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ESTIMATES FOR FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS OF DEGREE TWO, II

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Introduction

Let F be a Siegel cusp form of integral weight k on $\Gamma_2 := Sp_2(\boldsymbol{Z})$ and denote by $a(T)$ (T a positive definite symmetric half-integral (2,2)-matrix) its Fourier coefficients. In [2] Kitaoka proved that

(1) $$a(T) \ll_{\varepsilon,F} (\det T)^{k/2-1/4+\varepsilon} \qquad (\varepsilon > 0)$$

(the result is actually stated only under the assumption that k is even).

In our previous paper [3] it was shown that one can attain

(2) $$a(T) \ll_{\varepsilon,F} (\det T)^{k/2-13/36+\varepsilon} \qquad (\varepsilon > 0)$$

provided that $-4\det T$ is a fundamental discriminant, i.e. is the discriminant of an imaginary quadratic field. Unlike Kitaoka's proof of (1) which was based on estimates for generalized matrix argument Kloosterman sums which occur in the Fourier coefficients of Poincaré series of exponential type on Γ_2, the method of [3] used a combination of appropriate bounds both for the Fourier coefficients $c(n, r)$ of Jacobi cusp forms ($n, r \in \mathbf{Z}$; $D := r^2 - 4mn$ a negative fundamental discriminant) and for the Petersson norms of the Fourier-Jacobi coefficients of F.

The purpose of the present paper is to prove that (2), in fact, is true for all positive definite T (§ 2, Theorem). To do so we shall show that the estimate for the coefficients $c(n, r)$ referred to above, is true for all indices (n, r) with $D < 0$ (§ 1, Proposition). The proof which is rather different from that given in [3] in the special case where D is fundamental, is based on a closer direct investigation of the linear combination of Kloosterman sums occurring in the Fourier coefficients of Poincaré series on the Jacobi group. In particular, we shall connect the latter with Salié sums and shall use similar arguments as those given in § 3 of

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Iwaniec's paper [1].

The rest of the proof of (2) in the general case follows the same pattern as in [3]. We shall recall it briefly in § 2, for the reader's convenience.

§ 1. Estimates for Fourier coefficients of Jacobi forms

We let $\mathscr{H}$ be the upper half-plane. We put $\Gamma_1 = SL_2(\mathbf{Z})$ and denote by $\Gamma_1^J := \Gamma_1 \ltimes \mathbf{Z}^2$ the Jacobi group which in the usual way operates on $\mathscr{H} \times \mathbf{C}$.

If $\phi(\tau, z)$ $(\tau \in \mathscr{H}, z \in \mathbf{C})$ is a Jacobi cusp form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}$, we write

$$\| \phi \|^2 = \int_{\Gamma_1^J \backslash \mathscr{H} \times \mathbf{C}} | \phi(\tau, z) |^2 e^{-4\pi m y^2/v} v^{k-3} \, dudvdxdy \qquad (\tau = u + iv,\ z = x + iy)$$

for the square of the Petersson norm of ϕ.

We shall prove

Proposition. *Let ϕ be a Jacobi cusp form of weight $k > 2$ and index m on Γ_1^J and let $c(n, r)$ $(n, r \in \mathbf{Z};\ D := r^2 - 4mn < 0)$ be its Fourier coefficients. Then*

$$c(n, r) \ll_{\varepsilon,k} (m + |D|^{1/2+\varepsilon})^{1/2} \cdot \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} \cdot \| \phi \| \qquad (\varepsilon > 0)$$

where the constant implied in $\ll$ depends only on ε and k.

