_\lambda)\{F_\lambda * \Delta(\zeta_\lambda v)\} \|_1 + t \| \zeta_\lambda \{F_\lambda * \Delta(D(\zeta_\lambda) v)\} \|_1.$$

Exactly as in (5.9) we have (5.18). Q. E. D.

Proof of Proposition 5.1. Multiplying (3.17) by $u_{n,\lambda}$ and integrating over $\Omega \times (0, t)$, we obtain, keeping in mind (3.11), that

$$(5.19)\quad \int_0^t \langle u_n', u_{n,\lambda} \rangle \, d\tau = \int_0^t \langle f_n, u_{n,\lambda} \rangle \, d\tau - \frac{1}{n} \int_0^t \langle \mathscr{L}_n(u_n), (\mathscr{L}_n(u_n))_\lambda \rangle \, d\tau - \int_0^t \langle B(v_n) + e_n(v_n) + \partial\psi_p(v_n) + w_n, v_{n,\lambda} \rangle \, d\tau,$$

where $w_n(t) \in \partial\varphi_1(v_n(t))$. Since

$$\langle u_n', u_{n,\lambda} \rangle = \frac{1}{4} \frac{d}{dt} \langle \zeta_\lambda \operatorname{rot} u_n, F_\lambda * (\zeta_\lambda \operatorname{rot} u_n) \rangle,$$

we have

$$(5.20)\qquad 2 \int_0^t \langle u_n', u_{n,\lambda} \rangle \, d\tau = \langle u_n(t), u_{n,\lambda}(t) \rangle - \langle u_n, u_{n,\lambda} \rangle.$$

On the other hand we obtain from (5.4), (5.6) and (5.7) that

$$(5.21)\quad - \langle u_n, u_{n,\lambda} \rangle + \| \zeta_\lambda u_n \|^2 = \langle u_n - v_n + v_n, \zeta_\lambda(\lambda F_\lambda * (\zeta_\lambda u_n)) \rangle - \langle u_n, R_\lambda u_n \rangle \leq \| u_n - v_n \| \| u_n \| + C\lambda^{1/2} \| u_n \|^2 + \| \zeta_\lambda v_n * \lambda F_\lambda \| \| u_n \|.$$

Therefore, we get, using (P.4),

$$(5.22)\quad \| \zeta_\lambda u_n \|^2 \leq 2 \int_0^t \langle u_n', u_{n,\lambda} \rangle \, ds + \| \zeta_\lambda u_n \|^2 + C\lambda^{1/2} \| u_0 \|^2 + K_T(\| u_n(t) - v_n(t) \| + \| \zeta_\lambda v_n(t) * \lambda F_\lambda \|) + CK_T \lambda^{1/2}$$

for all $t \leq T$.

For the proof of the proposition it is sufficient to establish

$$(5.23)\qquad \limsup_{n \to \infty} \int_0^T \| \zeta_\lambda u_n(t) \|^2 \, dt \to 0 \quad \text{as } \lambda \to 0.$$

Applying (1.4) with $r = 2$, $p = 3/2$ and $q = 6/5$, we obtain, keeping in mind

(1.2),

$$\| \zeta_\lambda v_n * \lambda F_\lambda \| \leq \| v_n \|_{3/2} \| \lambda F_\lambda \|_{6/5} \leq C\lambda^{1/10} \| D(v_n) \|_1. \tag{5.24}$$

Thus, we have only to pay attention to each term of the RHS of (5.19). From (5.7) it immediately follows that

$$\begin{aligned} \int_0^t \langle f_n, u_{n,\lambda} \rangle \, ds &\leq 2 \int_0^T \| \zeta_\lambda f_n \| \zeta_\lambda u_n \| \, ds + C\lambda^{1/2} \int_0^T \| f_n \| \|u_n\| \, ds \\ &\leq 2K_T \int_0^T (\| f_n - f \| + \| \zeta_\lambda f \|) ds + CK_T \lambda^{1/2} \int_0^T \| f_n \| \, ds, \end{aligned} \tag{5.25}$$

