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# NEWTON POLYGONS AND GEVREY INDICES FOR LINEAR PARTIAL DIFFERENTIAL OPERATORS

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## 0. Introduction

This paper is a continuation of Miyake [7] by the first named author. \ shall study the unique solvability of an integro-differential equation in the ca gory of formal or convergent power series with Gevrey estimate for the coel cients, and our results give some analogue in partial differential equations Ramis [10, 11] in ordinary differential equations.

In the study of analytic ordinary differential equations, the notion of irreg larity was first introduced by Malgrange [3] as a difference of indices of differential operator in the categories of formal power series and converge power series. After that, Ramis extended his theory to the category of formal convergent power series with Gevrey estimate for the coefficients. In these studi Ramis revealed a significant meaning of a Newton polygon associated with differential operator.

We define a Newton polygon of a partial differential operator following idea of Yonemura [13] which is an extension of Ramis' one. Let

$$(0.1)\qquad P=\sum_{\sigma\in\mathbf{N}^p}\sum_{j\in\mathbf{N}^p}\sum_{\alpha\in\mathbf{N}^q}a_{\sigma j\alpha}(x)t^{\sigma}D_t^jD_x^{\alpha}\quad(|j|+|\alpha|<+\infty)$$

be a partial differential operator of finite order with holomorphic coefficients ir neighbourhood of the origin, where $t=(t_1,\cdots,t_p)\in\mathbf{C}^p$ $(p\geq 1)$, $x=(x_1,\cdots,$ $\in\mathbf{C}^q$ $(q\geq 0)$, $D_t=(\partial/\partial t_1,\cdots,\partial/\partial t_p)$, etc.

For $(\sigma, j, \alpha)\in\mathbf{N}^p\times\mathbf{N}^p\times\mathbf{N}^q$, we define a left half line $Q(\sigma, j, \alpha)$ in plane $\mathbf{R}^2$ by

$$(0.2)\qquad Q(\sigma,j,\alpha):=\{(u,|\sigma|-|j|)\in\mathbf{R}^2;\ u\leq|j|+|\alpha|\}.$$

Now a Newton polygon $\mathrm{N}(P)$ of the operator $P$ is defined by

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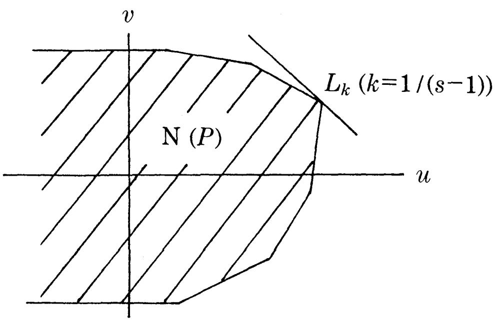
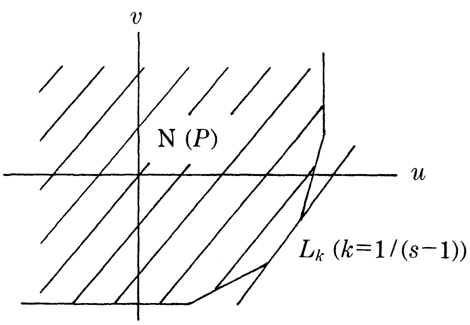
$$(0.3) \qquad \mathrm{N}(P) := \mathrm{Ch}\{Q(\sigma, j, \alpha)\ ;\ (\sigma, j, \alpha) \text{ with } a_{\sigma j \alpha}(x) \not\equiv 0\},$$

where $\mathrm{Ch}\{\cdot\}$ denotes the convex hull of sets in $\{\cdot\}$. By the definition, Newton polygon $\mathrm{N}(P)$ looks like as follows.

(i) The case of polynomial coefficients in $t$.




(ii) The case of non polynomial coefficients in $t$.




Ramis made clear the meaning of sides and vertices of $\mathrm{N}(P)$ in the case of ordinary differential operator (i.e., $(p, q) = (1,0)$) from a view point of index theorems of the mappings,

$$(0.4) \qquad P: G^s \to G^s,$$

$$(0.5) \qquad P: G^{(s)} \to G^{(s)} \ (s \in \mathbf{R}),$$

as follows. For the definitions of $G^s$ and $G^{(s)}$, see § 1.2.

