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# FOURIER-EISENSTEIN TRANSFORM AND PLANCHEREL FORMULA FOR RATIONAL BINARY QUADRATIC FORMS

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## § 0. Introduction

**0.1.** Let $\boldsymbol{X}$ be the space of nondegenerate rational symmetric matrices of size 2 and put

$$G = \{g \in GL_2(\mathbf{Q}) \mid \det g > 0\} \text{ and } \Gamma = SL_2(\mathbf{Z}).$$

The group $G$ acts on $\boldsymbol{X}$ by

$$g * x = (\det g)^{-1} \cdot g x {}^t g.$$

We are interested in the space $\mathscr{C}^\infty(\Gamma \backslash \boldsymbol{X})$ of $\Gamma$-invariant $\mathbf{C}$-valued functions on $\boldsymbol{X}$ and its subspace $\mathscr{S}(\Gamma \backslash \boldsymbol{X})$ of functions whose supports consist of a finite number of $\Gamma$-orbits. The Hecke algebra $\mathscr{H}(G, \Gamma)$ of $G$ with respect to $\Gamma$ acts naturally on these spaces.

For an $x \in \boldsymbol{X}$, let $K = \mathbf{Q}(\sqrt{-\det x})$ or $\mathbf{Q} \oplus \mathbf{Q}$ according as $-\det x \notin (\mathbf{Q}^\times)^2$ or $\in (\mathbf{Q}^\times)^2$. Take a positive rational number $r$ such that $rx$ is primitive half-integral and let $\mathfrak{f}(x)$ be the conductor of $rx$. For any positive integer $f$, denote by $\mathcal{O}_f^1$ the group of units with positive norm of the order of conductor $f$ of $K$. We define the Eisenstein series (zeta functions of binary quadratic forms) on $\boldsymbol{X}$ by

$$E(x; s_1, s_2) = \frac{1}{\mu(x)} \cdot \sum_{\substack{v \in \mathbf{Z}^2 / SO(x)_{\mathbf{Z}} \\ v x {}^t v > 0}} \frac{1}{| v x {}^t v |^{s_1 + \frac{1}{2}} \, | \det x |^{s_2 - \frac{1}{4}}},$$

where $\mu(x) = [\mathcal{O}_1^1 : \mathcal{O}_{\mathfrak{f}(x)}^1]$. As a function of $x$, the series $E(x; s_1, s_2)$ is in $\mathscr{C}^\infty(\Gamma \backslash \boldsymbol{X})$ and will turn out to be a $\mathscr{H}(G, \Gamma)$-eigenfunction.

The purpose of the present paper is analysing the structure of $\mathscr{S}(\Gamma \backslash \boldsymbol{X})$ as $\mathscr{H}(G, \Gamma)$-module through an integral transform with kernel function $E(x; s_1, s_2)$, which we call the *Fourier-Eisenstein transform.*

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**0.2.** Let $K = \mathbf{Q} \oplus \mathbf{Q}$ or a quadratic number field and $D = D_K$ the discriminant of $K$. We understand that $D = 1$ if $K = \mathbf{Q} \oplus \mathbf{Q}$. For $r \in \mathbf{Q}$, $r > 0$, we put

$$X_{D,r} = \left\{x \in \boldsymbol{X} \,\middle|\, \det x = -\frac{r^2 D}{4}\right\}$$

and, if $D < 0$, we further put

$$X_{D,r}^+ = \{x \in \boldsymbol{X} \mid \det x = -\frac{r^2 D}{4},\ x \text{ is positive definite}\},$$

$$X_{D,r}^- = \{x \in \boldsymbol{X} \mid \det x = -\frac{r^2 D}{4},\ x \text{ is negative definite}\}.$$

Then the $G$-orbit decomposition of $\boldsymbol{X}$ is given by

$$\boldsymbol{X} = \left\{\bigsqcup_{D>0} \bigsqcup_{\substack{r \in \mathbf{Q}\\ r>0}} X_{D,r}\right\} \sqcup \left\{\bigsqcup_{D<0} \bigsqcup_{\substack{r \in \mathbf{Q}\\ r>0}} (X_{D,r}^+ \sqcup X_{D,r}^-)\right\}.$$

This yields the decomposition

$$\mathcal{S}(\Gamma\backslash \boldsymbol{X}) = \left\{\bigoplus_{D>0} \bigoplus_{\substack{r \in \mathbf{Q}\\ r>0}} \mathcal{S}(\Gamma\backslash X_{D,r})\right\} \oplus \left\{\bigoplus_{D<0} \bigoplus_{\substack{r \in \mathbf{Q}\\ r>0}} (\mathcal{S}(\Gamma\backslash X_{D,r}^+) \oplus \mathcal{S}(\Gamma\backslash X_{D,r}^-))\right\}$$

into direct sum of $\mathcal{H}(G, \Gamma)$-submodules. Here we denote by $\mathcal{S}(\Gamma\backslash X_{D,r}^{(\pm)})$ the subspace of $\mathcal{S}(\Gamma\backslash \boldsymbol{X})$ consisting of functions whose supports are contained in $X_{D,r}^{(\pm)}$. For a fixed $D > 0$ (resp. $D < 0$), all $\mathcal{S}(\Gamma\backslash X_{D,r})$ (resp. $\mathcal{S}(\Gamma\backslash X_{D,r}^{\pm})$) ($r \in \mathbf{Q}$, $r > 0$) are isomorphic $\mathcal{H}(G, \Gamma)$-modules. Therefore it suffices to consider $\mathcal{S}(\Gamma\backslash X)$, where $X = X_{D,1}$ or $X_{D,1}^{\pm}$.

Let $\mathfrak{X}^{\mathrm{pr}}$ be the set of all primitive characters of the narrow ideal class groups of (not necessarily maximal) orders of $K$. Then we can define an orthogonal family of projections $\{p_\chi \mid \chi \in \mathfrak{X}^{\mathrm{pr}}\}$ of the $\mathcal{H}(G,\Gamma)$-module $\mathcal{S}(\Gamma\backslash X)$ and we have the direct sum decomposition

$$\mathcal{S}(\Gamma\backslash X) = \bigoplus_{\chi \in \mathfrak{X}^{\mathrm{pr}}} \mathcal{S}(\Gamma\backslash X)_\chi, \tag{0.1}$$

where $\mathcal{S}(\Gamma\backslash X)_\chi = p_\chi(\mathcal{S}(\Gamma\backslash X))$.

Let

$$\mathfrak{R} = \mathbf{C}\,[2^t + 2^{-t}, 3^t + 3^{-t}, \dots, p^t + p^{-t}, \dots],$$

where $p$ runs over all rational primes. Define a homomorphism $\hat{\ }: \mathcal{H}(G,\Gamma) \to \mathfrak{R}$ by

$$T(p, p)\hat{\ } = 1,\ T(p, 1)\hat{\ } = p^{1/2}(p^t + p^{-t}) \text{ for any rational prime } p,$$

where $T(p, p)$ and $T(p, 1)$ are the characteristic functions of $\Gamma\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}\Gamma$ and

$\Gamma\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\Gamma$, respectively. We consider the ring $\mathfrak{R}$ as an $\mathscr{H}(G, \Gamma)$-module through this homomorphism. Then our main result (Theorem 3) is as follows:

THEOREM. (i) *For* $\varphi \in \mathcal{S}(\Gamma\backslash X)_\chi$, *put*

$$F_\chi(\varphi)(t) = a_\chi \{ \sum_{x\in\Gamma\backslash X} \varphi(x)\mu(x)E(x;\, t,\, 0)\} / (f_\chi)^t L\Big(\chi;\, t+\frac{1}{2}\Big),$$

*where* $L(\chi; s)$ *is the Hecke* $L$*-function attached to the class character* $\chi$ *and* $a_\chi$ *is a normalizing constant. Then* $F_\chi(\varphi)(t)$ *is contained in* $\mathfrak{R}$ *and the mapping*

$$F_\chi : \mathcal{S}(\Gamma\backslash X)_\chi \to \mathfrak{R}$$

*is an isomorphism of* $\mathscr{H}(G, \Gamma)$*-modules.*

(ii) *We have an* $\mathscr{H}(G, \Gamma)$*-module isomorphism*

$$\mathcal{S}(\Gamma\backslash X)_\chi \cong \mathscr{H}(G, \Gamma)/\mathscr{I},$$

*where* $\mathscr{I}$ *is the ideal of* $\mathscr{H}(G, \Gamma)$ *generated by*

$$\{T(p, p) - 1 \mid p : \text{rational primes}\}.$$

We define a structure of pre-Hilbert space on $\mathcal{S}(\Gamma\backslash X)_\chi$ via the inner product

$$\langle\varphi, \psi\rangle_{\mathcal{S}} = \sum_{x\in\Gamma\backslash X} \mu(x)\varphi(x)\overline{\psi(x)}.$$

Let $L^2(\Gamma\backslash X)_\chi$ be the completion of $\mathcal{S}(\Gamma\backslash X)_\chi$. Moreover we construct a Hilbert space $\mathscr{L}^2_\chi$ which is a completion of $\mathfrak{R}$ with respect to an explicitly given inner product $\langle\ ,\ \rangle_\chi$ and prove that the mapping $F_\chi$ can be extended to an isometry of $L^2(\Gamma\backslash X)$ onto $\mathscr{L}^2_\chi$ (Theorem 4). This result may be considered as the Plancherel formula for the (normalized) Fourier-Eisenstein transform $F_\chi$.

An explicit form of the inverse transformation of $F_\chi$ follows quite easily from the Plancherel formula (Theorem 5). Furthermore, using the main theorem, we can determine all $\mathscr{H}(G, \Gamma)$-eigenfunctions in $\mathscr{C}^\infty(\Gamma\backslash X)$ (Theorem 6).

**0.3.** Let $K$ be a real quadratic field. Then the set $K - \mathbf{Q}$ can naturally be identified with the space $X = X_{D,1}$, where $D$ is the discriminant of $K$. The action of $G$ on $K - \mathbf{Q}$ is given by linear fractional transformation. Arakawa [A] and Lu [Lu] constructed certain $\mathscr{H}(G, \Gamma)$-eigenfunctions by arithmetic means. In §5, we shall dicuss these eigenfunctions from our point of view.

**0.4.** In [M1] and [M2], Mautner took up the same problem for positive definite forms and obtained the decomposition (0.1). He further noted that $\mathscr{H}(G, \Gamma)$-eigenfunctions are products of local eigenfunctions. Our investigation can be viewed as a development of his work. We complete his results with the Plancherel formula, an explicit formula for eigenfunctions, the relation between eigenfunctions and zeta functions of binary quadratic forms, and a generalization to the case of indefinite forms.

**0.5.** In [SH], we have defined Eisenstein series and Fourier-Eisenstein transforms for reductive symmetric spaces and showed that analogous results can be obtained at least for the symmetric spaces $GL(n) \times GL(n)/\Delta(GL(n))$, $GL(2n)/Sp(n)$ and $GL(m+n)/GL(m) \times GL(n)$. Thus it is quite natural to expect that the results in the present paper will turn out to be one of the simplest examples of a general phenomenon.