$$-\frac{1}{n} \int_0^t \langle \mathscr{L}_n(u_n), (\mathscr{L}_n(u_n))_\lambda \rangle \, ds \leq C\lambda^{1/2} \frac{1}{n} \int_0^T \| \mathscr{L}_n(u_n) \|^2 \, ds \leq CK_T^2 \lambda^{1/2}, \tag{5.26}$$

and

$$-\int_0^t \langle e(v_n), v_{n,\lambda} \rangle \, ds \leq C\lambda \int_0^T \xi_n \| v_n \| \| \nabla v_n \| \exp(\lambda_n \| \nabla v_n \|^c) \, ds. \tag{5.27}$$

Here, we used the positively of $\delta - \lambda F_\lambda$:

$$\langle h, (\delta - \lambda F_\lambda) * h \rangle \geq 0, \quad h \in L^2.$$

From Lemma 5.2 it follows that

$$-\int_0^t \langle B(v_n), v_{n,\lambda} \rangle \, ds \leq C\lambda^{1/2} \int_0^T \| v_n \|^2 \| \nabla v_n \|_q^b \, ds \leq CC_T \lambda^{1/2}.$$

Lemmas 5.3 and 5.4 lead to

$$\begin{aligned} &-\int_0^t \langle \partial\varphi(v_n) + w_n, v_{n,\lambda} \rangle \, ds \\ &\leq C\lambda^{1/2} \int_0^T (\| \nabla v_n \|_p^p + \| v_n \| \| \nabla v_n \|_p^{p-1} + \| D(v_n) \|_1) \, ds \leq CC_T \lambda^{1/2}. \end{aligned} \tag{5.29}$$

Thanks to (5.22), we can prove (5.23) by virtue of (5.24)~(5.29).

§6. Proof of Theorem 3

We first observe that functional $\varphi_t(u) = \varphi(t, u)$ defined by (2.23) satisfies (A.1) ~ (A.3) with $p = 2$ if $\mu \in \mathscr{M}$ and $g \in \mathscr{G}$. Applying Proposition 3.2 with $a_n = u_0 + \frac{\chi}{n}$ and $f_n = f$, we can find sequences $\{\lambda_n\}$, $\{T_n\}$, $\{\xi_n\}$, $\{Y_n\}$ and $\{M_n\}$ satisfying (3.9) and that for any $u \in H_n = \{u \in H \, ; \, \| u \| \leq M_n\}$ and any $t \geq 0$

there exists exactly one $v \in V$ such that $u \in (1 + \frac{1}{n} L_n(t, \cdot))(v)$ and $\|\nabla v\| \leq Y_n$, where

$$L_n(t, v) = B(v) + e_n(v) + \partial\varphi_n(t, v), \tag{6.1}$$
$$\varphi_n(t, v) = \varphi(t, v) - \varepsilon_n \| D(v) \|^2 \quad \text{with } \varepsilon_n = \xi_n \exp(\lambda_n \| \nabla u_0 \|^c).$$

Moreover, setting

$$\mathscr{L}_n(t,u) = n\left\{1 - \left(1 + \frac{1}{n} L_n(t, \cdot)\right)^{-1}\right\}(u) : H_n \to H,$$

we obtain one and only one function $u_n \in C^1([0, T_n] ; H_n)$ satisfying

$$u_n'(t) + \mathscr{L}_n(t, u_n(t)) = f(t) \quad \text{in } t \in (0, T_n), \tag{6.2}$$
$$u_n(0) = a_n.$$

We then define $v_n(t)$ as in (3.16):

$$v_n(t) = \left\{1 + \frac{1}{n} L_n (t, \cdot)\right\}^{-1}(u_n(t)). \tag{6.3}$$

From (3.15) it immediately follows that $v_n \in C([0, T_n] ; V)$ for all n. We can further prove that

$$v_n(0) = u_0 \quad \text{and} \quad \mathscr{L}_n(0, u_n(0)) = \chi. \tag{6.4}$$

In fact, observing (2.30) and $\partial\varphi(t, u_0) = e_n(u_0) + \partial\varphi_n(t, u_0)$, we have $\chi \in L_n(0, u_0)$ and hence $u_n(0) = u_0 + \frac{1}{n}\chi \in \left(1 + \frac{1}{n} L_n(0, \cdot)\right)(u_0)$.