For $s \in \mathbf{R}$, we put $k = 1/(s-1) \in \mathbf{R} \cup \{\infty\}$. We draw a line $L_k$ with slope $k$ such that $L_k$ contacts with $\mathrm{N}(P)$ at a vertex or on a side of $\mathrm{N}(P)$. Then the index $\chi(P; G^s)$ (resp. $\chi(P; G^{(s)})$) of the mapping (0.4) (resp. (0.5)) is given by $\chi(P; G^s) = -\min\{v\ ;\ (u, v) \in \mathrm{N}(P) \cap L_k\}$ (resp. $\chi(P; G^{(s)}) = -\max\{v\ ;\ (u, v) \in \mathrm{N}(P) \cap L_k\}$) (see Ramis [10, 11] for the detailed descriptions).

The aim of this paper is to give some analogue of these results. For a partial differential operator, dimensions of kernel and cokernel, however, are infinite in general. Therefore, we shall study the unique solvability of the Cauchy-Goursat problem,

$$(0.6)\qquad \begin{cases} Pu(t,x) = f(t,x), \\ u(t,x) - w(t,x) = O(t^l x^\beta) \ (l \in \mathrm{N}^p, \beta \in \mathrm{N}^q), \end{cases}$$

in the category $G^s$ or $G^{(s)}$ ($s \in \mathbf{R} \cup \{\pm \infty\}$). For this purpose, it is convenient to convert the problem to the bijectivity of the mapping for an integro-differential operator $L := PD_t^{-l} D_x^{-\beta}$,

$$(0.7)\qquad L : G^s \to G^s \quad \text{or} \quad L : G^{(s)} \to G^{(s)},$$

by replacing the unknown function $u(t,x)$ to $U(t,x)$ by $u = D_t^{-l} D_x^{-\beta} U + w$.

In Chapter 1 (§1–§4), we shall study an integro-differential operator of the form,

$$(0.8)\qquad L = I - \sum_{\sigma,j,\alpha}^{\text{finite}} a_{\sigma j \alpha}(x) t^\sigma D_t^j D_x^\alpha \quad (\sigma \in \mathbf{N}^p, j \in \mathbf{Z}^p, \alpha \in \mathbf{Z}^q),$$

where $I$ denotes the identity map. We call this operator of standard type, because such an operator is derived from the Cauchy-Goursat problem of usual type (see, for exampe, §1.6 and Wagschal [12]). Under an assumption that the origin is a vertex or is in a side of the Newton polygon $\mathrm{N}(L)$, which will be defined in §1.4, we shall study the bijectivity of the mapping (0.7) (Theorems A and B).

In Chapter 2 (§5–§7), we shall study an integro-differential operator of the form,

$$(0.9)\qquad L = P_m(\delta_t) - \sum_{\sigma,j,\alpha}^{\text{finite}} a_{\sigma j \alpha}(x) t^\sigma D_t^j D_x^\alpha \quad (\sigma \in \mathbf{N}^p, j \in \mathbf{Z}^p, \alpha \in \mathbf{Z}^q),$$

where $P_m(\delta_t)$ is a multi-dimensional Euler type operator of order $m$. Such an operator was called of Cauchy-Goursat-Fuchs type in Miyake [7], because such an operator is derived from the characteristic Cauchy problem of Fuchs type or a multi-dimensional singular operator of Fuchs type. Under an assumption that a point $(m, 0)$ is a vertex of the Newton polygon $\mathrm{N}(L)$, we shall study the bijectivity of the mapping (0.7) (Theorem C), and also we shall characterize such number $s \in \mathbf{R}$ that the mapping,

$$(0.10)\qquad L : G^{+\infty}/G^s \to G^{+\infty}/G^s,$$

is bijective (Theorem D), which is known as Maillet's type theorem in ordinary differential equations (see Malgrange [4], Gérard and Tahara [1] and references cited there).

We note that in Yonemura [13] and Miyake [7] only the first positive slope among the sides of $N(L)$ was analyzed, and hence only the case $s \geq 1$ was studied. We also note that the case $0 < s < 1$ was essentially studied in Miyake [6]. Therefore, the most interesting part in this paper is in the treatment of the spaces $G^s$ and $G^{(s)}$ for $s \leq 0$, and we shall see that this case is completely different from the case $s > 0$ (see Theorem B and Miyake [8]).