If D is a fundamental discriminant, this was proved in [3, §1]. Inspecting the proof one checks that the hypothesis that D be fundamental was only used in showing the estimate

$$(3) \qquad H_{m,c}(n, r, n, \pm r) \ll_\varepsilon c^{-1/2+\varepsilon}(D, c) \qquad (\varepsilon > 0)$$

where

$$H_{m,c}(n, r, n, \pm r) := c^{-3/2} \sum_{x(c), y(c)^*} e_c((mx^2 + rx + n)\bar{y} + ny \pm rx) e_{2mc}(\pm r^2) \qquad (c \in \mathbf{N})$$

(notation: x resp. y run through $\mathbf{Z}/c\mathbf{Z}$ resp. $(\mathbf{Z}/c\mathbf{Z})^*$, we have written $\bar{y}$ for an inverse of $y \pmod c$ and $e_a(b) := e^{2\pi i b/a}$ for $a \in \mathbf{N}$, $b \in \mathbf{Z}/a\mathbf{Z}$), cf. [3, §1, Lemma].

To *prove* the proposition it is therefore sufficient to show that (3) is valid for an arbitrary negative discriminant D. This is implied by the following

LEMMA. *For m, $c \in \mathbf{N}$ and n, $r \in \mathbf{Z}$ put*

$$H^{\pm}_{m,c}(n, r) := \sum_{x(c),y(c)*} e_c((mx^2+rx+n)\bar{y}+ny \pm rx)$$

(*same notation as above*). *Put $D := r^2 - 4mn$. Then*

$$H^{\pm}_{m,c}(n, r) \ll_{\varepsilon} c^{1+\varepsilon}(D, c) \qquad (\varepsilon > 0). \tag{4}$$

Proof. Suppose that $c = c_1c_2$ with c_1, $c_2 \in \mathbf{N}$ and $(c_1, c_2) = 1$. Write $x \equiv x_1c_2 + x_2c_1 \pmod{c}$ with $x_i \in \mathbf{Z}/c_1\mathbf{Z}$ $(i = 1, 2)$ and $y \equiv y_1c_2\bar{c_2} + y_2c_1\bar{c_1} \pmod{c}$ with $y_i \in (\mathbf{Z}/c_i\mathbf{Z})^*$ $(i = 1, 2)$ and where $\bar{c_1}$ resp. $\bar{c_2}$ denote inverses of c_1 resp. c_2 mod c_2 resp. mod c_1 (thus $\bar{y} \equiv \bar{y_1}\,\bar{c_2}\,c_2 + \bar{y_2}\,\bar{c_1}\,c_1 \pmod{c}$). Then it is easily checked that

$$H^{\pm}_{m,c}(n, r) = H^{\pm}_{mc_1,c_2}(n\bar{c_1}, r)\, H^{\pm}_{mc_2,c_1}(n\bar{c_2}, r).$$

Since $D \equiv r^2 - 4mc_1n\bar{c_1} \pmod{c_2}$ and $D \equiv r^2 - 4mc_2n\bar{c_2} \pmod{c_1}$, it is therefore sufficient to prove (4) for c a prime power, $c = p^{\nu}$ $(\nu \geq 1)$.

We shall distinguish several cases. Denote by p^{μ} $(\mu \geq 0)$ resp. p^{ρ} $(\rho \geq 0)$ the exact p-power of m resp. r (we put $\mu = \infty$ resp. $\rho = \infty$ if $m = 0$ resp. $r = 0$).

We shall first look at the trivial case where $\mu \geq \nu$. Then

$$\begin{aligned} H^{\pm}_{m,c}(n, r) &= \sum_{x(p^{\nu}),y(p^{\nu})*} e_{p^{\nu}}((rx+n)\,\bar{y}+ny \pm rx) \\ &= \sum_{y(p^{\nu})*} e_{p^{\nu}}(n(y+\bar{y})) \sum_{x(p^{\nu})} e_{p^{\nu}}((\bar{y}\pm 1)x) \\ &= p^{\nu} \sum_{y(p^{\nu})*,\, r(\bar{y}\pm 1)\equiv 0(p^{\nu})} e_{p^{\nu}}(n(y+\bar{y})). \end{aligned}$$

The number of elements y in $(\mathbf{Z}/p^{\nu}\mathbf{Z})^*$ with $r(\bar{y}\pm 1) \equiv 0(p^{\nu})$ is $\leq p^{\min\{\nu,\rho\}}$, and $p^{\min\{\nu,\rho\}} \leq p^{\min\{\mu,2\rho,\nu\}} \leq (D, p^{\nu})$ since $\nu \leq \mu$, so (4) follows.