Analogously as in Theorem 1 we can find a weak solution u of (2.25)-(2.26). Corollary 1 says that u is a strong solution of (2.25)-(2.26) as well if it satisfies (2.32). So we have only to establish the regularity properties (2.32) and (2.33).

We first consider a solution $u \in V$ of a stationary problem:

$$\langle B(u), v - u\rangle + \varphi(t, v) - \varphi(t, u) \geq \langle h, v - u\rangle, \quad v \in V \tag{6.5}$$

for $t \geq 0$ and $h \in L^\infty(\Omega)^3$. It is easily seen from the Hahn-Banach theorem and Temam [17, p.14] that there exist $\pi \in L^2(\Omega)$, a constant $c = c(\Omega)$ and $m = (m_{ij})_{i,j=1}^3$ with $m_{ij} \in L^\infty(\Omega)$ and $|m| \leq g_1$ such that

$$-\nabla\cdot(2\mu D(u) + m) + B(u) + \nabla\pi = h, \tag{6.6}$$

$$\| \pi \| \leq c(\| h \| + \| B(u) \|_{V'} + \| \mu\nabla u \| + g_1). \tag{6.7}$$

Moreover, we can establish the regularity of u as in Kim [8], making use of Cattabriga's result concerning the regularity of solutions of the Stokes equation (see [4]).

Lemma 6.1. *Let $u \in V$ be a solution of* (6.5) *and assume that a satisfies* (2.27). *Then, there exists a positive constant C_0 depending only on a and Ω such that*

$$\|\nabla u\|_a \leq C_0\nu_0(\|\nu\nabla\mu(t)\|_\alpha + 1)(\|h\| + \|u\|_\alpha\|\nabla u\| + g_1 + \mu_0\|\nabla u\|), \tag{6.8}$$

where $\nu = 1/\mu(t)$ and $\nu_0 = 1/\mu_0$.

Proof. We begin by rewriting (6.6) as

$$-\Delta u + \nabla(\nu\pi) = \nu\nabla\mu\cdot(2D(u) - \nu\pi I_d + \nu m) + \nabla\cdot(\nu m) + \nu h - \nu B(u),$$

where I_d denotes the identity tensor. The inequality (6.8) is then an easy consequence of (6.7) and the inequality due to [4] (see also [17, p. 35]):

$$\begin{aligned}\|\nabla u\|_a + \|\nu\pi\|_a \leq C\|\nu\nabla\mu\|_\alpha(\|\nabla u\| + \|\nu\pi\| + \|\nu m\|)\\ + C(\|\nu m\|_a + \|\nu h\| + \nu_0\|u\|_\alpha\|\nabla u\|).\end{aligned} \tag{6.9}$$

Q. E. D.

Lemma 6.2. *Let N be the largest integer in the set of integers $< b/2$ and let us define finite sequences $\{a_n\}_{n=0}^N$ and $\{r_n\}_{n=0}^N$ by*

$$\frac{1}{a_n} = \frac{1}{2} - \frac{n}{b} \quad \text{and} \quad \frac{1}{r_n} = \frac{1}{a_n} + \frac{1}{3} \quad \text{for} \quad n \leq N. \tag{6.10}$$