For the simplicity of descriptions, we restrict ourselves to the operator with polynomial coefficients in the variables $t$, but the results obtained in the case $s \geq 1$ hold under the assumption that the coefficients belong to $G^s$ or $G^{(s)}$ according to the mapping (see Miyake [7] and Remark 3.2).

At the end of the introduction, we wish to mention that we can see another analogy between the studies of ordinary and partial differential equations in the problems of characterization of regular singular points for systems of ordinary differential equations and that of Kowalevskian systems for partial differential equations (compare Miyake [5] with Moser [9] and Kitagawa [2]).

# Chapter 1. Operators of standard type

## 1. Statement of results

### 1.1. Integro-differential operators of standard type

Let $t = (t_1, \ldots, t_p) \in \mathbf{C}^p$ $(p \geq 1)$ and $x = (x_1, \ldots, x_q) \in \mathbf{C}^q$ $(q \geq 1)$. We shall study the following integro-differential operator with holomorphic coefficients in a neighbourhood of the origin:

$$L = I - \sum_{\sigma, j, \alpha}^{\text{finite}} a_{\sigma j \alpha}(x) t^\sigma D_t^j D_x^\alpha \ (\sigma \in \mathbf{N}^p,\ j \in \mathbf{Z}^p,\ \alpha \in \mathbf{Z}^q), \tag{1.1}$$

where $I$ denotes the identity map and $\mathbf{N}$ (resp. $\mathbf{Z}$) denotes the set of non negative integers (resp. integers).

For $j = (j_1, \cdots, j_p) \in \mathbf{Z}^p$, we define $D_t^j = D_{t_1}^{j_1} \cdots D_{t_p}^{j_p}$ as follows:

$$D_{t_k} := \partial/\partial t_k \text{ and } D_{t_k}^{-1} := \int_0^{t_k} \text{(integration in the variable } t_k \text{ from } 0 \text{ to } t_k).$$

We define $|j| := j_1 + j_2 + \cdots + j_p \in \mathbf{Z}$ for $j \in \mathbf{Z}^p$ as usually. It is the same for $D_x^\alpha$ for $\alpha \in \mathbf{Z}^q$.

### 1.2. Gevrey spaces $G^s$ and $G^{(s)}$

We denote by $G^{+\infty}$ the set of formal power series of the form,

$$U(t, x) = \sum_{l \in \mathbf{N}^p} U_l(x) \frac{t^l}{l!},$$

where $U_l(x)$ are holomorphic in a common neighbourhood of $x = 0$ for all $l \in \mathbf{N}^p$.

Definition 1.1. Let $s \in \mathbf{R} \cup \{\pm \infty\}$ and $U(t, x) \in G^{+\infty}$.

(i) $U(t, x) \in G^s$ $(s \in \mathbf{R})$ if there are positive constants $X$ and $T$ such that

$$\max_{\|x\| \le X} |U_l(x)| \le C \frac{|l|!^s}{T^{|l|}} \quad (l \in \mathbf{N}^p) \tag{1.2}$$

holds for some non negative constant $C$. Here, $\|x\| := \sum_{k=1}^q |x_k|$.

(ii) $U(t, x) \in G^{(s)}$ $(s \in \mathbf{R})$ if there is a positive constant $X$ such that

$$\max_{\|x\| \le X} |U_l(x)| \le C(T) \frac{|l|!^s}{T^{|l|}} \quad (l \in \mathbf{N}^p) \tag{1.3}$$

holds for any $T > 0$ and some non negative constant $C(T)$ depending on $T$.

(iii) $G^{-\infty} := \bigcap_{s \in \mathbf{R}} G^s,\quad G^{(-\infty)} := \bigcap_{s \in \mathbf{R}} G^{(s)},\ G^{(+\infty)} := \bigcup_{s \in \mathbf{R}} G^{(s)}.$

By the definition, $G^1$ is the set of holomorphic functions in a neighbourhood of the origin and $G^s$ $(s < 1)$ is the set of locally holomorphic functions in the variables $x$ and entire functions of exponential order $1/(1 - s)$ in the variable $t$. The other function spaces are now easily understood.