Now suppose that $\mu < \nu$ and write

$$H^{\pm}_{m,c}(n, r) = \sum_{y(p^{\nu})*}\Big(\sum_{x(p^{\nu})} e_{p^{\nu}}(\bar{y}mx^2 + r(\bar{y}\pm 1)x + n(y+\bar{y}))\Big).$$

If in the inner sum x is replaced by $x + p^{\nu-\mu}$, then the inner sum is multiplied with $e_{p^{\mu}}(r(\bar{y}\pm 1))$, hence vanishes unless $p^{\mu} \nmid r(\bar{y}\pm 1)$. Therefore

$$H^{\pm}_{m,c}(n, r) = p^{\mu} \sum_{y(p^{\nu})*,\, r(\bar{y}\pm 1)\equiv 0(p^{\mu})} e_{p^{\nu}}(n(y+\bar{y})) \sum_{x(p^{\nu-\mu})} e_{p^{\nu-\mu}}\Big(\bar{y}\frac{m}{p^{\mu}}x^2 + \frac{r(\bar{y}\pm 1)}{p^{\mu}}x\Big).$$

In the following we shall assume that $p \neq 2$ (the case $p = 2$ which is similar, will be left to the reader). If we use the standard formula (Gauss sum)

(5) $$\sum_{x(p^\gamma)} e_{p^\gamma}(ax^2+bx) = p^{\gamma/2}\varepsilon(p^\gamma)\left(\frac{a}{p^\gamma}\right) e_{p^\gamma}(-b^2\cdot\overline{4a})$$

(p odd, $p \nmid a$; $\varepsilon(p^\gamma)=1$ or i according as $p^\gamma \equiv 1 (\mathrm{mod}\ 4)$ or $\equiv 3\ (\mathrm{mod}\ 4)$; $\overline{4a}\cdot 4a \equiv 1\ (\mathrm{mod}\ p^\gamma)$) with $\gamma = \nu-\mu$, $a = \bar{y}\dfrac{m}{p^\mu}$ and $b = \dfrac{r(\bar{y}\pm 1)}{p^\mu}$ we find

$$H^{\pm}_{m,c}(n,r) = p^{(\nu+\mu)/2}\varepsilon(p^{\nu-\mu})\left(\frac{m/p^\mu}{p^{\nu-\mu}}\right)\sum\nolimits_{y(p^\nu)^*, r(\bar{y}\pm1)\equiv 0(p^\mu)}\left(\frac{y}{p^{\nu-\mu}}\right)e_{p^\nu}(n(y+\bar{y}))$$
$$\cdot e_{p^{\nu-\mu}}\left(-\frac{r^2(\bar{y}\pm1)^2}{p^{2\mu}}\cdot\overline{4\frac{m}{p^\mu}\cdot y}\right).$$

If m_1 denotes an inverse of $4\dfrac{m}{p^\mu}$ (mod $p^{\nu+\mu}$), the latter equality can more conveniently be written as

(6) $$H^{\pm}_{m,c}(n,r) = p^{(\nu-\mu)/2}\varepsilon(p^{\nu-\mu})\left(\frac{m/p^\mu}{p^{\nu-\mu}}\right)e_{p^{\nu+\mu}}(\mp 2r^2m_1)\sum\nolimits_{y(p^{\nu+\mu})^*, r(y\pm1)\equiv 0(p^\mu)}\left(\frac{y}{p}\right)^{\nu+\mu}$$
$$e_{p^{\nu+\mu}}(-m_1D(y+\bar{y})).$$

The number of summands in the sum over y in (6) is $\leq p^{\nu+\mu}$. Hence

$$|H^{\pm}_{m,c}(n,r)| \leq p^{(\nu-\mu)/2}\cdot p^{\nu+\mu},$$

and this is less than $(p^\nu)^{1+\varepsilon}(D,p^\nu)$ if $p^\nu \mid D$, since $\mu < \nu$ by assumption.