Let $q > a$, and assume that $a_{n_0-1} < q \leq a_{n_0}$ (or $a_N < q$) and $1/r = 1/q + 1/3$. Then, for any solution u of (6.5) *the following estimates hold.*

$$\|\nabla u\|_q + \|\nu\pi\|_q \leq c_l\{P^l(\|\nabla u\| + \|\nu\pi\|) + \frac{P^l - 1}{P-1}Q_r\}, \tag{6.11}$$

where $l = n_0$ or $N+1$, c_l is a positive constant depending only on α, l and Ω, and

$$P = \|\nu\nabla\mu(t)\|_\alpha + \nu_0\|u\|_\alpha, \quad Q_r = \nu_0\{g_1(1 + \|\nu\nabla\mu(t)\|_\alpha) + \|h\|_r\}.$$

Proof. Since $1/\alpha + 1/b = 1/3$, it follows that $1/\alpha + 1/a_{n-1} = 1/r_n$ for all $n \geq N$. Hence

$$L^{r_n}(\Omega) \subset W^{-1,a_n}(\Omega) \quad \text{and} \quad \|\nu B(u)\|_{r_n} \leq \nu_0\|u\|_\alpha\|\nabla u\|_{a_{n-1}}.$$

Like (6.9), we obtain

$$\| \nabla u \|_{a_n} + \| \nu\pi \|_{a_n} \leq C_n \| \nu \nabla \mu \|_\alpha (\| \nabla u \|_{a_{n-1}} + \| \nu\pi \|_{a_{n-1}} + \| \nu m \|_{a_{n-1}}) + C_n (\| \nu m \|_{\alpha_n} + \| \nu h \|_{r_n} + \nu_0 \| u \|_\alpha \| \nabla u \|_{a_{n-1}})$$

for all $n \leq N$, where C_n is a positive constant depending only on α, n and Ω. Therefore, we have

$$\| \nabla u \|_{a_n} + \| \nu\pi \|_{a_n} \leq C_n' \{P (\| \nabla u \|_{a_{n-1}} + \| \nu\pi \|_{a_{n-1}}) + Q_{r_n}\},$$

from which it follows by induction on n that

$$\| \nabla u \|_{a_n} + \| \nu\pi \|_{a_n} \leq c_n \left\{P^n (\| \nabla u \| + \| \nu\pi \|) + \frac{P^n - 1}{P - 1} Q_{r_n}\right\}.$$

The proof of (6.11) is readily achieved. Q. E. D.

We now return to (6.2) and (6.3).

PROPOSITION 6.1. *Let $T > 0$. Suppose that there exists a positive constant E satisfying one of the following conditions*

$$(6.12) \qquad \text{(i)} \begin{cases} \gamma_0^5/\gamma_0^4 > c_0 A_T E \\ \mu_0 \| \nabla u_0 \|^2 < E \end{cases} \quad \textit{and} \quad \text{(ii)} \begin{cases} \mu_0^3 > T^{1/2} E \\ \mu_0 \| \nabla u_0 \|^2 < E \end{cases}$$

and define

$$(6.13) \qquad T_n(E) = \sup \{T^* ; \mu_0 \| \nabla v_n(t) \|^2 < E,\ 0 \leq t < T^* \leq T\}.$$

Then, there exists a positive integer n_0 such that $T_n(E) > 0$ and

$$(6.14) \qquad \| u_n'(t) \|^2 + \frac{\mu_{n,0}}{4} \int_0^t \| \nabla v_n' \|^2 dt \leq I_T + J_T(\mu_0 E + \mu_0^{\lambda-2} A_T^\lambda E^{2-\lambda}),$$

for all $t \leq T_n(E)$ and all $n \geq n_0$, where $\mu_{n,0} = \mu_0 - \varepsilon_n$, and A_T, I_T, J_T are the same as in Theorem 3.