### 1.3. Problem

We shall study the following mappings,

$$(L)_s \qquad L : G^s \to G^s,$$

$$(L)_{(s)} \qquad L : G^{(s)} \to G^{(s)},$$

for $s \in \mathbf{R} \cup \{\pm \infty\}$.

The purpose of this chapter is to characterize the number $s$ such that the above mapping will be bijective. In conclusion, we shall see that such a number is characterized from the slopes of sides of a Newton polygon $\mathrm{N}(L)$ of the operator $L$ defined below.

### 1.4. Newton polygon N(*L*)

For $(\sigma, j, \alpha) \in \mathbf{N}^p \times \mathbf{Z}^p \times \mathbf{Z}^q$, we define a left half line $Q(\sigma, j, \alpha)$ in a plane $\mathbf{R}^2$ by

$$Q(\sigma, j, \alpha) := \{(u, |\sigma| - |j|) \in \mathbf{R}^2 \, ; u \leq |j| + |\alpha|\}.$$

Then a Newton polygon $\mathrm{N}(L)$ of $L$ is defined by

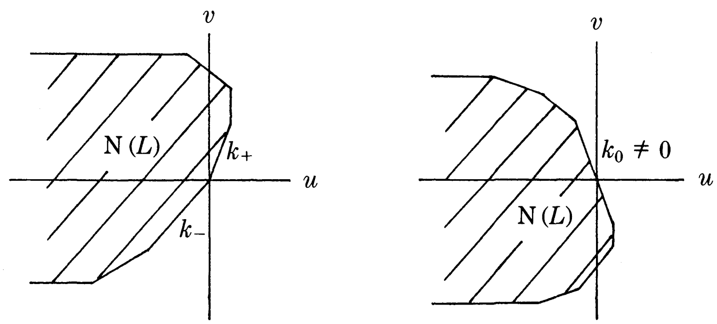
$$\mathrm{N}(L) := \mathrm{Ch}\{Q(\sigma, j, \alpha) \, ; (\sigma, j, \alpha) \text{ with } a_{\sigma j \alpha}(x) \not\equiv 0\}. \tag{1.4}$$

Here $\mathrm{Ch}\{\cdot\}$ denotes the convex hull of sets in $\{\cdot\}$. We note that $Q(o, o, o)$ (which corresponds to the identity map $I$) is included in this set.

### 1.5. Results

First of all, we assume the following fundamental assumption.

(A) The origin is a vertex or is in a side with non zero slope of $\mathrm{N}(L)$.




First, we study the case where the origin is a vertex of $\mathrm{N}(L)$.

Let $k_+ \in \mathbf{R} \cup \{+\infty\}$ (resp. $k_- \in \mathbf{R} \cup \{-\infty\}$) be the slope of the side of $\mathrm{N}(L)$ with the origin as an end point which is lying in an upper (resp. a lower) half plane.

We define two numbers $s_i \in \mathbf{R} \cup \{\pm\infty\}$ $(i = \pm, \, s_+ < s_-)$ by

$$s_i = 1 + \frac{1}{k_i}. \tag{1.5}$$

Here we make the following rule:

(i) If $k_- = 0$, then $s_- := \lim_{k \downarrow 0} 1 + (1/k) = +\infty$.

(ii) If $k_+ = +\infty$, then $s_+ = 1$.

(iii) If $k_- = -\infty$, then $s_- = 1$.

(iv) If $k_+ = 0$, then $s_+ := \lim_{k \uparrow 0} 1 + (1/k) = -\infty$.

Now our first result is stated as follows.

THEOREM A. *We assume the following additional condition,*

$$\sum_{|j|=|\alpha|=0} |a_{oj\alpha}(0)| < 1 \quad \text{(Spectral condition)}, \tag{1.6}$$

*where* $o = (0,\cdots,0) \in \mathbf{N}^p$. *Then we have:*

(i) *If* $s_+ < s < s_-$, *then* $(L)_s$ *and* $(L)_{(s)}$ *are bijective.*

(ii) $(L)_{s_+}$ *is bijective.*

(iii) $(L)_{(s_-)}$ *is bijective.*

(iv) *If* $s_- = +\infty$, *then* $(L)_{+\infty}$ *is bijective.*

*Remark* 1.2. The assumption (1.6) can be weakened as follows.