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## § 1. Function spaces and the invariant measure on the set of rational binary quadratic forms

**1.1.** Let

$$\boldsymbol{X} = \{g \in GL_2(\mathbf{Q}) \mid {}^t g = g\} \text{ and } G = GL_2^+(\mathbf{Q}) = \{g \in GL_2(\mathbf{Q}) \mid \det g > 0\}.$$

The group $G$ acts on $\boldsymbol{X}$ by

$$g * x = (\det g)^{-1} \cdot g x\, {}^t g.$$

Put $\Gamma = SL_2(\mathbf{Z})$ and consider the following function spaces:

$$\begin{aligned}
\mathscr{C}^\infty(\Gamma \backslash \boldsymbol{X}) &= \{\varphi : \boldsymbol{X} \to \mathbf{C} \mid \varphi(\gamma * x) = \varphi(x), \text{ for every } \gamma \in \Gamma\}, \\
\mathscr{S}(\Gamma \backslash \boldsymbol{X}) &= \{\varphi \in \mathscr{C}^\infty(\Gamma \backslash \boldsymbol{X}) \mid \varphi = 0 \text{ outside a finite union of } \Gamma\text{-orbits}\}, \\
\mathscr{C}^\infty(\Gamma \backslash Y) &= \{\varphi \in \mathscr{C}^\infty(\Gamma \backslash \boldsymbol{X}) \mid \operatorname{Supp} \varphi \subset Y\}, \\
\mathscr{S}(\Gamma \backslash Y) &= \{\varphi \in \mathscr{S}(\Gamma \backslash \boldsymbol{X}) \mid \operatorname{Supp} \varphi \subset Y\},
\end{aligned}$$

where $Y$ is a $G$-stable subset of $\boldsymbol{X}$. Denote by $\mathrm{ch}_{\Gamma g \Gamma}$ $(g \in G)$ the characteristic function of the double coset $\Gamma g \Gamma$. As usual, the Hecke algebra $\mathscr{H}(G, \Gamma)$ of $G$ with respect to $\Gamma$ is defined to be the $\mathbf{C}$-vector space spanned by $\{\mathrm{ch}_{\Gamma g \Gamma} \mid g \in G\}$ with product

$$\mathrm{ch}_{\Gamma g\Gamma}\cdot\mathrm{ch}_{\Gamma h\Gamma}=\sum_{\Gamma k\Gamma\in\Gamma\backslash G/\Gamma} m_k\mathrm{ch}_{\Gamma k\Gamma},$$

where

$$m_k=\#\{(i,j)\mid g_ih_j\in k\Gamma\},\ \Gamma_g\Gamma=\bigsqcup_i g_i\Gamma,\ \Gamma h\Gamma=\bigsqcup_j h_j\Gamma.$$

Here we use the symbol $\bigsqcup$ to indicate disjoint union. We define an action of $\mathscr{H}(G,\Gamma)$ on $\mathscr{C}^\infty(\Gamma\backslash\boldsymbol{X})$ by

$$(\mathrm{ch}_{\Gamma g\Gamma}*\varphi)(x)=\sum_i\varphi(g_i^{-1}*x),\ \text{where}\ \Gamma g\Gamma=\bigsqcup_i g_i\Gamma.$$

Then, for any $G$-stable subset $Y$, the spaces $\mathscr{C}^\infty(\Gamma\backslash Y)$ and $\mathscr{S}(\Gamma\backslash Y)$ are $\mathscr{H}(G,\Gamma)$-submodules of $\mathscr{C}^\infty(\Gamma\backslash\boldsymbol{X})$.

Our aim is to determine the $\mathscr{H}(G,\Gamma)$-module structure of $\mathscr{S}(\Gamma\backslash\boldsymbol{X})$. For the discriminant $D$ of a quadratic field or $\mathbf{Q}$ and $r\in\mathbf{Q}$, $r>0$, put

$$\begin{aligned}
X_{D,r}&=\{x\in\boldsymbol{X}\mid\det x=-r^2D/4\} && \text{if } D>0,\\
X^+_{D,r}&=\{x\in\boldsymbol{X}\mid\det x=-r^2D/4,\ x \text{ is positive definite}\} && \text{if } D<0,\\
X^-_{D,r}&=\{x\in\boldsymbol{X}\mid\det x=-r^2D/4,\ x \text{ is negative definite}\} && \text{if } D<0.
\end{aligned}$$

Then $G$ acts on these subsets transitively and we get the orbit decomposition

$$\boldsymbol{X}=\Big\{\bigsqcup_{D>0}\bigsqcup_{\substack{r\in\mathbf{Q}\\ r>0}}X_{D,r}\Big\}\sqcup\Big\{\bigsqcup_{D<0}\bigsqcup_{\substack{r\in\mathbf{Q}\\ r>0}}(X^+_{D,r}\sqcup X^-_{D,r})\Big\},$$

and the direct sum decomposition

$$\mathscr{S}(\Gamma\backslash\boldsymbol{X})=\Big\{\bigoplus_{D>0}\bigoplus_{\substack{r\in\mathbf{Q}\\ r>0}}\mathscr{S}(\Gamma\backslash X_{D,r})\Big\}\oplus\Big\{\bigoplus_{D<0}\bigoplus_{\substack{r\in\mathbf{Q}\\ r>0}}(\mathscr{S}(\Gamma\backslash X^+_{D,r})\oplus\mathscr{S}(\Gamma\backslash X^-_{D,r}))\Big\}$$

as $\mathscr{H}(G,\Gamma)$-module. Since $X_{D,r}=\{rx\mid x\in X_{D,1}\}$ for $D>0$ and $X^\pm_{D,r}=\{\pm rx\mid x\in X^+_{D,1}\}$ for $D<0$, we have the following isomorphisms of $\mathscr{H}(G,\Gamma)$-modules:

$$\begin{aligned}
\mathscr{S}(\Gamma\backslash X_{D,r})&\cong\mathscr{S}(\Gamma\backslash X_{D,1})\quad(D>0),\\
\mathscr{S}(\Gamma\backslash X^\pm_{D,r})&\cong\mathscr{S}(\Gamma\backslash X^+_{D,1})\quad(D<0).
\end{aligned}$$

Hence it suffices to consider only $\mathscr{S}(\Gamma\backslash X_{D,1})$ $(D>0)$ and $\mathscr{S}(\Gamma\backslash X^+_{D,1})$ $(D<0)$.

**1.2.** In the following, we always fix the discriminant $D$ of a quadratic field or $\mathbf{Q}$ and put $X=X_{D,1}$ (resp. $X^+_{D,1}$) if $D>0$ (resp. $D<0$). We also put

$$K=K_D=\begin{cases}\mathbf{Q}\oplus\mathbf{Q} & \text{if } D=1\\ \mathbf{Q}(\sqrt{D}) & \text{if } D\neq 1.\end{cases}$$

We define the norm $N:K\to\mathbf{Q}$ by

$$N(x) = \begin{cases} x_1 x_2 & \text{if } D = 1 \text{ and } x = (x_1, x_2) \\ N_{K/\mathbf{Q}}(x) & \text{if } D \neq 1. \end{cases}$$

Let $\mathbf{P}$ be the set of rational primes. We define Dirichlet characters $\chi_K$ and $\chi_{K,f}$ with $f \in \mathbf{N}$ as follows: for $p \in \mathbf{P}$,

$$\chi_K(p) = \quad 1 \quad \text{if } D = 1,$$

$$\chi_K(p) = \begin{cases} 1 & \text{if } p \text{ splits in } K \\ -1 & \text{if } p \text{ is inert in } K \\ 0 & \text{if } p \text{ ramifies in } K \end{cases} \quad \text{if } D \neq 1,$$

$$\chi_{K,f}(p) = \begin{cases} \chi_K(p) & \text{if } p \nmid f \\ 0 & \text{if } p \mid f. \end{cases}$$

For each natural number $f$, let $\mathcal{O}_f$ be the $\mathbf{Z}$-order in $K$ of conductor $f$, i.e.

$$\mathcal{O}_f = \begin{cases} \{(x, y) \in \mathbf{Z}^2 \mid x \equiv y \ (\mathrm{mod}\ f)\} & \text{if } D = 1 \\ \left[1, \dfrac{f(D + \sqrt{D})}{2}\right] & \text{if } D \neq 1, \end{cases}$$

and let

$$\mathcal{O}_f^1 = \{x \in \mathcal{O}_f \mid N(x) = 1\}.$$

We have used the symbol $[\alpha, \beta]$ to denote the $\mathbf{Z}$-lattice in $K$ with $\mathbf{Z}$-basis $\{\alpha, \beta\}$. For simplicity, we write $\mathcal{O} = \mathcal{O}_1$ and $\mathcal{O}^1 = \mathcal{O}_1^1$.

For an $\mathcal{O}_f$-ideal $\mathfrak{a}$, we define its norm by $N_f(\mathfrak{a}) = [\mathcal{O}_f : \mathfrak{a}]$. Then , for $\alpha \in \mathcal{O}_f$, we have $N_f(\alpha\mathcal{O}_f) = |N(\alpha)|$.

A full $\mathbf{Z}$-lattice $\mathfrak{a}$ in $\mathcal{O}$ is called an *$\mathcal{O}_f$-proper ideal* if $\{x \in K \mid \mathfrak{a}x \subseteq \mathfrak{a}\} = \mathcal{O}_f$. Let $I_f$ be the multiplicative semigroup of all $\mathcal{O}_f$-proper ideals. As usual, we write $\mathfrak{a} \sim \mathfrak{b}$ if $\mathfrak{b} = \mathfrak{a}x$ for some $x \in K$ with $N(x) > 0$. Then the narrow ideal class group $Cl_f$ is defined by $Cl_f = I_f/\sim$. We denote by $h_f$ the order of $Cl_f$. It is known that the class number $h_f$ is given explicitly by

$$h_f = \frac{f h_K}{[\mathcal{O}^1 : \mathcal{O}_f^1]} \prod_{p \mid f} (1 - \chi_K(p) p^{-1}), \tag{1.1}$$

where $h_K = h_1$ (cf. [L, Chapter 8, Theorem 7], for example). For $D = 1$, it is easy to see that

$$I_f = \{[(n, mt), (0, ft)] \mid n, t > 0,\ 0 \leq m < f,\ (f, m) = 1,\ n \equiv mt (\mathrm{mod}\ f)\}$$

and

$$Cl_f \cong (\mathbf{Z}/f\mathbf{Z})^\times.$$

Now we recall the correspondence between the set of ideal classes and the set of equivalence classes of primitive binary quadratic forms. For $S, T \in \boldsymbol{X}$, we say that $S$ and $T$ are equivalent and write $S \sim T$ if $T = \gamma * S$ for some $\gamma \in \Gamma$. Put

$$X_f^{\mathrm{pr}} = \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \middle| \, a, b, c \in \mathbf{Z}, (a, b, c) = 1, b^2 - 4ac = f^2 D \right\}.$$

Then

$$X = \bigsqcup_{f \in \mathbf{N}} \frac{1}{f} X_f^{\mathrm{pr}}, \quad \frac{1}{f} X_f^{\mathrm{pr}} = \left\{ \frac{1}{f} x \, \middle| \, x \in X_f^{\mathrm{pr}} \right\}.$$

We say that $x$ is *of conductor* $f$ if $x \in \frac{1}{f} X_f^{\mathrm{pr}}$. If $D = 1$, a complete set of representatives of $X_f^{\mathrm{pr}}/\sim$ can be chosen as

$$\left\{ \begin{pmatrix} m & f/2 \\ f/2 & 0 \end{pmatrix} \middle| \, 0 \leq m < f, (f, m) = 1 \right\}.$$

Then, as is well known, there is a bijective correspondence between $X_f^{\mathrm{pr}}/\sim$ and $Cl_f$ induced by

$$\begin{pmatrix} m & f/2 \\ f/2 & 0 \end{pmatrix} \mapsto [(m, m), (0, f)] \quad \text{if } D = 1.$$

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \mapsto \left[ a, \frac{b + f\sqrt{D}}{2} \right] \quad \text{if } D \neq 1,$$

By this bijection we identify the both sets and use the following notation to indicate the corresponding classes:

$$\begin{array}{ccc} X_f^{\mathrm{pr}}/\sim & \leftrightarrow & Cl_f \\ \Psi & & \Psi \\ [S] & \mapsto & [\mathfrak{a}_S] \\ [S_{\mathfrak{a}}] & \leftarrow\!\shortmid & [\mathfrak{a}]. \end{array}$$

If $T \in X_f^{\mathrm{pr}}$, $S \in X_{f_1}^{\mathrm{pr}}$ and $f \mid f_1$, then $\mathfrak{a}_T \mathfrak{a}_S$ is an $\mathcal{O}_f$-proper ideal. We denote by $T * S \in X_f^{\mathrm{pr}}$ the matrix corresponding to $\mathfrak{a}_T \mathfrak{a}_S$, which is determined up to $\Gamma$-equivalence.

**1.3.** We recall the definition of the completions of $G$ and $X$ (cf. [SH]). Let $\Gamma(n)$ be the principal congruence subgroup of level $n \in \mathbf{N}$:

$$\Gamma(n) = \{\gamma \in \Gamma \mid \gamma \equiv 1 (\mathrm{mod}\, n)\}.$$

We define the completions $\tilde{G}$ and $\tilde{X}$ of $G$ and $X$, respectively, by

$$\tilde{G} = \varprojlim_n G/\Gamma(n) \text{ and } \tilde{X} = \varprojlim_n \Gamma(n)/X.$$

Then $\tilde{G}$ is a locally compact totally disconnected unimodular topological group and $\tilde{X}$ is a locally compact totally disconnected topological space. Since the action of $G$ on $X$ is uniquely extended to a continuous action of $\tilde{G}$ on $\tilde{X}$, we use the same symbol $*$ to denote the extended action.