Proof. From (6.2) and (6.3) it follows that

$$(6.15) \quad \langle e_n(v_n(t)) + B(v_n(t)),\ v - v_n(t)\rangle + 2 \langle \mu_n(t) D(v_n(t)),\ D(v - v_n(t))\rangle + \int_\Omega g(t) (| D(v) | - | D(v_n(t)) |)\, dx \geq \langle f(t) - u_n'(t),\ v - v_n(t)\rangle, \quad v \in V,$$

where $\mu_n(t) = \mu(t) - \varepsilon_n$. Inserting $v = v_n(t + h)$, we obtain after a simple calculation

$$\langle \delta_h e_n(v_n) + \delta_h B(v_n), \delta_h v_n\rangle + 2\langle \delta_h(\mu_n D(v_n)), D(\delta_h v_n)\rangle \\ \leq \langle \delta_h(f - u_n'), \delta_h v_n\rangle - \langle \delta_h g, D(\delta_h v_n)\rangle,$$

where $\delta_h u = \{u(t+h) - u(t)\}/h$. Keeping in mind $f - u_n' = \mathscr{L}_n(t, u_n)$ and $\delta_h v_n = \delta_h u_n - \frac{1}{n}\delta_h \mathscr{L}_n(t, u_n)$ and using Schwarz' inequality, we get

$$\tag{6.16} \frac{d}{dt}\| \delta_h u_n \|^2 + \| \sqrt{\mu_n}\, D(\delta_h v_n) \|^2 - 2\langle B(\delta_h(v_n)), v_n(t)\rangle \\ \leq 2\| \sqrt{\nu\mu_n}\, \delta_h \mu \cdot D(v_n) \|^2 + 2\langle \delta_h f, \delta_h u_n\rangle + \| \sqrt{\nu_n}\, \delta_h g \|^2.$$

We first suppose (i) of (6.12) to hold. Then, (6.16), together with (2.27) and (2.28), leads to

$$\tag{6.17} \frac{d}{dt}\| \delta_h u_n \|^2 + \frac{1}{4}(2\mu_{n,0} - \gamma_0 \| v_n \|_3) \| \nabla \delta_h v_n \|^2 \\ \leq \| \delta_h f_n \| + 2\| \nu_n \delta_h \mu \|_b^2 \| \sqrt{\mu_n}\, \nabla v_n \|_a^2 + \| \sqrt{\nu_n}\, \delta_h g \|^2 + \| \delta_h f \| \| \delta_h u_n \|^2,$$

where $v_n = 1/\mu_n$.

On the other hand, from (6.15) with $v = 0$ it immediately follows that

$$\tag{6.18} \frac{1}{2}\frac{d}{dt}\| u_n \|^2 + \varphi_n(t, v_n) \leq \langle f, u_n\rangle.$$

Hence, the use of Gronwall's lemma implies $\| u_n(t) \|^2 \leq A_T$ for all $t \leq T$. Moreover, observing (2.28), (6.4) and (6.12), we readily obtain $T_n(E) > 0$ and

$$\| v_n(t) \|_3^4 \leq c_0 \| u_n(t) \|^2 \| \nabla v_n(t) \|^2 \leq c_0 A_T \nu_0 E, \quad t \leq T_n(E)$$

for all $n \geq n_0$. So that $2\mu_{n,0} - \gamma_0 \| v_n \|_3 \geq \mu_{n,0}$. Integrating (6.17) over the interval $(0, t)$, applying Gronwall's lemma and letting $h \to 0$, we obtain

$$\tag{6.19} \| u_n'(t) \|^2 + \frac{\mu_{n,0}}{4}\int_0^t \| v_n' \|^2 dt \\ \leq \{\| f(0) - \chi \|^2 + \int_0^T \{\| f' \| + 2\| \nu\mu' \|_b^2 \| \sqrt{\mu}\, \nabla v_n \|_a^2 + \| \sqrt{\nu} g' \|^2)\, dt\} \\ \times \exp\left(\int_0^T \| f' \|\, dt\right)$$

for all $t \leq T_n(E)$ and all $n \geq n_0$.

Exactly as in Lemma 6.1 we can derive

$$\tag{6.20} \| \nabla v_n(t) \|_a^2 \leq C_1 \nu_0^2 (\| \nu \nabla \mu(t) \|_a^2 + 1)(\| u_n'(t) \|^2 + \| f(t) \|^2 + g_1^2 \\ + \mu_0^2 \| \nabla v_n(t) \|^2 + \| v_n(t) \|_a^2 \| \nabla v_n(t) \|^2).$$

Employing again Gronwall's lemma after substitution of (6.20) into (6.19), we get (6.14), since $\| v \|_\alpha \le \| v \|^\lambda \| v \|_6^{1-\lambda}$.