(1.6)′ There are $\tau \in \mathbf{R}_+^p$ and $\xi \in \mathbf{R}_+^q$ $(\mathbf{R}_+ := (0, +\infty))$ such that

$$\sum_{|j|=|\alpha|=0} |a_{oj\alpha}(0)| \tau^j \xi^\alpha < 1.$$

Indeed, if we transform the variables $(t, x)$ to $(s, y)$ by $t = (\tau_1 s_1, \cdots, \tau_p s_p)$ and $x = (\xi_1 y_1, \cdots, \xi_q y_q)$, then the condition (1.6)′ is reduced to (1.6).

Next, we study the case where the origin is in a side of $\mathrm{N}(L)$. Let $k_0$ $(\neq 0)$ be the slope of this side and put

$$s_0 := 1 + \frac{1}{k_0} \in \mathbf{R}. \tag{1.7}$$

Then we can prove the following,

THEOREM B. (i) *Let* $s_0 > 0$. *Then under the condition,*

$$\sum_{s_0|j|+|\alpha|=0} |a_{oj\alpha}(0)| < 1 \quad \text{(Spectral condition)}, \tag{1.8}$$

*the mapping* $(L)_{s_0}$ *is bijective.*

(ii) *Let* $s_0 \leq 0$ *and assume the following condition,*

$$\kappa = \sum_{|j|+|\alpha| \geq 0}^{(1)} e^{|j|+|\alpha|} |a_{oj\alpha}(0)| + \sum_{|j|+|\alpha|<0}^{(2)} e^{s_0(|j|+|\alpha|)} |a_{oj\alpha}(0)| < 1 \quad \text{(Spectral condition)}, \tag{1.9}$$

*where the summations are taken over* $(j, \alpha)$ *such that* $s_0|j| + |\alpha| = 0$. *Then the mapping* $(L)_{(s_0)}$ *is injective and the mapping,* $L: G^{s_0} \to G^{(s_0)}$, *is surjective.*

*Remark* 1.3. (i) The conditions (1.8) and (1.9) can be weakened as the same manner as Remark 1.2.

(ii) The condition (1.8) seems to be best in the general framework, but the possibility of improvement of the condition (1.9) will be a problem leaving in the future.

The following proposition is an immediate consequence of the definitions $s_\pm$ and $s_0$, but it will play an important role in the proofs of the theorems.

PROPOSITION 1.4. *Let* $s_i$ $(i = \pm, 0)$ *be as above, and* $s_+ \leq s \leq s_-$ *or* $s = s_0$. *Then it holds that*

$$(1.10) \quad s|j| + (1-s)|\sigma| + |\alpha| \equiv s(|j| - |\sigma|) + |\sigma| + |\alpha| \underset{\text{put}}{=} -\delta \leq 0,$$

*for any* $(\sigma, j, \alpha)$ *with* $a_{\sigma j \alpha}(x) \not\equiv 0$.

*If* $s_+ < s < s_-$, *then* $\delta = 0$ *if and only if* $(|\alpha + |j|, |\sigma| - |j|) = (0,0)$. *If* $s = s_i$ $(i = \pm, 0)$, *then* $\delta = 0$ *if and only if* $(|\alpha + |j|, |\sigma| - |j|)$ *is on a side of* $\mathrm{N}(L)$ *with the slope* $k_i$. *Therefore, if* $s = s_+$ *(resp.* $s = s_-$*), then* $\delta = 0$ *only if* $|\sigma| \geq |j|$ *(resp.* $|\sigma| \leq |j|$*).*

### 1.6. A simple application and a remark

(A) Let $P = P(t, x; D_t, D_x)$ be a partial differential operator given by (0.1) and $\mathrm{N}(P)$ be its Newton polygon defined by (0.3). We study the surjectivity of mappings,

$$(P)_s \qquad P: G^s \to G^s,$$

$$(P)_{(s)} \qquad P: G^{(s)} \to G^{(s)}$$

for $s \in \mathbf{R} \cup \{\pm \infty\}$.