We may identify $\tilde{G}$ with the closure of $G$ in $GL_2(\mathbf{A}_f)$, where $\mathbf{A}_f = \prod'_p \mathbf{Q}_p$, the finite part of the adele ring of $\mathbf{Q}$. In the present case, since the group $SL_2$ satisfies the strong approximation theorem, we have

$$\tilde{G} = \{g \in GL_2(\mathbf{A}_f) \mid \det g \in \mathbf{Q},\ \det g > 0\}.$$

We denote by $\bar{\Gamma}$ the closure of $\Gamma$ in $\tilde{G}$. Then we get natural bijective correspondences between $\Gamma\backslash X$ and $\bar{\Gamma}\backslash\tilde{X}$, and between $\Gamma\backslash G/\Gamma$ and $\bar{\Gamma}\backslash\tilde{G}/\bar{\Gamma}$, so we may identify $\mathscr{C}^\infty(\Gamma\backslash X)$ with

$$\mathscr{C}^\infty(\bar{\Gamma}\backslash\tilde{X}) = \{\varphi : \tilde{X} \to \mathbf{C} \mid \varphi(\gamma * x) = \varphi(x),\ \gamma \in \bar{\Gamma}\},$$

$\mathscr{S}(\Gamma\backslash X)$ with

$$\mathscr{S}(\bar{\Gamma}\backslash\tilde{X}) = \{\varphi \in \mathscr{C}^\infty(\bar{\Gamma}\backslash\tilde{X}) \mid \varphi : \text{compactly supported}\}$$

and $\mathscr{H}(G, \Gamma)$ with

$$\mathscr{H}(\tilde{G}, \bar{\Gamma}) = \left\{ f : \tilde{G} \to \mathbf{C} \,\middle|\, \begin{array}{l} f : \text{compactly supported}, \\ f(\gamma_1 x \gamma_2) = f(x)\ (\gamma_1, \gamma_2 \in \Gamma) \end{array} \right\}.$$

We normalize the Haar measure $dg$ on $\tilde{G}$ by $\int_{\bar{\Gamma}} dg = 1$. Then the multiplication of $\mathscr{H}(G, \Gamma)$ can be expressed as

$$(f_1 \cdot f_2)(h) = \int_{\tilde{G}} f_1(g) f_2(g^{-1}h)\, dg, \quad f_1, f_2 \in \mathscr{H}(G, \Gamma)$$

and the action of $\mathscr{H}(G, \Gamma)$ on $\mathscr{C}^\infty(\Gamma\backslash X)$ can be expressed as

$$(f * \varphi)(x) = \int_{\tilde{G}} f(g) \varphi(g^{-1} * x)\, dg, \quad f \in \mathscr{H}(G, \Gamma),\ \varphi \in \mathscr{C}^\infty(\Gamma\backslash X).$$

By Proposition 2.6 of [SH], the space $\tilde{X}$ carries a $\tilde{G}$-invariant measure $d\mu$. For $x \in X$, denote by $\Gamma_x$ the isotropy subgroup of $\Gamma$ at $x$. We fix a base point $x_0 \in X_1^{\mathrm{pr}}$ $(\subset X)$ and we normalize $d\mu$ by setting

$$\int_{\overline{\Gamma * x_0}} d\mu = 1.$$

Then, by Proposition 1.9 of [SH], we have

$$\int_{\overline{\Gamma * x}} d\mu = [\Gamma_{x_0} : g_x \Gamma_x g_x^{-1}], \tag{1.2}$$

where $g_x \in G$ for which $x_0 = g_x * x$. For simplicity, we write $\mu(x) = \int_{\overline{\Gamma * x}} d\mu$. For later use, we compute the value of $\mu(x)$.

Lemma 1.1. *If $x \in X$ is of conductor $f$, then $\mu(x) = [\mathcal{O}^1 : \mathcal{O}_f^1]$. In particular, if $D = 1$, then $\mu(x) = 1$ for every $x \in X$.*

*Proof.* Let $D = 1$ and $x \in X$. Denote by $G_x^{(1)}$ the isotropy subgroup of $G^{(1)} = SL_2(\mathbf{Q})$ at $x$. We may take $x_0 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ as base point. Then we get

$$\Gamma_x = \Gamma \cap g_x^{-1} G_{x_0}^{(1)} g_x = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Thus we get $\mu(x) = 1$ by (1.2).

Let $D \neq 1$, $x = \frac{1}{f} S$, $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in X_f^{\mathrm{pr}}$ and $x_0 = T \in X_1^{\mathrm{pr}}$. Then

$$G_x^{(1)} = G_S^{(1)} = \left\{ \begin{pmatrix} s & t \\ -\dfrac{c}{a} t & s + \dfrac{b}{a} t \end{pmatrix} \in G^{(1)} \,\middle|\, (s, t) \in \mathbf{Q}^2 - \{(0, 0)\} \right\}.$$

So we obtain an isomorphism

$$\begin{array}{ccc} \mathcal{O}_f^1 & \longrightarrow & \Gamma_x \\ \cup\!\!\!| & & \cup\!\!\!| \\ \dfrac{t + uf\sqrt{D}}{2} & \mapsto & \begin{pmatrix} \dfrac{t - bu}{2} & au \\ -cu & \dfrac{t + bu}{2} \end{pmatrix} = S \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} + \begin{pmatrix} t/2 & 0 \\ 0 & t/2 \end{pmatrix}. \end{array}$$

Take a $g \in G$ such that $g * x = x_0$, equivalently $(f \det g) \cdot T = gS{}^t g$. Then we see

$$g\Gamma_x g^{-1} = \left\{ T\begin{pmatrix} 0 & uf \\ -uf & 0 \end{pmatrix} + \begin{pmatrix} t/2 & 0 \\ 0 & t/2 \end{pmatrix} \middle| \frac{t+uf\sqrt{D}}{2} \in \mathcal{O}_f^1 \right\},$$

while

$$\Gamma_{x_0} = \left\{ T\begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} + \begin{pmatrix} t/2 & 0 \\ 0 & t/2 \end{pmatrix} \middle| \frac{t+u\sqrt{D}}{2} \in \mathcal{O}^1 \right\}.$$

Now, by (1.2), it is obvious that $\mu(x) = [\mathcal{O}^1 : \mathcal{O}_f^1]$. ■

## § 2. Decomposition of $\mathcal{S}(\Gamma\backslash X)$ by characters of class groups

For a positive integer $f$, let $\mathfrak{X}(f)$ be the character group of $Cl_f$. If $f_1 \mid f$, then there exists a canonical surjective homomorphism $p_{f_1}^f : Cl_f \to Cl_{f_1}$ induced by $\mathfrak{a} \mapsto \mathfrak{a}\mathcal{O}_{f_1}$; hence we have a natural injective map

$$\begin{array}{cccc} \mathrm{Ind}_f^{f_1} & \mathfrak{X}(f_1) & \to & \mathfrak{X}(f) \\ & \cup & & \cup \\ & \chi & \mapsto & \chi\circ p_{f_1}^f. \end{array}$$

The conductor $f_\chi$ of $\chi \in \mathfrak{X}(f)$ is defined by

$$f_\chi = \min\{f_1 \in \mathbf{N} \mid f_1 \text{ devides } f,\ \chi \in \mathrm{Ind}_f^{f_1}(\mathfrak{X}(f_1))\}.$$

If $f = f_\chi$, then $\chi \in \mathfrak{X}(f)$ is called *primitive*. Let $\mathfrak{X}^{\mathrm{pr}}$ be the set of all primitive characters of arbitrary conductor.

Denote by $\mathrm{ch}_x$ the characteristic function of $\Gamma * x$ for $x \in X$. Let $\chi \in \mathfrak{X}^{\mathrm{pr}}$ and $T \in X_f^{\mathrm{pr}}$. Taking $f_1$ satisfying $f_\chi \mid f_1$ and $f \mid f_1$, we put

$$p_\chi(\mathrm{ch}_{\frac{1}{f}T}) = \frac{1}{h_{f_1}} \sum_{[S]\in Cl_{f_1}} \chi(p_{f_\chi}^{f_1}([S])\,\mathrm{ch}_{\frac{1}{f}(T*S)}.$$

It is easy to see that the right hand side is independent of the choice of such an $f_1$, hence we get a linear operator $p_\chi$ on $\mathscr{C}^\infty(\Gamma\backslash X)$. The operator $p_\chi$ stabilizes $\mathcal{S}(\Gamma\backslash X)$.

For a $\chi \in \mathfrak{X}^{\mathrm{pr}}$ and a positive integer $f$ such that $f_\chi \mid f$, set

$$c_{\chi,f} = \frac{1}{h_f} \sum_{[S]\in Cl_f} \chi(p_{f_\chi}^f([S])\,\mathrm{ch}_{\frac{1}{f}S}.$$

The purpose of this section is to show the following proposition.

PROPOSITION 2.1. (i) *Let* $\mathcal{S}(\Gamma\backslash X)_\chi = p_\chi(\mathcal{S}(\Gamma\backslash X))$. *Then the space* $\mathcal{S}(\Gamma\backslash X)_\chi$ *is spanned by* $c_{\chi,f}$ $(f \in \mathbf{N}, f_\chi \mid f)$.

(ii) *The operators* $p_\chi$ *commute with the action of* $\mathcal{H}(G, \Gamma)$ *and we have an* $\mathcal{H}(G, \Gamma)$*-module isomorphism*

$$\mathcal{S}(\Gamma\backslash X) \cong \bigoplus_{\chi\in\mathfrak{X}^{pr}} \mathcal{S}(\Gamma\backslash X)_\chi.$$

For the proof of Proposition 2.1, we need the following lemmas.

LEMMA 2.2. *For any* $T \in X_f^{\mathrm{pr}}$, *we have*

$$p_\chi(\mathrm{ch}_{\frac{1}{f}T}) = \begin{cases} 0 & \text{if } f_\chi \nmid f \\ \bar{\chi}([T])c_{\chi,f} & \text{if } f_\chi \mid f, \end{cases}$$

*where*

$$\bar{\chi}([T]) = \overline{\chi([T])}.$$

*Proof.* It is easy to see that the identity holds for the case $f_\chi \mid f$. Let $f_\chi \nmid f$ and take a common multiple $f_1$ of $f$ and $f_\chi$. Then we have

$$p_\chi(\mathrm{ch}_{\frac{1}{f}T}) = \frac{1}{h_{f_1}} \sum_{[S]\in Cl_f} \mathrm{ch}_{\frac{1}{f}(T*S)} \sum_{\substack{[U]\in Cl_{f_1} \\ p_f^{f_1}([U])=[S]}} \chi(p_{f_\chi}^{f_1}([U]))$$

$$= \frac{1}{h_{f_1}} \sum_{[S]\in Cl_f} \mathrm{ch}_{\frac{1}{f}(T*S)}\chi(p_{f_\chi}^{f_1}([U_S])) \sum_{[V]\in\mathrm{Ker}(p_f^{f_1})} \chi(p_{f_\chi}^{f_1}([V])),$$

where $[U_S] \in Cl_{f_1}$ with $p_f^{f_1}[U_S] = [S]$. Since $f_\chi \nmid f$, we get

$$\sum_{[V]\in\mathrm{Ker}(p_f^{f_1})} \chi(p_{f_\chi}^{f_1}([V])) = 0,$$

hence

$$p_\chi(\mathrm{ch}_{\frac{1}{f}T}) = 0 \text{ if } f_\chi \nmid f. \qquad \blacksquare$$

LEMMA 2.3. (i) *For any characters* $\chi$ *and* $\psi$ *in* $\mathfrak{X}^{\mathrm{pr}}$, *we have*

$$p_\chi \circ p_\psi = p_\psi \circ p_\chi = \delta_{\chi,\psi} p_\chi,$$

*where* $\delta_{\chi,\psi}$ *is the Kronecker delta.*

(ii) *For any* $S \in X_f^{\mathrm{pr}}$, *we have*

$$\mathrm{ch}_{\frac{1}{f}S} = \sum_{\substack{\chi\in\mathfrak{X}^{\mathrm{pr}} \\ f_\chi \mid f}} \bar{\chi}(p_{f_\chi}^f([S]))c_{\chi,f}.$$

*Proof.* Trivial from the orthogonality relation of characters. ∎

Two lemmas above show that $\mathcal{S}(\Gamma\backslash X)_\chi = p_\chi(\mathcal{S}(\Gamma\backslash X))$ is spanned by $\{c_{\chi,f} \mid f \in \mathbf{N}, f_\chi \mid f\}$ and $\mathcal{S}(\Gamma\backslash X)$ is a direct sum of $\mathcal{S}(\Gamma\backslash X)_\chi$ $(\chi \in \mathfrak{X}^{\mathrm{pr}})$ as $\mathbf{C}$-vector space. Therefore, to prove Proposition 2.1, it suffices to show that the operators $p_\chi$ commute with the action of $\mathcal{H}(G, \Gamma)$. For this purpose it is convenient to introduce another $\mathcal{H}(G, \Gamma)$-action on $\mathcal{C}^\infty(\Gamma\backslash X)$. For $\varphi = \mathrm{ch}_x \in \mathcal{C}^\infty(\Gamma\backslash X)$ and $f = \mathrm{ch}_{\Gamma g \Gamma} \in \mathcal{H}(G, \Gamma)$ with $\Gamma g \Gamma = \bigsqcup_j \Gamma h_j$, put

$$f \ast \varphi = \sum_j \mathrm{ch}_{h_j \ast x}. \tag{2.1}$$

It is easy to see that (2.1) induces an $\mathcal{H}(G, \Gamma)$-action on $\mathcal{C}^\infty(\Gamma\backslash X)$. Define a $\mathbf{C}$-linear map $V : \mathcal{C}^\infty(\Gamma\backslash X) \to \mathcal{C}^\infty(\Gamma\backslash X)$ by $V(\mathrm{ch}_x) = \mu(x)\mathrm{ch}_x$.