Now suppose that $p^\lambda (\lambda \geq 0, \lambda < \nu)$ exactly divides D. We then see from (6) that

$$|H^{\pm}_{m,c}(n,r)| \leq p^{(\nu-\mu)/2+\lambda}\,|S^{\pm}_{p^\alpha,p^\kappa}(\Delta)|$$

where

$$S^{\pm}_{p^\alpha,p^\kappa}(\Delta) := \sum_{y(p^\alpha)^*, y\equiv\mp1(p^\kappa)}\left(\frac{y}{p}\right)^{\nu+\mu}e_{p^\alpha}(\Delta(y+\bar{y})),$$
$$\alpha := \nu-\lambda+\mu,\quad \kappa := \max\{\mu-\rho, 0\},\ \Delta := -m_1\frac{D}{p^\lambda}.$$

Note that $p \nmid \Delta$. We shall simply write S for $S^{\pm}_{p^\alpha,p^\kappa}(\Delta)$.

To prove (4) we have to show that

(7) $$S \ll_\varepsilon p^{\varepsilon\nu}\cdot p^{(\nu+\mu)/2}.$$

The number of summands in S is $\leq p^{\alpha-\kappa} = p^{(\nu+\mu)/2+\alpha/2-(\kappa+\lambda/2)}$, hence is $\leq p^{(\nu+\mu)/2}$ if $\alpha/2 \leq \kappa+\lambda/2$. We may therefore suppose that

(8) $$\alpha/2 > \kappa+\lambda/2.$$

If $\alpha=1$ (so $\mu=\kappa=\lambda=0$, $\nu=1$), then S is a Salié sum, hence

$$S \ll_{\varepsilon} p^{1/2+\varepsilon},$$

by Lemma 4 in [1] with $\alpha=1$ and (7) follows.

Let us now suppose that $\alpha \geq 2$. To prove (7) we shall use similar arguments as those given in the proof of Lemma 4 in [1] in the case $\alpha \geq 2$.

First assume that α is even, $\alpha=2\beta$, $\beta \geq 1$. Write $y \equiv u+vp^{\beta} \pmod{p^{2\beta}}$ with u resp. v running over $(\mathbf{Z}/p^{\beta}\mathbf{Z})^{*}$ resp. $\mathbf{Z}/p^{\beta}\mathbf{Z}$. Then $\bar{y} \equiv \bar{u}-\bar{u}^{2}vp^{\beta} \pmod{p^{2\beta}}$, where $\bar{u}u \equiv 1 \pmod{p^{2\beta}}$, hence using (8) we find

$$\begin{aligned} S &= \sum_{u(p^{\beta})^{*}, u \equiv \mp 1(p^{\kappa})} \left(\frac{u}{p}\right)^{\nu+\mu} e_{p^{\alpha}}(\Delta(u+\bar{u})) \sum_{v(p^{\beta})} e_{p^{\beta}}(\Delta(u^{2}-1)v) \\ &= p^{\beta} \sum_{u(p^{\beta})^{*}, u^{2} \equiv 1(p^{\beta}), u \equiv \mp 1(p^{\kappa})} \left(\frac{u}{p}\right)^{\nu+\mu} e_{p^{\alpha}}(\Delta(u+\bar{u})). \end{aligned}$$

The congruence $u^{2} \equiv 1 \pmod{p^{\beta}}$ has two solutions ± 1 in $\mathbf{Z}/p^{\beta}\mathbf{Z}$, whence

$$|S| \leq 2p^{\beta} = 2p^{(\nu-\lambda+\mu)/2} \ll_{\varepsilon} p^{\varepsilon\nu} \cdot p^{(\nu+\mu)/2}.$$