Secondly, we suppose (ii) of (6.12) to hold. The use of (2.29) in the LHS of (6.16) implies

$$\begin{aligned}(6.17) \qquad & \frac{d}{dt} \|\delta_h u_n \|^2 + \frac{1}{4}(2\mu_{n,0} - \eta \| \nabla v_n \|) \| \nabla \delta_h v_n \|^2 \\ & \le \| \delta_h f_n \| + \left(1 + \frac{2}{n}\right)(2 \| \nu_n \delta_h \mu \|_b^2 \| \sqrt{\mu_n} \nabla v_n \|_a^2 + \| \sqrt{\nu_n}\, \delta_h g \|^2) \\ & \quad + (\| \delta_h f \| + 2\gamma_1 \eta^{-3} \| \nabla v_n \|) \| \delta_h u_n \|^2,\end{aligned}$$

where $\eta^4 = T$ and we used the inequality:

$$(6.21) \qquad \| \delta_h v_n \|^2 \le 2 \| \delta_h u_n \|^2 + \frac{2}{n}(2 \| \nu_n \delta_h \mu \|_b^2 \| \sqrt{\mu_n} \nabla v_n \|_a^2 + \| \sqrt{\nu_n} \delta_h g \|^2),$$

which is easily derived from (3.14) by observing that

$$\text{the RHS of (3.14)} \le \int_\Omega \{2\mu(t_i) \mid D(v_j) \mid + g(t_i)\} (\mid D(v_j) \mid - \mid D(v_i) \mid)\, dx.$$

Therefore, we have

$$\begin{aligned} & \| u_n'(t) \|^2 + \frac{\mu_{n,0}}{4} \int_0^t \| v_n' \|^2 dt \\ & \le \{\| f(0) - \chi \|^2 + \int_0^T \left(\| f' \| + 2\left(1 + \frac{2}{n}\right) \| \nu\mu' \|_b^2 \| \sqrt{\mu}\, \nabla v_n \|_a^2 + \| \sqrt{\nu} g' \|^2\right) dt\} \\ & \qquad \times \exp\left(\int_0^T \| f' \| dt + \gamma_1 \mu_0\right)\end{aligned}$$

for all $t \le T_n(E)$ and all $n \ge n_0$. By the same argument as above we arrive at (6.14). Q. E. D.

Our next task is to find E such that $T_n(E) = T$. From (6.18) it easily follows that

$$(6.22) \qquad \varphi_n(t, v_n(t))^2 \le 2 \| u_n(t) \|^2 (\| f(t) \|^2 + \| u_n'(t) \|^2).$$

Accordingly, if E is chosen so that

$$(6.23) \qquad 9A_T(\max_{0 \le t \le T} \| f(t) \|^2 + I_T) + 9A_T J_T(\mu_0 E + A_T^\lambda \mu_0^{\lambda-2} E^{2-\lambda}) < E^2,$$

then we can derive from (6.22) and Proposition 6.1 that

$$\mu_0 \| \nabla v_n(t) \|^2 \le \sqrt{9/2}\, \varphi_n(t, v_n(t)) < E$$

for all $t \leq T_n(E)$ and all $n \geq n_0$. Hence, it is concluded that $T_n(E) = T$. In fact, this contradicts the definition (6.13) if $T_n(E) < T$. For the sake of simplicity we write

$$(6.23) \quad \text{as} \quad B_0 + B_1 E + B_2 E^{2-\lambda} < E^2.$$

Set

$$E_1 = (2B_2)^{1/\lambda} \quad \text{and} \quad E_2 = 2B_1 + \sqrt{2B_0}.$$

Then, $B_2 E_1^{2-\lambda} = E_1^2/2$ and $B_0 + B_1 E_2 \leq E_2^2/2$. It is easily verified that $E_T = E_1 + E_2$ satisfies (6.23).