LEMMA 2.4. *For every $f \in \mathcal{H}(G, \Gamma)$ and $\mathcal{C}^\infty(\Gamma\backslash X)$, the following identity holds:*

$$f \ast \varphi = V(f \ast (V^{-1}\varphi)).$$

*Proof.* We have only to show the identity for $f = \mathrm{ch}_{\Gamma g \Gamma} \in \mathcal{H}(G, \Gamma)$ and $\varphi = \mathrm{ch}_x \in \mathcal{C}^\infty(\Gamma\backslash X)$. Let

$$\Gamma g \Gamma = \bigsqcup_i g_i \Gamma = \bigsqcup_j \Gamma h_j = \bigsqcup_l \Gamma m_l \Gamma_x,$$

where $\Gamma_x = \{\gamma \in \Gamma \mid \gamma \ast x = x\}$. Then we get

$$\begin{aligned} f \ast \varphi(y) &= \#\{i \mid y \in g_i \Gamma \ast x\} \\ &= \#\{i \mid \text{there exists } k\Gamma_x \in g_i\Gamma/\Gamma_x \text{ such that } k \ast x = y\} \\ &= \#\{k\Gamma_x \in \Gamma g \Gamma/\Gamma_x \mid k \ast x = y\}. \end{aligned}$$

We see that the number of left $\Gamma_x$-cosets in $\Gamma m \Gamma_x$ which give the same element $\gamma m \ast x$ in $X$ is equal to $[\Gamma_{m \ast x} : m\Gamma_x m^{-1} \cap \Gamma]$. So we obtain

$$f \ast \varphi = \sum_l [\Gamma_{m_l \ast x} : m_l \Gamma_x m_l^{-1} \cap \Gamma]\mathrm{ch}_{m_l \ast x}.$$

On the other hand we get

$$f \ast \varphi = \sum_l [\Gamma_x : \Gamma_x \cap m_l^{-1}\Gamma m_l]\mathrm{ch}_{m_l \ast x},$$

since the number of left $\Gamma$-cosets $\Gamma k$ satisfying $\Gamma h \Gamma_x = \Gamma k \Gamma_x$ is equal to $[\Gamma_x : \Gamma_x \cap h^{-1}\Gamma h]$. We obtain

$$[\Gamma_x : \Gamma_x \cap m^{-1}\Gamma m] = [m\Gamma_x m^{-1} : \Gamma_{m*x}][\Gamma_{m*x} : m\Gamma_x m^{-1} \cap \Gamma]$$

and, by (1.2),

$$[m\Gamma_x m^{-1} : \Gamma_{m*x}] = \mu(m * x)/\mu(x).$$

Hence we get the required density. ∎

By Lemma 1.1 and Lemma 2.2, we see that

$$p_\chi \circ V = V \circ p_\chi.$$

Hence the proof of the commutativity reduces to the proof of the identity

$$p_\chi(f * \varphi) = f * (p_\chi(\varphi)), \quad f \in \mathscr{H}(G, \Gamma), \ \varphi \in \mathscr{C}^\infty(\Gamma \backslash X). \tag{2.2}$$

In the rest of this section, we consider the case $D \neq 1$, since the proof for $D = 1$ is much easier. We need the following lemma due to Shintani (cf. [Sn, Lemmas 2.3 and 2.5]).

LEMMA 2.5. *Let $\mathfrak{a}$ be an $\mathcal{O}_f$-proper ideal and $p$ be a rational prime.*

(i) *Among $p+1$ sublattices in $\mathfrak{a}$ of index $p$, there are $p - \chi_K(p)$ $\mathcal{O}_{fp}$-proper ideals and $1 + \chi_K(p)$ $\mathcal{O}_f$-proper ideals if $p \nmid f$, and there are $p$ $\mathcal{O}_{fp}$-proper ideals and one $\mathcal{O}_{f/p}$-proper ideal if $p \mid f$.*

*Let $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{h_f}\}$ be a complete set of representatives of ideal classes in $Cl_f$ and $\mathscr{B}$ the set of all sublattices of $\mathfrak{a}_i$ $(1 \leq i \leq h_f)$ of index $p$.*

(ii) *For every $C \in Cl_{fp}$, there are $[\mathcal{O}_f^1 : \mathcal{O}_{fp}^1]$ lattices $\mathfrak{b}$ in $\mathscr{B}$ such that $\mathfrak{b}$ is $\mathcal{O}_{fp}$-proper and $\mathfrak{b} \in C$.*

(iii) *If $p \nmid f$, then for every $C \in Cl_f$, there are $1 + \chi_K(p)$ lattices $\mathfrak{b}$ in $\mathscr{B}$ such that $\mathfrak{b}$ is $\mathcal{O}_f$-proper and $\mathfrak{b} \in C$.*

(iv) *If $p \mid f$, then for every $C \in Cl_{f/p}$, there are $h_f/h_{f/p}$ lattices $\mathfrak{b}$ in $\mathscr{B}$ such that is $\mathcal{O}_{f/p}$-proper and $\mathfrak{b} \in C$.*

Recall that $\mathscr{H}(G, \Gamma)$ is generated as a $\mathbf{C}$-algebra by the elements

$$\{T(p, 1), \ T(p, p)^{\pm 1} \mid p \in \mathbf{P}\},$$

where $T(p, p)^{\pm 1}$ (resp. $T(p, 1)$) is the characteristic function of the double $\Gamma$-coset containing $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{\pm 1}$ $\left(\text{resp.} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right)$. Since it is clear that the identity (2.2) holds for every $T(p, p)^{\pm 1} \in \mathscr{H}(G, \Gamma)$, it suffices to show the following identity for every $p \in \mathbf{P}$ and $x \in \dfrac{1}{f} X_f^{\mathrm{pr}}$ :

$$p_\chi(T(p,1) * \mathrm{ch}_x) = T(p,1) * p_\chi(\mathrm{ch}_x). \tag{2.3}$$

We denote by $(R)$ (resp. $(L)$) the right (resp. left) hand side of (2.3).

Write $x = \frac{1}{f} S$, $S \in X_f^{\mathrm{pr}}$. First we consider the case where $f_\chi \nmid f$. Then $(R) = 0$ by Lemma 2.2. If $f_\chi \nmid fp$, then clearly $(L) = 0$. If $f_\chi \nmid fp$, then we get by Lemma 2.5,

$$\begin{aligned}(L) &= p_\chi\Big([\mathcal{O}_f^1 : \mathcal{O}_{fp}^1] \sum_{[T]} \mathrm{ch}_{\frac{1}{fp}T})\Big) \\ &= [\mathcal{O}_f^1 : \mathcal{O}_{fp}^1] \Big(\overline{\sum_{[T]} \chi(p_{f_\chi}^{fp}([T]))}\Big) \cdot c_{\chi, fp},\end{aligned}$$

where the summation is taken over all $[T] \in Cl_{fp}$ satisfying $p_f^{fp}([T]) = [S]$. Since $f_\chi \nmid f$, we have $(L) = 0$. Thus, we see that $(L) = (R) = 0$ if $f_\chi \nmid f$.

Now we assume that $f_\chi \mid f$. The conductor of a lattice $\mathfrak{b}$ in $K$ is, by definition, a positive integer $f$ for which $\mathfrak{b}$ is an $\mathcal{O}_f$-proper ideal and we denote it by $\mathfrak{f}(\mathfrak{b})$. We may choose an ideal $\mathfrak{a}_S$ coprime to $pf$ from the ideal class corresponding to $S$. We get

$$(L) = \sum_{\mathfrak{b}} \bar{\chi}(\mathfrak{b}) c_{\chi, \mathfrak{f}(\mathfrak{b})},$$

where $\mathfrak{b}$ runs over all sublattices of $\mathfrak{a}_S$ of index $p$ and

$$\chi(\mathfrak{b}) = \begin{cases} \chi(p_{f_\chi}^{\mathfrak{f}(\mathfrak{b})}([\mathfrak{b}])) & \text{if } f_\chi \mid \mathfrak{f}(\mathfrak{b}) \\ 0 & \text{if } f_\chi \nmid \mathfrak{f}(\mathfrak{b}) \end{cases}$$

We consider the right hand side $(R)$. For $[T] \in Cl_{f'}$ with $f_\chi \mid f'$, we simply write $\chi(T)$ for $\chi(p_{f_\chi}^{f'}[T])$. Let

$$(R)_m = \frac{\bar{\chi}(S)}{h_f} \sum_{[T] \in Cl_f} \chi(T) \sum_{\mathfrak{b}} \mathrm{ch}_{\frac{1}{m}S_\mathfrak{b}},$$

where the summation with respect to $\mathfrak{b}$ is taken over all sublattices $\mathfrak{b}$ of $\mathfrak{a}_T$ satisfying $[\mathfrak{a}_T : \mathfrak{b}] = p$ and $\mathfrak{f}(\mathfrak{b}) = m$. Then we see by Lemma 2.5 (i) that

$$(R) = \begin{cases} (R)_{fp} + (R)_f & \text{if } p \nmid f \\ (R)_{fp} + (R)_{f/p} & \text{if } p \mid f. \end{cases}$$

If $\mathfrak{b}$ is a sublattice of $\mathfrak{a}_T$ of index $p$ and $\mathfrak{f}(\mathfrak{b}) = fp$, then $\mathfrak{b}\mathcal{O}_f = \mathfrak{a}_T$, and so we get by Lemma 2.5 (ii)

$$\begin{aligned}(R)_{fp} &= \bar{\chi}(S) \frac{h_{fp}[\mathcal{O}_f^1 : \mathcal{O}^1{}_{fp}]}{h_f} c_{\chi, fp} \\ &= (p - \chi_{K,f}(p)) \bar{\chi}(S) \cdot c_{\chi, fp}\end{aligned}$$

(for the definition of $\chi_{K,f}$, see §1.2). If $\mathfrak{b}$ is a sublattice of $\mathfrak{a}_T$ of index $p$ and $\mathfrak{f}(\mathfrak{b}) = f/p$, then $\mathfrak{b} \sim \mathfrak{a}_T \mathcal{O}_{f/p}$. Hence, if $p \mid f$, we get by Lemma 2.5 (iv)

$$\begin{aligned}(R)_{f/p} &= \frac{\bar{\chi}(S)}{h_f} \sum_{[U] \in Cl_{f/p}} \mathrm{ch}_{\frac{p}{f}U} \sum_{[T]} \chi(T) \\ &= \begin{cases} 0 & \text{if } f_\chi \nmid \dfrac{f}{p} \\ \bar{\chi}(S) c_{\chi, f/p} & \text{if } f_\chi \mid \dfrac{f}{p}, \end{cases}\end{aligned}$$

where the summation is taken over all $[T] \in Cl_f$ such that $p^f_{f/p}([T]) = [U]$. If $p \nmid f$, then we obtain, by Lemma 2.5 (iii),

$$\begin{aligned}(R)_f &= \frac{\bar{\chi}(S)}{h_f} \sum_{[T] \in Cl_f} \chi(T) \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_f \\ [\mathcal{O}_f : \mathfrak{b}] = p, \mathfrak{f}(\mathfrak{b}) = f}} \mathrm{ch}_{\frac{1}{f}(T * S_{\mathfrak{b}})} \\ &= \sum_{\substack{\mathfrak{b} \subset \mathfrak{a}_S \\ [\mathfrak{a}_S : \mathfrak{b}] = p, \mathfrak{f}(\mathfrak{b}) = f}} \bar{\chi}(\mathfrak{b}) c_{\chi, f}.\end{aligned}$$

Hence we see that $(R) = (L)$, and this completes the proof of the commutativity, and so we finish the proof of Proposition 2.1.

## §3. Eisenstein series

**3.1.** We define the Eisenstein series on $X$, which is a slight modification of the zeta functions of binary quadratic forms, by the following formula:

$$E_\varepsilon(x; s_1, s_2) = \mu(x)^{-1} \sum_{\substack{v \in \mathbf{Z}^2/\Gamma_x \\ vx^tv \neq 0}} \frac{\mathrm{sgn}^\varepsilon(vx^tv)}{|vx^tv|^{s_1 + \frac{1}{2}} |\det x|^{s_2 - \frac{1}{4}}}, \tag{3.1}$$

where $\varepsilon = 0$ or $1$, $\mathrm{sgn}^\varepsilon(\ ) = \{\mathrm{sgn}(\ )\}^\varepsilon$ and $\Gamma_x = \{\gamma \in \Gamma \mid \gamma * x = x\}$.