We draw a line $L_k$ with slope $k := 1/(s-1)$ which contacts to $\mathrm{N}(P)$ as in introduction. We define two vertices $V_s = (u_0, v_0)$ and $V_{(s)} = (u_1, v_1)$ of $\mathrm{N}(P)$ by

$$(1.11) \qquad \begin{aligned} v_0 &= \min\{v; (u, v) \in \mathrm{N}(P) \cap L_k\}, \\ v_1 &= \max\{v; (u, v) \in \mathrm{N}(P) \cap L_k\}. \end{aligned}$$

We put

$$\overset{\circ}{V}_s = \{(o, j, \alpha) \in \mathbf{N}^p \times \mathbf{N}^p \times \mathbf{N}^q \,;\, a_{oj\alpha}(0) \neq 0,\ (|j| + |\alpha|, -|j|) = V_s\},$$
$$\overset{\circ}{V}_{(s)} = \{(o, j, \alpha) \in \mathbf{N}^p \times \mathbf{N}^p \times \mathbf{N}^q \,;\, a_{oj\alpha}(0) \neq 0,\ (|j| + |\alpha|, -|j|) = V_{(s)}\}.$$

Then Theorem A implies:

(i) If $\overset{\circ}{V}_s \neq \phi$ then the mapping $(P)_s$ is surjective.

(ii) If $\overset{\circ}{V}_{(s)} \neq \phi$ then the mapping $(P)_{(s)}$ is surjective.

Indeed, to prove the statement (i) it is sufficient to show the existence of $(o, l, \beta) \in \overset{\circ}{V}_s$ such that the Goursat problem,

$$Pu(t, x) = f(t, x) \in G^s, \quad u(t, x) = O(t^l x^\beta)$$

is uniquely solvable in $G^s$. Theorem A shows that it is sufficient if we can take $(l, \beta)$ such that

$$|a_{ol\beta}(0)| > \sum_{(o,j,\alpha) \in \overset{\circ}{V}_s \setminus (o,l,\beta)} |a_{oj\alpha}(0)|\, \tau^{j-l} \xi^{\alpha-\beta} \tag{1.12}$$

holds for some $(\tau, \xi) \in \mathbf{R}_+^p \times \mathbf{R}_+^q$. Now the existence of such $(l, \beta)$ is easily proved as follows: if $\#\{j\,;\,(o, j, \alpha) \in \overset{\circ}{V}_s\} \geq 2$, then we take $(o, l, \beta) \in \overset{\circ}{V}_s$ such that $l$ is a vertex of $\mathrm{Ch}\{j \in \mathbf{N}^p\,;\,(o, j, \alpha) \in \overset{\circ}{V}_s\}$, where $\#\{\cdot\}$ denotes the cardinal number of $\{\cdot\}$. If $\#\{j\,;\,(o, j, \alpha) \in \overset{\circ}{V}_s\} = 1$, then we take $(o, l, \beta) \in \overset{\circ}{V}_s$ such that $\beta$ is a vertex of $\mathrm{Ch}\{\alpha \in \mathbf{N}^q\,;\,(o, j, \alpha) \in \overset{\circ}{V}_s\}$.

It is the same for the statement (ii).

In a special case of an operator $P$ with constant coefficients, the mappings $(P)_s$ and $(P)_{(s)}$ are surjective for all $s \in \mathbf{R} \cup \{\pm\infty\}$.

(B) We consider the following Goursat problem,

$$\begin{cases} Pu(t, x) := \{aD_t^l D_x^\beta - D_t^{l-j} D_x^{\beta+j+\alpha} - D_t^{l+j} D_x^{\beta-j-\alpha}\}u = f(t, x), \\ u(t, x) - w(t,x) = O(t^l x^\beta), \end{cases} \tag{1.13}$$

where $t, x \in \mathbf{C}$, $1 \leq j \leq l$, $-\beta - j \leq \alpha \leq \beta - j$ and $a \in \mathbf{C}\setminus\{0\}$. In this operator, $-\beta/j \leq s_0 = 1 + (\alpha/j) \leq \beta/j$. Theorem B implies:

(i) Let $\alpha > -j$. Then $s_0 > 0$ and the problem (1.13) is uniquely solvable in $G^{s_0}$ if $|a| > 2$. The condition $|a| > 2$ is the well known spectral condition in the case $\alpha = 0$ (i.e. $s_0 = 1$).

(ii) If $\alpha \leq -j$, then $s_0 \leq 0$. Therefore, the problem (1.13) is solvable in $G^{s_0}$ for any $f, w \in G^{(s_0)}$ if $|a| > \exp(s_0\alpha) + \exp(-\alpha)$.