Suppose that $\alpha=2\beta+1$, $\beta \geq 1$. Write $y \equiv u+vp^{\beta+1} \pmod{p^{2\beta+1}}$ with $u \in (\mathbf{Z}/p^{\beta+1}\mathbf{Z})^{*}$, $v \in \mathbf{Z}/p^{\beta}\mathbf{Z}$. Then $\bar{y} \equiv \bar{u}-\bar{u}^{2}vp^{\beta+1} \pmod{p^{2\beta+1}}$, where $\bar{u}u \equiv 1 \pmod{p^{2\beta+1}}$, so by (8) we get

$$\begin{aligned} S &= \sum_{u(p^{\beta+1})^{*}, u \equiv \mp 1(p^{\kappa})} \left(\frac{u}{p}\right)^{\nu+\mu} e_{p^{\alpha}}(\Delta(u+\bar{u})) \sum_{v(p^{\beta})} e_{p^{\beta}}(\Delta(u^{2}-1)v) \\ &= p^{\beta} \sum_{u(p^{\beta+1})^{*}, u^{2} \equiv 1(p^{\beta}), u \equiv \mp 1(p^{\kappa})} \left(\frac{u}{p}\right)^{\nu+\mu} e_{p^{\alpha}}(\Delta(u+\bar{u})). \end{aligned}$$

The solutions to $u^{2} \equiv 1 \pmod{p^{\beta}}$ in $\mathbf{Z}/p^{\beta+1}\mathbf{Z}$ are given by $u \equiv \pm 1+wp^{\beta}$ with w ranging over $\mathbf{Z}/p\mathbf{Z}$, and then $\bar{u} \equiv \pm 1-wp^{\beta} \pm w^{2}p^{2\beta} \pmod{p^{2\beta+1}}$.

Note that (8) implies that $\beta \geq \kappa$. Therefore

$$S = \begin{cases} p^{\beta} e_{p^{\alpha}}(2\Delta) \sum_{w(p)} e_{p}(\Delta w^{2}) + p^{\beta} e_{p^{\alpha}}(-2\Delta) \left(\frac{-1}{p}\right)^{\nu+\mu} \sum_{w(p)} e_{p}(-\Delta w^{2}) & (\kappa=0) \\ p^{\beta} e_{p^{\alpha}}(\mp 2\Delta) \left(\frac{\mp 1}{p}\right)^{\nu+\mu} \sum_{w(p)} e_{p}(-\Delta w^{2}) & (\kappa=0), \end{cases}$$

hence by (5) with $\gamma=1$ we find

$$|S| \leq 2p^{\beta+1/2} = 2p^{(\nu-\lambda+\mu)/2} \ll_{\varepsilon} p^{\varepsilon\nu} \cdot p^{(\nu+\mu)/2}.$$

This proves the Lemma.

§ 2. Estimates for Fourier coefficients of Siegel cusp forms

Using the Proposition of §1 we shall show in the same way as in [3]:

THEOREM. *Let F be a Siegel cusp form of integral weight k on Γ_2 and denote by $a(T)$ (T a positive definite symmetric half-integral $(2,2)$-matrix) its Fourier coefficients. Then*

$$a(T) \ll_{\varepsilon,F} (\det T)^{k/2-13/36+\varepsilon} \qquad (\varepsilon > 0). \tag{9}$$

We briefly recall the idea of proof. We may suppose that $F \neq 0$ (hence $k \geq 10$ as is well-known). Both sides of (9) are invariant if T is replaced by $U'TU$ ($U \in GL_2(\mathbf{Z})$), so we may assume that $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ with $m = \min T$ where $\min T$ denotes the least positive integer represented by T.

If ϕ_m ($m \in \mathbf{N}$) denote the Fourier-Jacobi coefficients of F, then it was proved in [3, §2, Theorem 2] that

$$\| \phi_m \| \ll_{\varepsilon,F} m^{k/2-2/9+\varepsilon} \qquad (\varepsilon > 0). \tag{10}$$

Observing that $a(T)$ is the (n,r)-th Fourier coefficient of ϕ_m and combining the Proposition in §1, the estimate (10) and the bound $m = \min T \ll (\det T)^{1/2}$ (which is well-known from reduction theory), we obtain (9).

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