This Eisenstein series coincides with the one introduced in [SH, §3.1, (3.7)] up to the factor $\zeta(2s_1 + 1)$ (see also [SH, §3.2]). The right hand side of (3.1) is absolutely convergent if $\mathrm{Re}(s_1) > \dfrac{1}{2}$, has a meromorphic continuation to the whole $\mathbf{C}^2$ and satisfies the following functional equation (cf. [S]):

$$\Lambda_\varepsilon(x; z_2, z_1) = \Lambda_\varepsilon(x; z_1, z_2),$$

where

$$\Lambda_\varepsilon(x; z_1, z_2) = \pi^{z_1 - z_2} \Gamma\Big(z_2 - z_1 + \frac{1}{2}\Big) \eta_{D,\varepsilon}\Big(z_2 - z_1 + \frac{1}{2}\Big) E_\varepsilon(x; z_2 - z_1, -z_2),$$

$$\eta_{D,\varepsilon}(s) = \begin{cases} 1 & \text{if } D < 0 \\ \cos(s\pi/2) & \text{if } D > 0 \text{ and } \varepsilon = 0 \\ \sin(s\pi/2) & \text{if } D > 0 \text{ and } \varepsilon = 1. \end{cases}$$

**3.2.** As usual, for $S \in X_f^{\mathrm{pr}}$ and $s \in \mathbf{C}$, we define

$$\zeta_S(s) = \sum_{\substack{v \in \mathbf{Z}^2/\Gamma_S \\ vS^tv > 0}} \frac{1}{(vS^tv)^s} = \sum_{\substack{\alpha \in \mathfrak{a}_S/\mathcal{O}_f^1 \\ N(\alpha) > 0}} \frac{N_f(\mathfrak{a}_S)^s}{N(\alpha)^s} \tag{3.2}$$

and, for $C \in Cl_f$ and $s \in \mathbf{C}$,

$$\zeta^{(f)}(C; s) = \sum_{\substack{\mathfrak{a} \in C \\ \mathfrak{a} + f\mathcal{O}_f = \mathcal{O}_f}} \frac{1}{N(\mathfrak{a})^s}. \tag{3.3}$$

The series $\zeta_S(s)$ and $\zeta^{(f)}(C; s)$ are absolutely convergent for $\mathrm{Re}(s) > 1$. It is obvious that $\zeta_S(s)$ depends only on $[S] \in Cl_f$.

THEOREM 1. *Let $x = \frac{1}{f} S$, $S$, $\in X_f^{\mathrm{pr}}$. Then we have*

$$E_\varepsilon(x; s) = f^{-(s_1+\frac{1}{2})}\left(\frac{D}{4}\right)^{-s_2+\frac{1}{4}}$$
$$\times \begin{cases} \displaystyle\sum_{d \mid f} \frac{d^{2s_1+1}}{[\mathcal{O}^1 : \mathcal{O}_d^1]}\Big(\zeta^{(d)}\Big(p_d^f([S]); s_1 + \frac{1}{2}\Big) \\ \qquad\qquad + (-1)^\varepsilon \zeta^{(d)}\Big(p_d^f(J_f \cdot [S]); s_1 + \frac{1}{2}\Big)\Big) & \text{if } D > 0 \\ \displaystyle\sum_{d \mid f} \frac{d^{2s_1+1}}{[\mathcal{O}^1 : \mathcal{O}_d^1]}\zeta^{(d)}\Big(p_d^f([S]); s_1 + \frac{1}{2}\Big) & \text{if } D < 0, \end{cases}$$

*where $J_f$ is the ideal class in $Cl_f$ containing the ideal*

$$\begin{array}{ll} (f\sqrt{D}) & \text{if } D > 1 \\ ((f, -f)) & \text{if } D = 1. \end{array}$$

*Proof.* From (3.1), (3.2) and Lemma 1.1, it is easy to see that

$$E_\varepsilon(x; s) = \frac{1}{[\mathcal{O}^1 : \mathcal{O}_f^1]} f^{s_1+\frac{1}{2}}\left(\frac{D}{4}\right)^{-s_2+\frac{1}{4}} \times \begin{cases} \Big(\zeta_S\Big(s_1 + \frac{1}{2}\Big) + (-1)^\varepsilon \zeta_{S'}\Big(s_1 + \frac{1}{2}\Big)\Big) & \\ & \text{if } D > 0 \\ \zeta_S\Big(s_1 + \frac{1}{2}\Big) & \text{if } D < 0 \end{cases}$$

where $[S'] = J_f \cdot [S]$. Hence the theorem is an immediate consequence of the following lemma. ∎

LEMMA 3.1. (i) *Let $C \in Cl_f$. If $\mathfrak{a} \in C$ satisfies $\mathfrak{a} + f\mathcal{O}_f = \mathcal{O}_f$, then*

$$\zeta^{(f)}(C; s) = \sum_{\substack{\alpha \in \mathfrak{a}/\mathcal{O}_f^1 \\ N(\alpha)>0, (\alpha, f)=1}} \frac{N_f(\mathfrak{a})^s}{N(\alpha)^s},$$

*where $(\alpha, f) = 1$ means $\alpha\mathcal{O} + f\mathcal{O} = \mathcal{O}$.*
(ii) *For $S \in X_f^{\mathrm{pr}}$,*

$$\zeta_S(s) = \sum_{d \mid f} [\mathcal{O}_d^1 : \mathcal{O}_f^1] \left(\frac{f}{d}\right)^{-2s} \zeta^{(d)}(p_d^f([S]); s).$$

*Proof.* (i) We put

$$\bar{\mathfrak{a}} = \{\bar{\alpha} \mid \alpha \in \mathfrak{a}\},$$

where for $x \in K$,

$$\bar{x} = \begin{cases} (b, a) & \text{if } D = 1 \text{ and } x = (a, b) \\ a - b\sqrt{D} & \text{if } D \neq 1 \text{ and } x = a + b\sqrt{D},\ a, b \in \mathbf{Q}. \end{cases}$$

Then we get

$$N(\mathfrak{a}) = N(\bar{\mathfrak{a}}) \text{ and } \mathfrak{a}\bar{\mathfrak{a}} = N(\mathfrak{a})\mathcal{O}_f,$$

and so

$$\zeta^{(f)}(C; s) = \zeta^{(f)}(C^{-1}; s).$$

There is a bijection

$$\begin{array}{ccc} \{\alpha \in \mathfrak{a} \mid N(\alpha) > 0,\ (\alpha, f) = 1\}/\mathcal{O}_f^1 & \rightarrow & \{\mathfrak{b} \in C^{-1} \mid \mathfrak{b} + f\mathcal{O}_f = \mathcal{O}_f\}, \\ \Psi & & \Psi \\ \alpha & \mapsto & \dfrac{\alpha\bar{\mathfrak{a}}}{N(\mathfrak{a})} \end{array}$$

and so we obtain the identity.

(ii) Let $\mathfrak{a}$ be an ideal belonging to the class $[\mathfrak{a}_S]$ such that $\mathfrak{a} + f\mathcal{O}_f = \mathcal{O}_f$. Then we see that

$$\mathfrak{a}\mathcal{O}_{f_1} \cap \mathcal{O}_{f_2} = \mathfrak{a}\mathcal{O}_{f_2} \text{ if } f_1 \mid f_2 \text{ and } f_2 \mid f,$$

and

$$N_f(\mathfrak{a}) = N_{f_1}(\mathfrak{a}\mathcal{O}_{f_1}) \text{ if } f_1 \mid f.$$

We see that, for $\alpha \in \mathfrak{a} - \{0\}$ and $d$ which divides $f$,

$$(\alpha, f)_{\mathcal{O}} \subseteq d\mathcal{O} \text{ if and only if } d^{-1}\alpha \in \mathfrak{a}\mathcal{O}_{\frac{f}{d}}.$$

For $d \mid f$, put

$$\mathfrak{a}^{(d)} = \{\alpha \in \mathfrak{a} \mid (\alpha, f)_{\mathcal{O}} = d\mathcal{O}\},$$

then

$$\mathfrak{a} - \{0\} = \bigsqcup_{d \mid f} \mathfrak{a}^{(d)}.$$

Now we get

$$\begin{aligned}
\zeta_S(s) &= \sum_{d \mid f} \sum_{\substack{\alpha \in \mathfrak{a}^{(d)}/\mathcal{O}_f^1 \\ N(\alpha) > 0}} \frac{N_f(\mathfrak{a})^s}{N(\alpha)^s} \\
&= N_f(\mathfrak{a})^s \sum_{d \mid f} \sum_{\substack{\beta \in \mathfrak{a}\mathcal{O}_{f/d}/\mathcal{O}_f^1 \\ (\beta, f/d) = 1, N(\beta) > 0}} \frac{1}{N(d\beta)^s} \\
&= \sum_{d \mid f} [\mathcal{O}_{f/d}^1 : \mathcal{O}_f^1] d^{-2s} \zeta^{(f/d)}([\mathfrak{a}\mathcal{O}_{f/d}]; s) \\
&= \sum_{d \mid f} [\mathcal{O}_d^1 : \mathcal{O}_f^1] \left(\frac{f}{d}\right)^{-2s} \zeta^{(d)}([\mathfrak{a}\mathcal{O}_d]; s).
\end{aligned}$$

■

## § 4. Fourier-Eisenstein transform and Plancherel formula

**4.1.** Let $\mathfrak{D}\mathcal{S}_\varepsilon$ be the $\mathbf{C}$-vector space of Dirichlet series

$$\xi(z_1, z_2) = \sum_{m_1, m_2 \in \mathbf{Q}_+^\times} c(m_1, m_2) m_1^{-z_1} m_2^{-z_2}$$

which converge absolutely for $\mathrm{Re}(z_2) - \mathrm{Re}(z_1) > \frac{3}{2}$, have meromorphic continuations to the whole $\mathbf{C}^2$ and satisfy the functional equation

$$\Xi(z_2, z_1) = \Xi(z_1, z_2),$$

where

$$\Xi(z_1, z_2) = \pi^{z_1 - z_2} \Gamma\left(z_2 - z_1 + \frac{1}{2}\right) \eta_{D,\varepsilon}\left(z_2 - z_1 + \frac{1}{2}\right) \xi(z_1, z_2).$$

We define the Fourier-Eisenstein transform on $\mathcal{S}(\Gamma \backslash X)$ as follows:

$$
\begin{array}{lcl}
F_\varepsilon : \mathcal{S}(\Gamma \backslash X) & \to & \mathcal{D}\mathcal{S}_\varepsilon \\
\quad\ \ \cup & & \ \cup \\
\quad\ \ \varphi & \mapsto & F_\varepsilon(\varphi)(s) = \int_{\tilde{X}} \varphi(x) E_\varepsilon(x; s_1, s_2)\, d\mu(x).
\end{array}
\tag{4.1}
$$

Here we consider $E_\varepsilon(x; s)$ as a function in $\mathcal{C}^\infty(\Gamma \backslash X)$. Note that $F_\varepsilon(\varphi)(s)$ is a finite linear combination of the Eisenstein series. In fact, by Lemma 1.1, we have

$$F_\varepsilon(\varphi)(s) = \sum_{x \in \Gamma \backslash X} \varphi(x)\ [\mathcal{O}^1 : \mathcal{O}^1_{\mathfrak{f}(x)}]\ E_\varepsilon(x; s_1, s_2),$$

where $\mathfrak{f}(x)$ is the conductor of $x$. Hence $F_\varepsilon(\varphi)$ is in $\mathcal{D}\mathcal{S}_\varepsilon$.

Let

$$\mathfrak{R} = \mathbf{C}[x_2, x_3, \dots, x_p, \dots], \quad x_p = p^t + p^{-t}\ \ (p \in \mathbf{P}).$$

PROPOSITION 4.1. (i) *There is a surjective* $\mathbf{C}$*-algebra homomorphism*

$$
\begin{array}{lcl}
\mathcal{H}(G, \Gamma) & \to & \mathfrak{R} \\
\quad \cup & & \cup \\
\quad f & \mapsto & \hat{f}(t) = \int_{\tilde{G}} \left| \dfrac{a(p(g))}{d(p(g))} \right|_{\mathbf{A}_f}^{t+1/2} f(g)\, dg,
\end{array}
$$

*where* $p(g)$ *is an element in* $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{G} \,\middle|\, b = 0 \right\}$ *such that* $gp(g)^{-1} \in \bar{\Gamma}$ *and* $a(p(g))$ *and* $d(p(g))$ *are the* (1,1)*-entry and the* (2,2)*-entry of* $p(g)$*, respectively.*

(ii) *The following identities hold for any* $f \in \mathcal{H}(G, \Gamma)$ *and any* $\varphi \in \mathcal{S}(\Gamma \backslash X)$:

$$
\begin{aligned}
F_\varepsilon(f * \varphi)(s) &= \hat{f}(s_1) F_\varepsilon(\varphi)(s) \\
(f * E_\varepsilon)(x; s_1, s_2) &= \hat{f}(s_1) E_\varepsilon(x; s_1, s_2).
\end{aligned}
\tag{4.2}
$$

*Proof.* (i) By the Iwasawa decomposition of $GL(2)$, we see that $f \mapsto \hat{f}$ is a $\mathbf{C}$-algebra homomorphism. By direct computation, we get

$$T(p, p)^\wedge(t) = 1 \text{ and } T(p, 1)^\wedge(t) = p^{1/2}(p^t + p^{-t}).$$

Thus we obtain the result.