The inequality $\mu_0 \| \nabla u_0 \|^2 < E_T$ is then trivial. Making use of the compactness argument, we thus arrive at (2.32). Evidently, u is a solution of (2.25)-(2.26). Moreover, with the aid of Lemma 6.2 we can prove that (2.33) are bounded. Let l be the integer mentioned in Lemma 6.2. Then, (6.11) implies

$$\| \nabla u \|_q \leq c_l \left\{ P^l(\| \nabla u \| + \| \nu\pi \|) + \frac{P^l - 1}{P - 1} Q_r \right\},$$

where $P(t)$ is bounded and $Q_r(t)$ is the sum of the bounded function and $\| f(t) - u'(t) \|_r$. If $2 \leq q \leq 6$, then $6/5 \leq r \leq 2$. We now suppose $q > 6$. Then, $2 < r < 3$. By (1.10) and Sobolev's inequality we have

$$\| u' \|_r \leq \text{const.} \| u' \|^{1-\delta} \| \nabla u' \|^{\delta},$$

where $\delta = 3(1/2 - 1/r)$ and $1/r = 1/q + 1/3$. Therefore, $\| \nabla u \|_q^p$ is integrable for $p = 2/\delta$, which completes the proof of the fact mentioned above. The uniqueness easily follows from (ii) of Corollary 2.

REFERENCES

[1] G. Astarita and G. Marrucci, Principles of Non-Newtonian Fluid Mechanics, McGraw-Hill, London, 1974.
[2] M. S. Berger, Nonlinear and Functional Analysis, Academic Press, New York-San Francisco-London, 1977.
[3] H. Brezis, Problèmes unilateraux, J. Math. Pures Appl., **51** (1972), 1–168.
[4] L. Cattabriga, Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Mat. Univ. Padova, **31** (1961), 308–340.
[5] G. Duvaut and J. L. Lions, Ecoulement d'un fluide rigide viscoplastique incompressible, C. R. Acad. Sc. Paris, **270** (1970), 58–61.
[6] ——, Les inéquations en méchanique et en physique (English translation), Springer-Verlag, Berlin-Heiderberg-New York, 1976.
[7] J. U. Kim, On the Cauchy problem associated with the motion of a Bingham fluid in

the plane, Trans. Amer. Math. Soc., **298** (1986), 371–400.

[8] ——, On the initial boundary value problem for a Bingham fluid in a three dimensional domain, Trans. Amer. Math. Soc., **304** (1987), 751–770.

[9] J. L. Lions, Quelque méthodes de résolution des problèmes aux limites non linéaires, Dunod/Gauthier-Villars, Paris, 1969.

[10] K. Masuda, Weak solutions of Navier-Stokes equations, Tôhoku Math. J., **36** (1984), 623–646.

[11] T. Miyakawa and H. Sohr, On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains, Math. Z., **199** (1988), 455–478.

[12] P. P. Mosolov and V. P. Mjasnikov, On the correctness of boundary value problems in the mechanics of continuous media, Math. USSR Sbornik, **17** (1972), 257–268.

[13] J. Naumann and Wulst, On evolution inequalities of Bingham type in three dimensions, I, J. Math. Anal. Appl., **68** (1979), 211–227.

[14] ——, On evolution inequalities of Bingham type in three dimensions, II, J. Math. Anal. Appl., **70** (1979), 309–325.

[15] M. Renardy, Dense imbedding of test funtions in certain function spaces, Trans. Amer. Math. Soc., **298** (1986), 241–243.

[16] M. J. Strauss, Variations of Korn's and Sobolev's inequalities, Berkeley symposium on partial differential equations, A.M.S. Symposia, **23** (1971).

[17] R. Temam, Navier-Stokes equations, North-Holland, 1977.

[18] K. Yosida, Functional Analysis, Springer-Verlag, Berlin-Heiderberg-New York, 1966.

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