(ii) The former identity is an immediate consequence of [SH, Theorem 2]. Since

$$F_\varepsilon(\mathrm{ch}_x)(s) = \mu(x) E_\varepsilon(x, s),$$

we obtain

$$(f * E_\varepsilon)(x\,; s) = \mu(x)^{-1} F_\varepsilon(f' * \mathrm{ch}_x)(s),$$

where $f'(g) = f(g^{-1})$. It is easy to see that $f' * \varphi = f * \varphi$ for any $f \in \mathcal{H}(G,$

$\Gamma$) and $\varphi \in \mathcal{S}(\Gamma\backslash X)$. Hence

$$\begin{aligned}(f * E_\varepsilon)(x\,;\,s) &= \mu(x)^{-1}\hat{f}(s_1)F_\varepsilon(\mathrm{ch}_x)(s)\\ &= \hat{f}(s_1)E_\varepsilon(x;\,s).\end{aligned}$$

This concludes the proof. ∎

*Remark.* The homomorphism given in the first part of the proposition above is nothing but a specialization of (the tensor product of) the Satake transform on $GL(2)$. We call the homomorphism the *restricted Satake transform.*

Let $\chi \in \mathfrak{X}^{\mathrm{pr}}$ and $\varphi \in \mathcal{S}(\Gamma\backslash X)_\chi$, and define the normalized Fourier-Eisenstein transform $F_\chi$ by

$$F_\chi(\varphi)(t) = \frac{\sum_{\varepsilon=0,1} F_\varepsilon(\varphi)(t,\,s_2)}{\sum_{\varepsilon=0,1} F_\varepsilon(c_{\chi,f_\chi})(t,\,s_2)}. \tag{4.3}$$

It is obvious that the right hand side of the identity is independent of $s_2$.

THEOREM 2. *For an $m \in \mathbf{N}$, define a function $\psi_{\chi,m}(t) \in \mathfrak{R}$ by setting*

$$\psi_{\chi,m}(t) = \prod_{p\mid m} \psi_{\chi,p^{e_p}}(t), \quad e_p = \mathrm{ord}_p(m),$$

$$\psi_{\chi,p^e}(t) = \begin{cases} p^{-\frac{e}{2}}\dfrac{p^{(e+1)t}-p^{-(e+1)t}}{p^t-p^{-t}} & \text{if } \chi_{K,f_\chi}(p)=0\\[2ex] \dfrac{p^{-\frac{e}{2}}}{(1+p^{-1})(p^t-p^{-t})}\{p^{(e-1)t}(p^{2t}-p^{-1})-p^{-(e-1)t}(p^{-2t}-p^{-1})\} & \\ & \text{if } \chi_{K,f_\chi}(p)=-1\\[1ex] \dfrac{p^{-\frac{e}{2}}}{(1-p^{-1})(p^t-p^{-t})}\{p^{et}(p^t+p^{-1-t}-(\chi(\mathfrak{p})+\bar{\chi}(\mathfrak{p}))p^{-\frac{1}{2}}) & \\ \qquad -p^{-et}(p^{-t}+p^{-1+t}-(\chi(\mathfrak{p})+\bar{\chi}(\mathfrak{p}))p^{-\frac{1}{2}})\} & \text{if } \chi_{K,f_\chi}(p)=1,\end{cases}$$

*where*

$$\chi(\mathfrak{p}) = \begin{cases}\chi([(p,\,p),\,(1,\,f_\chi)]) & \text{if } D=1\\ \chi([\mathfrak{p}\cap\mathcal{O}_{f_\chi}]) & \text{if } D\neq 1 \text{ and } (p)=\mathfrak{p}\bar{\mathfrak{p}} \text{ in } K.\end{cases}$$

*Then, for $f_\chi \mid f$, we have*

$$F_\chi(c_{\chi,f})(t) = [\mathcal{O}^1_{f_\chi} : \mathcal{O}^1_f]\,\psi_{\chi,f/f_\chi}(t). \tag{4.4}$$

*In particular, for any $\varphi \in \mathcal{S}(\Gamma\backslash X)_\chi$, $F_\chi(\varphi)$ is contained in the ring $\mathfrak{R}$.*

For the proof of the theorem above, we prepare some notation on $L$-functions of quadratic fields. For $\chi \in \mathfrak{X}(f)$ and $s \in \mathbf{C}$, let

$$L^{(f)}(\chi; s) = \sum_{C \in Cl_f} \chi(C)\zeta^{(f)}(C; s) \tag{4.5}$$

and

$$L(\chi; s) = L^{(f_\chi)}(\chi; s).$$

Then, if $\mathrm{Re}(s) > 1$, we obtain

$$L(\chi; s) = \prod_p L_p(\chi; s) \text{ and } L^{(f)}(\chi; s) = L(\chi; s) / \prod_{p|f} L_p(\chi; s),$$

where

$$L_p(\chi; s) = \begin{cases} \displaystyle\prod_{\substack{\mathfrak{p}:\text{prime in } K \\ p \in \mathfrak{p}}} \frac{1}{1 - \chi([\mathfrak{p} \cap \mathcal{O}_f])\, N(\mathfrak{p})^{-s}} & \text{if } p \nmid f_\chi \text{ and } D \neq 1 \\ \dfrac{1}{(1 - \chi([p])p^{-s})(1 - \overline{\chi([p])p^{-s}})} & \text{if } p \nmid f_\chi \text{ and } D = 1 \\ 1 & \text{if } p \mid f_\chi. \end{cases} \tag{4.6}$$

Here we write $\chi([p])$ for $\chi([(p, p), (1, f_\chi)])$.

*Proof of Theorem* 2. Let $f_\chi \mid f$ and put $\sigma = 0$ or $1$ according as $D < 0$ or $D > 0$. Then, by Theorem 1, we obtain

$$\begin{aligned} F_\varepsilon(c_{\chi,f})(s) &= \frac{[\mathcal{O}^1 : \mathcal{O}_f^1]}{h_f} \sum_{[S] \in Cl_f} \chi(p^f_{f_\chi}([S]) E_\varepsilon\Big(\frac{1}{f} S; s\Big) \\ &= \frac{f^{-(s_1+\frac{1}{2})}\left(\frac{D}{4}\right)^{-s_2+\frac{1}{4}}}{h_f} \sum_{d|f} | \mathcal{O}_d^1 : \mathcal{O}_f^1] d^{2s_1+1} \\ &\quad \times \sum_{[S] \in Cl_f} \chi(p^f_{f_\chi}([S]) \Big\{\zeta^{(d)}\Big(p^f_d([S]); s_1 + \frac{1}{2}\Big) \\ &\qquad\qquad + (-1)^\varepsilon \sigma \cdot \zeta^{(d)}\Big(p^f_d(J_f \cdot [S]); s_1 + \frac{1}{2}\Big)\Big\} \\ &= \frac{f^{-(s_1+\frac{1}{2})}\left(\frac{D}{4}\right)^{-s_2+\frac{1}{4}}}{h_f} \sum_{d|f} | \mathcal{O}_d^1 : \mathcal{O}_f^1] d^{2s_1+1} \sum_{[T] \in Cl_d} \zeta^{(d)}\Big([T]; s_1 + \frac{1}{2}\Big) \\ &\qquad \sum_{\substack{[S] \in Cl_f \\ p^f_d([S]) = [T]}} \{\chi(p^f_{f_\chi}([S])) + (-1)^\varepsilon \sigma \cdot \chi(p^f_{f_\chi}(J_f \cdot [S]))\} \end{aligned}$$

$$= (1 + (-1)^{\varepsilon}\sigma\cdot\bar{\chi}(J_{f_\chi}))\, f^{-s_1-\frac{1}{2}} \left(\frac{D}{4}\right)^{-s_2+\frac{1}{4}}$$

$$\times \sum_{\substack{d|f \\ f_\chi|d}} \frac{[\mathcal{O}_d^1 : \mathcal{O}_f^1]\, d^{2s_1+1}}{h_d} L^{(d)}\left(\mathrm{Ind}_d^{f_\chi}(\chi);\, s_1 + \frac{1}{2}\right).$$

By (1.1), we obtain

$$F_\varepsilon(c_{\chi,f})(s) = (1 + (-1)^{\varepsilon}\sigma\cdot\bar{\chi}(J_{f_\chi}))\, \frac{[\mathcal{O}^1 : \mathcal{O}_f^1]}{h_K} f^{-s_1-\frac{1}{2}} \left(\frac{D}{4}\right)^{-s_2+\frac{1}{4}}$$

$$\times L\left(\chi;\, s_1 + \frac{1}{2}\right) \sum_{\substack{d|f \\ f_\chi|d}} d^{2s_1} \prod_{p|d} \frac{L_p^{-1}\left(\chi;\, s_1 + \frac{1}{2}\right)}{1 - \chi_K(p)p^{-1}}.$$

Hence we get

$$F_\chi(c_{\chi,f})(t) = [\mathcal{O}_{f_\chi}^1 : \mathcal{O}_f^1] \left(\frac{f}{f_\chi}\right)^{-t-\frac{1}{2}} \sum_{d|\frac{f}{f_\chi}} d^{2t} \prod_{p|d} \frac{L_p^{-1}\left(\chi;\, t + \frac{1}{2}\right)}{1 - \chi_{K,f_\chi}(p)p^{-1}}$$

$$= [\mathcal{O}_{f_\chi}^1 : \mathcal{O}_f^1] \prod_{p|\frac{f}{f_\chi}} p^{-e_p\left(t+\frac{1}{2}\right)} \left(1 + \frac{L_p^{-1}\left(\chi;\, t + \frac{1}{2}\right)}{1 - \chi_{K,f_\chi}(p)p^{-1}} \sum_{n=1}^{e_p} p^{2tn}\right),$$

where $e_p = \mathrm{ord}_p(f/f_\chi)$. By (4.6), we obtain the identity (4.4). ∎

Through the restricted Satake transform $\widehat{\ } : \mathscr{H}(G, \Gamma) \to \mathfrak{R}$ given in Proposition 4.1 (i), we consider the ring $\mathfrak{R}$ as an $\mathscr{H}(G, \Gamma)$-module. Then, by (4.2) and Theorem 2, the normalized Fourier-Eisenstein transform $F_\chi$ defines an $\mathscr{H}(G, \Gamma)$-homomorphism of $\mathcal{S}(\Gamma\backslash X)_\chi$ into $\mathfrak{R}$ i.e., the following identity holds for any $f \in \mathscr{H}(G, \Gamma)$ and any $\varphi \in \mathcal{S}(\Gamma\backslash X)_\chi$:

$$F_\chi(f * \varphi)(t) = \hat{f}(t)\cdot F_\chi(\varphi)(t).$$

THEOREM 3. *Let* $\chi \in \mathfrak{X}^{\mathrm{pr}}$.

(i) *The normalized Fourier-Eisenstein transform*

$$F_\chi : \mathcal{S}(\Gamma\backslash X)_\chi \xrightarrow{\cong} \mathfrak{R}$$

*is an isomorphism of* $\mathscr{H}(G, \Gamma)$*-modules.*

(ii) *The space* $\mathcal{S}(\Gamma\backslash X)_\chi$ *is generated by* $c_{\chi,f_\chi}$ *as an* $\mathscr{H}(G, \Gamma)$*-module and we have an* $\mathscr{H}(G, \Gamma)$*-isomorphism*

$$\mathcal{S}(\Gamma\backslash X)_\chi \cong \mathscr{H}(G, \Gamma)/\mathscr{I},$$

*where* $\mathscr{I}$ *is the ideal of* $\mathscr{H}(G, \Gamma)$ *generated by* $\{T(p, p) - 1 \mid p \in \mathbf{P}\}$.

*Proof.* It follows from Theorem 2 that $F_\chi$ is bijective. This proves the first part. Since $F_\chi(c_{\chi,f_\chi}) = 1$, we get

$$\mathcal{S}(\Gamma\backslash X)_\chi = \mathcal{H}(G, \Gamma) * c_{\chi,f_\chi}.$$

This also implies that $\mathcal{S}(\Gamma\backslash X)_\chi \cong \mathcal{H}(G, \Gamma)/\mathcal{I}$, where $\mathcal{I}$ is the kernel of the restricted Satake transform. By the proof of Proposition 4.1 (i), we see that $\mathcal{I}$ is generated by $\{T(p, p) - 1 \mid p \in \mathbf{P}\}$. ■

**4.2.** We define a hermitian inner product on $\mathcal{S}(\Gamma\backslash X)$ as follows:

$$\langle\varphi,\psi\rangle_{\mathcal{S}} = \int_{\tilde{X}} \varphi(x)\overline{\psi(x)}d\mu(x) \quad (\varphi, \psi \in \mathcal{S}(\Gamma\backslash X)).$$

Thus $\mathcal{S}(\Gamma\backslash X)$ becomes a pre-Hilbert space. Let $L^2(\Gamma\backslash X)$ be the completion of $\mathcal{S}(\Gamma\backslash X)$:

$$L^2(\Gamma\backslash X) = \{\varphi \in \mathcal{C}^\infty(\Gamma\backslash X) \mid \sum_{x\in\Gamma\backslash X} \mu(x) \mid \varphi(x) \mid^2 < +\infty\}.$$

We denote by $L^2(\Gamma\backslash X)_\chi$ the closure of $\mathcal{S}(\Gamma\backslash X)_\chi$ in $L^2(\Gamma\backslash X)$.

Now we introduce a pre-Hilbert space structure on $\mathfrak{R}$. For $p \in \mathbf{P}$, put

$$\mathfrak{R}_p = \mathbf{C}[p^t + p^{-t}].$$

Then $\mathfrak{R}$ is canonically isomorphic to the restricted tensor product $\otimes'_{p\in\mathbf{P}} \mathfrak{R}_p$. First we define a hermitian inner product on $\mathfrak{R}_p$.

Let $\mathfrak{D}_p = \sqrt{-1}\left(\mathbf{R}/\frac{2\pi}{\log p}\mathbf{Z}\right)$ and let $d_pt$ be the Haar measure on $\mathfrak{D}_p$ normalized by $\int_{\mathfrak{D}_p} d_pt = 1$. Consider the measure $\omega_p(t)$ on $\mathfrak{D}_p$ given by

$$\omega_p(t) = \frac{1-\chi_K(p)p^{-1}}{2}\cdot\left|\frac{L_p\left(\chi;\, t+\frac{1}{2}\right)}{\zeta_p(2t)}\right|^2 d_pt, \tag{4.7}$$

where $\zeta_p(2t) = \dfrac{1}{1-p^{-2t}}$. Then we can define an inner product on $\mathfrak{R}_p$ by

$$\langle\varphi_p, \psi_p\rangle_{\chi,p} = \int_{\mathfrak{D}_p} \varphi_p(t)\overline{\psi_p(t)}\omega_p(t) \quad (\varphi_p, \psi_p \in \mathfrak{R}_p).$$

The inner product on $\mathfrak{R} \cong \bigotimes'_{p\in\mathbf{P}} \mathfrak{R}_p$ is now defined by

$$\langle\varphi, \psi\rangle_\chi = \frac{[\mathcal{O}^1 : \mathcal{O}^1_{f_\chi}]}{h_{f_\chi}} \sum_{i,j} a_i\bar{b}_j \prod_p \langle\varphi_{p,i}, \psi_{p,j}\rangle_{\chi,p}$$

for

$$\varphi = \sum_i a_i \left(\bigotimes_p \varphi_{p,i}\right) \text{ and } \psi = \sum_j b_j \left(\bigotimes_p \psi_{p,j}\right) \quad (a_i,\ b_j \in \mathbf{C},\ \varphi_{p,i},\ \psi_{p,j} \in \mathfrak{R}_p).$$

We denote by $\mathscr{L}^2_\chi$ (resp. $\mathscr{L}^2_{\chi,p}$) the completion of $\mathfrak{R}$ (resp. $\mathfrak{R}_p$) with respect to the inner product $\langle\ ,\ \rangle_\chi$ (resp. $\langle\ ,\ \rangle_{\chi,p}$). The Hilbert space $\mathscr{L}^2_\chi$ is the Hilbert restricted product of $\mathscr{L}^2_{\chi,P}$ ($p \in \mathbf{P}$).

THEOREM 4 (Plancherel formula). *The normalized Fourier-Eisenstein transform*

$$F_\chi : \mathcal{S}(\Gamma\backslash X)_\chi \to \mathfrak{R}$$

*can be extended to an isometry of* $L^2(\Gamma\backslash X)_\chi$ *onto* $\mathscr{L}^2_\chi$. *In particular, for every* $\varphi,\ \psi \in \mathcal{S}(\Gamma\backslash X)_\chi$, *the following identity holds:*

$$\langle\varphi,\ \psi\rangle_{\mathcal{S}} = \langle F_\chi(\varphi),\ F_\chi(\psi)\rangle_\chi. \tag{4.8}$$

First we prove the following result on local factors of the inner product.

LEMMA 4.2. *For any* $p \in \mathbf{P}$, *we have*

$$\langle\psi_{\chi,p^e},\ \psi_{\chi,p^d}\rangle_{\chi,p} = \begin{cases} 0 & \text{if } d \neq e \\ 1 & \text{if } d = e = 0 \\ \dfrac{p^{-e}}{1 - \chi_{K,f_\chi}(p)p^{-1}} & \text{if } d = e > 0. \end{cases}$$

*Proof.* By (4.6), we have

$$\left|\frac{L_p\left(\chi;\, t + \frac{1}{2}\right)}{\zeta_p(2t)}\right|^2 = \begin{cases} |p^t - p^{-t}|^2 & \text{if } \chi_{K,f_\chi}(p) = 0 \\ \left|\dfrac{p^t - p^{-t}}{p^{2t} - p^{-1}}\right|^2 & \text{if } \chi_{K,f_\chi}(p) = -1 \\ \left|\dfrac{p^t - p^{-t}}{(p^{\frac{t}{2}} - \chi(\mathfrak{p})p^{-\frac{1}{2}-\frac{t}{2}})(p^{\frac{t}{2}} - \bar{\chi}(\mathfrak{p})p^{-\frac{1}{2}-\frac{t}{2}})}\right|^2 & \text{if } \chi_{K,f_\chi}(p) = 1, \end{cases}$$

where

$$\chi(\mathfrak{p}) = \begin{cases} \chi([p]) & \text{if } D = 1 \\ \chi([\mathfrak{p} \cap \mathcal{O}_{f_\chi}]) & \text{if } D \neq 1 \text{ and } (p) = \mathfrak{p}\bar{\mathfrak{p}} \text{ in } K. \end{cases}$$

Let $\chi_{K,f_\chi}(p) = -1$. Then we get

$$\int_{\mathfrak{D}_p} \psi_{\chi,p^e}(t)\overline{\psi_{\chi,p^d}(t)}\,\omega_p(t)$$

$$= \frac{p^{-(d+e)/2}}{2(1+p^{-1})} \int_{\mathfrak{D}_p} \left( \frac{p^{(e-1)t}}{p^{-2t}-p^{-1}} - \frac{p^{-(e-1)t}}{p^{2t}-p^{-1}} \right)$$

$$\times \{p^{-(d-1)t}(p^{-2t}-p^{-1}) - p^{(d-1)t}(p^{2t}-p^{-1})\}\, d_p t$$

$$= \frac{p^{-(d+e)/2}}{2(1+p^{-1})} \int_{\mathfrak{D}_p} \left( \sum_{l \geq 0} p^{-l+(e+2l+1)t} - \sum_{m \geq 0} p^{-m-(e+2m+1)t} \right)$$

$$\times \{p^{-(d-1)t}(p^{-2t}-p^{-1}) - p^{(d-1)t}(p^{2t}-p^{-1})\}\, d_p t$$

$$= \begin{cases} 1 & \text{if } d = e = 0 \\ \dfrac{p^{-e}}{1+p^{-1}} & \text{if } d = e > 0 \\ 0 & \text{if } d \neq e. \end{cases}$$

We can prove the other cases similarly.

*Proof of Theorem* 4. We have only to show the identity (4.8) for $\varphi = c_{\chi,ef_\chi}$ and $\psi = c_{\chi,df_\chi}$ with $e, d \in \mathbf{N}$. It is easy to see that

$$\langle c_{\chi,ef_\chi}, c_{\chi,df_\chi} \rangle_{\mathcal{S}} = \delta_{e,d} \frac{[\mathcal{O}^1 : \mathcal{O}^1_{ef_\chi}]}{h_{ef_\chi}}.$$

On the other hand, we get

$$\langle F_\chi(\varphi), F_\chi(\psi) \rangle_\chi = \delta_{e,d} \frac{[\mathcal{O}^1 : \mathcal{O}^1_{f_\chi}]}{h_{f_\chi}} [\mathcal{O}^1_{f_\chi} : \mathcal{O}^1_{ef_\chi}]^2 \prod_{p \mid e} \frac{p^{-e_p}}{1 - \chi_{K,f_\chi}(p) p^{-1}},$$

where $e_p = \mathrm{ord}_p(e)$. By the class number formula (1.1), we obtain the result. ■

We define a function $\omega_{\chi,t}$ in $\mathscr{C}^\infty(\Gamma \backslash X) \otimes_{\mathbf{C}} \mathfrak{R}$ by

$$\omega_{\chi,t} = \frac{1}{[\mathcal{O}^1 : \mathcal{O}^1_{f_\chi}]} \sum_{f : f_\chi \mid f} h_f\, \psi_{\chi, f/f_\chi}(t)\, c_{\chi,f}.$$

THEOREM 5. *For every* $\varphi \in \mathcal{S}(\Gamma \backslash X)_\chi$, *we have*

$$\varphi(x) = \langle F_\chi(\varphi), \omega_{\bar{\chi},t}(x) \rangle_\chi,$$

*namely, the inverse transformation of* $F_\chi$ *is given by*

$$F_\chi(\varphi) \mapsto \langle F_\chi(\varphi), \omega_{\bar{\chi},t} \rangle_\chi.$$

*Proof.* By the definition of the inner product $\langle\ ,\ \rangle_{\mathcal{S}}$, we have

$$\varphi(x) = \frac{1}{\mu(x)} \langle \varphi, \mathrm{ch}_x \rangle_{\mathcal{S}}.$$

It is easy to see that

$$\langle p_\chi \varphi, \psi \rangle_{\mathcal{S}} = \langle \varphi, p_\chi \psi \rangle_{\mathcal{S}}$$

for any $\varphi, \psi \in \mathscr{C}^\infty(\Gamma \backslash X)$, if one of $\varphi$ and $\psi$ is in $\mathcal{S}(\Gamma \backslash X)$. If $\varphi$ is in $\mathcal{S}(\Gamma \backslash X)_\chi$, then $\varphi = p_\chi \varphi$. Hence, we get

$$\varphi(x) = \begin{cases} \mu(x)^{-1} \chi(p^f_{f_\chi}([S])) \langle \varphi, c_{\chi,f} \rangle_{\mathcal{S}} & \text{if } f_\chi \mid f \\ 0 & \text{if } f_\chi \nmid f, \end{cases}$$

where $f$ is the conductor of $x$ and $S = fx \in X_f^{\mathrm{pr}}$. By Theorem 4, we have

$$\langle \varphi, c_{\chi,f} \rangle_{\mathcal{S}} = \langle F_\chi(\varphi), F_\chi(c_{\chi,f}) \rangle_\chi.$$

Now the theorem follows immediately from (4.4). ∎

**4.3.** Theorem 3 enables us to determine all $\mathscr{H}(G, \Gamma)$-common eigenfunctions in $\mathscr{C}^\infty(\Gamma \backslash X)$. Since $\mathfrak{R} \cong \bigotimes'_p \mathfrak{R}_p$, we can define an algebra homomorphism $\lambda_{\boldsymbol{t}} : \mathfrak{R} \to \mathbf{C}$ for any $\boldsymbol{t} = (t_p)_{p \in \mathbf{P}} \in \mathbf{C}^{\mathbf{P}}$ by setting

$$\lambda_{\boldsymbol{t}}(\bigotimes_p \psi_p) = \prod_{p \in \mathbf{P}} \psi_p(t_p) \quad (\psi_p \in \mathfrak{R}_p,\ \psi_p = 1 \text{ for almost all } p).$$

Composing $\lambda_{\boldsymbol{t}}$ with the restricted Satake transform $\hat{\ } : \mathscr{H}(G, \Gamma) \to \mathfrak{R}$ given by Proposition 4.1 (i), we obtain an algebra homomorphism

$$\begin{array}{rcl} \mathscr{H}(G, \Gamma) & \to & \mathbf{C} \\ f & \mapsto & \hat{f}(\boldsymbol{t}) := \lambda_{\boldsymbol{t}}(\hat{f}). \end{array}$$

Any algebra homomorphism of $\mathscr{H}(G, \Gamma)$ into $\mathbf{C}$ can be obtained in this manner for some $\boldsymbol{t} \in \mathbf{C}^{\mathbf{P}}$.

For $\boldsymbol{t} = (t_p)_{p \in \mathbf{P}} \in \mathbf{C}^{\mathbf{P}}$, define a function $\omega_{\chi,\boldsymbol{t}} \in \mathscr{C}^\infty(\Gamma \backslash X)$ by

$$\omega_{\chi,\boldsymbol{t}} = \frac{1}{[\mathcal{O}^1 : \mathcal{O}^1_{f_\chi}]} \sum_{f : f_\chi \mid f} h_f\, \psi_{\chi, f/f_\chi}(\boldsymbol{t})\, c_{\chi,f},$$

where

$$\psi_{\chi,f/f_\chi}(\boldsymbol{t}) = \prod_{p \mid \frac{f}{f_\chi}} \psi_{\chi,p^{e_p}}(t_p), \quad e_p = \mathrm{ord}_p(f/f_\chi).$$

It is not hard to check the identity

$$f * \omega_{\chi,\boldsymbol{t}} = \hat{f}(\boldsymbol{t})\, \omega_{\chi,\boldsymbol{t}} \quad (f \in \mathscr{H}(G, \Gamma)). \tag{4.9}$$

THEOREM 6. *Let $\Psi$ be an $\mathscr{H}(G, \Gamma)$-common eigenfunction in $\mathscr{C}^\infty(\Gamma\backslash X)$ satisfying*

$$f * \Psi = \hat{f}(\boldsymbol{t})\,\Psi \text{ for all } f \in \mathscr{H}(G, \Gamma).$$

*Then $\Psi$ is a (not necessarily finite) linear combination of $\{\omega_{\chi,\boldsymbol{t}} \mid \chi \in \mathfrak{X}^{\mathrm{pr}}\}$, namely, $\Psi$ is of the form*

$$\Psi = \sum_{\chi} a_\chi \cdot \omega_{\chi,\boldsymbol{t}} \quad (a_\chi \in \mathbf{C}).$$

*Proof.* We identify $\mathscr{C}^\infty(\Gamma\backslash X)$ with $\mathrm{Hom}_{\mathbf{C}}(\mathscr{S}(\Gamma\backslash X), \mathbf{C})$ via the nondegenerate bilinear form

$$\begin{array}{rcl} \langle\, ,\, \rangle : \mathscr{C}^\infty(\Gamma\backslash X) \times \mathscr{S}(\Gamma\backslash X) & \rightarrow & \mathbf{C} \\ (\Psi, \varphi) & \mapsto & \langle \Psi, \varphi \rangle = \displaystyle\int_{\tilde{X}} \Psi(x)\varphi(x)\,d\mu(x). \end{array}$$

Since $\langle p_\chi(\Psi), \varphi\rangle = \langle \Psi, p_{\bar{\chi}}(\varphi)\rangle$ for any $\Psi \in \mathscr{C}^\infty(\Gamma\backslash X)$ and $\varphi \in \mathscr{S}(\Gamma\backslash X)$, the space $\mathscr{C}^\infty(\Gamma\backslash X)_\chi = p_\chi(\mathscr{C}^\infty(\Gamma\backslash X))$ can naturally be identified with $\mathrm{Hom}_{\mathbf{C}}(\mathscr{S}(\Gamma\backslash X)_{\bar{\chi}}, \mathbf{C})$. By Proposition 2.1 and Theorem 3, we have

$$\mathscr{C}^\infty(\Gamma\backslash X) = \prod_{\chi\in\mathfrak{X}^{\mathrm{pr}}} \mathscr{C}^\infty(\Gamma\backslash X)_\chi.$$

Let $\Psi$ be as in the theorem and denote by $\Psi_\chi$ the $\mathscr{C}^\infty(\Gamma\backslash X)_\chi$-component $p_\chi(\Psi)$ of $\Psi$. Then, for any $f \in \mathscr{H}(G, \Gamma)$, we have

$$\begin{aligned} \langle \Psi_\chi, f * c_{\bar{\chi},f_\chi}\rangle &= \langle \Psi, f * c_{\bar{\chi},f_\chi}\rangle \\ &= \langle f * \Psi, c_{\bar{\chi},f_\chi}\rangle \\ &= \hat{f}(\boldsymbol{t})\langle \Psi, c_{\bar{\chi},f_\chi}\rangle \\ &= \hat{f}(\boldsymbol{t})\langle \Psi_\chi, c_{\bar{\chi},f_\chi}\rangle. \end{aligned}$$

On the other hand, by (4.9), we have

$$\begin{aligned} \langle \omega_{\chi,\boldsymbol{t}}, f * c_{\bar{\chi},f_\chi}\rangle &= \hat{f}(\boldsymbol{t})\langle \omega_{\chi,\boldsymbol{t}}, c_{\bar{\chi},f_\chi}\rangle. \\ &= \hat{f}(\boldsymbol{t})\,\frac{h_{f_\chi}}{[\mathscr{O}^1 : \mathscr{O}^1_{f_\chi}]}\cdot\langle c_{\chi,f_\chi}, c_{\chi,f_\chi}\rangle_{\mathscr{S}} \\ &= \hat{f}(\boldsymbol{t}). \end{aligned}$$

Hence

$$\langle \Psi_\chi - a_\chi\cdot\omega_{\chi,\boldsymbol{t}}, f * c_{\bar{\chi},f_\chi}\rangle = 0 \quad (f \in \mathscr{H}(G,\Gamma)),$$

where we put

$$a_\chi = \langle \Psi_\chi, c_{\bar{\chi},f_\chi}\rangle.$$

Since $\mathcal{S}(\Gamma\backslash X)_\chi = \mathcal{H}(G, \Gamma) * c_{\chi, f_\chi}$ by Theorem 3, this implies that

$$\Psi_\chi = a_\chi \cdot \omega_{\chi, t}.$$

Thus we obtain

$$\Psi = \sum_\chi a_\chi \cdot \omega_{\chi, t}. \qquad \blacksquare$$

*Remark.* For the space of nondegenerate binary quadratic forms over $p$-adic fields, results analogous to Theorem 2-6 have been obtained in [H1], [H2].

## §5. Examples of Hecke eigenfunctions

Let $K$ be a real quadratic field with discriminant $D$. As in the previous sections, we put

$$X = X_{D,1} = \{x \in M(2, \mathbf{Q}) \mid {}^t x = x, \det x = -D/4\}.$$

Put $K' = K - \mathbf{Q}$ and consider the bijection

$$\begin{array}{ccc} K' & \to & X \\ \alpha & \mapsto & S_\alpha \end{array}$$

given by

$$S_\alpha = \frac{\sqrt{D}}{\alpha - \bar{\alpha}} \cdot \begin{pmatrix} 1 & -\operatorname{tr}(\alpha)/2 \\ -\operatorname{tr}(\alpha)/2 & N(\alpha) \end{pmatrix}.$$

Then, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = GL_2^+(\mathbf{Q})$, we have

$$g * S_\alpha = S_{g \cdot \alpha}, \quad g \cdot \alpha = \frac{d\alpha - c}{-b\alpha + a}.$$

Thus we can identify the space $\mathscr{C}^\infty(\Gamma\backslash X)$ with the space

$$\mathscr{C}^\infty(\Gamma\backslash K') = \{\varphi : K' \to \mathbf{C} \mid \varphi(\gamma \cdot \alpha) = \varphi(\alpha) (\gamma \in \Gamma)\}.$$

Hence the Hecke algebra $\mathcal{H}(G, \Gamma)$ acts on $\mathscr{C}^\infty(\Gamma\backslash K')$.

We give examples of Hecke eigenfunctions in $\mathscr{C}^\infty(\Gamma\backslash K')$.

EXAMPLE 1. In [A], Arakawa introduced the Dirichlet series

$$\xi(s, \alpha) = \sum_{n=1}^\infty \frac{\cot \pi n\alpha}{n^s} \quad (\alpha \in K')$$

and proved that

(1) $\xi(s, \alpha)$ converges absolutely for $\mathrm{Re}\, s > 1$;

(2) $\xi(s, \alpha)$ has an analytic continuation to a meromorphic function of $s$ on $\mathbf{C}$;

(3) $\xi(s, \alpha)$ has a simple pole at $s = 1$.

Let $c_{-1}(\alpha)$ be the residue of $\xi(s, \alpha)$ at $s = 1$. Then the following is a reformulation of [A, Theorem 2.16]:

THEOREM (Arakawa). *The function $c_{-1}(\alpha)$ belongs to $\mathscr{C}^\infty(\Gamma \backslash K')$ and satisfies the identity*

$$f * c_{-1} = \hat{f}\left(-\frac{1}{2}\right) c_{-1} \quad (f \in \mathscr{H}(G, \Gamma)).$$

EXAMPLE 2. For an $\alpha \in K'$, let

$$\alpha = c_0 + \cfrac{1}{c_1 + \cfrac{1}{\ddots \, c_s + \cfrac{1}{a_1 + \cfrac{1}{\ddots \, a_k + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}}}$$

be the expansion into periodic continued fraction. Using the block of periodic terms $a_1, \ldots, a_k$, we define the Hirzebruch sum $\Psi(\alpha)$ by

$$\Psi(\alpha) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \sum_{j=1}^{k} (-1)^{j+s} a_j & \text{if } k \text{ is even.} \end{cases}$$

In [Lu], Lu studied the behaviour of $\Psi(\alpha)$ under the action of the Hecke algebra.

Put

$$\Psi_0(\alpha) = \Psi(\alpha)/\mu(\alpha), \quad \mu(\alpha) = \mu(S_\alpha) = \int_{\overline{\Gamma * S_\alpha}} d\mu.$$

Then the following is a reformulation of [Lu, Theorem 7]:

THEOREM (Lu). *The function $\Psi_0(\alpha)$ belongs to $\mathscr{C}^\infty(\Gamma \backslash K')$ and satisfies the identity*

$$f * \Psi_0 = \hat{f}\left(-\frac{1}{2}\right) \Psi_0 \quad (f \in \mathscr{H}(G, \Gamma)).$$

*In other words,*

$$f * \Psi = \hat{f}\left(-\frac{1}{2}\right) \Psi \quad (f \in \mathscr{H}(G, \Gamma)).$$

Thus the functions $c_{-1}(\alpha)$ and $\Psi_0(\alpha)$ belong to the same eigen space of $\mathscr{H}(G, \Gamma)$. Arakawa proved that these two functions essentially coincide with each other.

PROPOSITION (Arakawa).

$$c_{-1}(\alpha) = -\frac{\pi}{6 \log \varepsilon} \Psi_0(\alpha),$$

*where $\varepsilon$ is the totally positive fundamental unit of $K$ with $\varepsilon > \bar{\varepsilon}$.*

In §4.3, we proved that any $\mathscr{H}(G, \Gamma)$-common eigenfunction in $\mathscr{C}^\infty(\Gamma \backslash X)$ is a linear combination of $\omega_{\chi, t}$'s. If all $t_p$ coincide with a fixed $t \in \mathbf{C}$, then, by (4.4), we have

$$\omega_{\chi, t} = \omega_{\chi, t} = \frac{h_K}{2^{1/2}[\mathcal{O}^1 : \mathcal{O}^1_{f_\chi}] D^{1/4}} \cdot \frac{(f_\chi)^{t+1/2}}{L\left(\chi; t + \frac{1}{2}\right)} \cdot p_\chi(E)(x; t, 0),$$

where

$$E(x; t, 0) = E_0(x; t, 0) + E_1(x: t, 0).$$

Hence if $L\left(\chi; t + \frac{1}{2}\right) \neq 0$, eigenfunctions of $\mathscr{H}(G, \Gamma)$ corresponding to the eigenvalue $f \mapsto \hat{f}(t)$ should have an expression in terms of special values of the Eisenstein series (zeta functions of binary quadratic forms) at $(t, 0)$.

For the function $c_{-1}(\alpha)$, such an expression has been obtained by Arakawa, if the conductor of $S_\alpha$ is equal to 1 ([A, Proposition 3.1]). Namely, under this assumption, he proved that

$$c_{-1}(\alpha) = -\frac{2\pi}{\log \varepsilon} E\left(S_\alpha; -\frac{1}{2}, 0\right). \tag{5.1}$$

By Theorem 6, the $\mathscr{C}^\infty(\Gamma \backslash K')_\chi$-component of an $\mathscr{H}(G, \Gamma)$-eigenfunction can be determined uniquely up to constant multiple by the corresponding eigenvalue. Hence, by (5.1), we have the following:

THEOREM 7. *For any character $\chi$ of $Cl_1$, the following identity holds:*

$$p_\chi(c_{-1})(\alpha) = -\frac{2\pi}{\log \varepsilon} p_\chi(E)\left(S_\alpha; -\frac{1}{2}, 0\right) \quad (\alpha \in K').$$

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