In Miyake [8], we shall study this operator and discuss more precisely the spectral condition on $a$, where we shall find that the circumstances are completely different between the cases $s_0 > 0$ and $s_0 \le 0$.

## 2. Gevrey spaces $G^s(T, X; k)$ $(s > 0)$ and $G^s_n(T, X; k)$ $(s \le 0)$

We denote by $\mathcal{O}(\|x\| < X)$ $(X > 0)$ the set of holomorphic functions in a domain $\|x\| := \sum_{i=1}^q |x_i| < X$ and by $\mathcal{O}(\|x\| \le X)$ the set of holomorphic functions in $\|x\| < X$ and continuous on $\|x\| \le X$.

DEFINITION 2.1. Let $U(t, x)$ be a formal power series written by

$$U(t, x) = \sum_{l,\beta} U_{l\beta} \frac{t^l x^\beta}{l!\beta!} \quad (U_{l\beta} \in \mathbf{C},\ l \in \mathbf{N}^p,\ \beta \in \mathbf{N}^q). \tag{2.1}$$

Then we define Banach spaces as follows.

(i) Let $s, T, X > 0$ and $k \in \mathbf{N}$. Then $U(t, x) \in G^s(T, X; k)$ if

$$\| U \|^{(s)}_{T,X;k} := \sup_{l,\beta} | U_{l\beta} | \frac{T^{|l|} X^{|\beta|}}{(s|l| + |\beta| + k)!} < \infty. \tag{2.2}$$

(ii) Let $s \le 0$, $T, X > 0$, $k \in \mathbf{N}$ and $n \ge 1$. Then $U(t, x) \in G^s_n(T, X; k)$ if

$$\| U \|^{(s)}_{n;T,X;k} := \sup_{l,\beta} | U_{l\beta} | T^{|l|} X^{|\beta|} \frac{\{(n-s)|l| + n|\beta| + k\}!}{\{n|l| + (n+1)|\beta| + k\}!} < \infty. \tag{2.3}$$

Here $y! := \Gamma(y+1)$ for $y \ge 0$.

From the above definition, it is easily proved that:

(i) If $s > 0$, then for any $k \in \mathbf{N}$, $0 < X' < X$ and $0 < T' < T$ it holds that

$$G^s(T, X) := G^s(T, X; 0) \subset G^s(T, X; k) \subset G^s(T', X'). \tag{2.4}$$

(ii) If $s \le 0$, then for any $k \in \mathbf{N}$, $0 < X' < X$ and $0 < T' < T$ it holds that

$$G^s_n(T, X) := G^s_n(T, X; 0) = G^s_n(T, X; k) \subset G^s_n(T', X'). \tag{2.5}$$

Indeed, it is sufficient to check the following inequalities,

$$\left(\frac{n}{n-s}\right)^k \le \frac{\{n|l| + (n+1)|\beta| + k\}!\{(n-s)|l| + n|\beta|\}!}{\{(n-s)|l| + n|\beta| + k\}!\{n|l| + (n+1)|\beta|\}!} \le \left(\frac{n+1}{n}\right)^k.$$

For $U(t, x) = \sum U_{l\beta} t^l x^\beta / l!\beta!$, we set

$$U_l(x) = \sum_{\beta\in\mathbf{N}^q} U_{l\beta}\frac{x^\beta}{\beta!} \quad (l \in \mathbf{N}^p). \tag{2.6}$$

Then we have the following,

LEMMA 2.2. (i) *Let $U(t, x) \in G^s(T, X)$ $(s > 0)$. Then $U_l(x) \in \mathcal{O}(\|x\| < X)$ $(l \in \mathbf{N}^p)$ and for any $Y$ with $0 < Y < X$, there is a positive constant $R$ depending only on $s$ and $Y/X\,(< 1)$ such that*

$$\max_{\|x\|\le Y} |U_l(x)| \le C\frac{|l|!^s}{(RT)^{|l|}} \tag{2.7}$$

*holds for some non negative $C$.*

(ii) *Let $U(t, x) \in G^s_n(T, X)$ $(s \le 0)$. Then $U_l(x) \in \mathcal{O}(\|x\| < X/e\,(n+1))$ $(l \in \mathbf{N}^p)$, and for any $Y$ with $0 < Y < X/e(n+1)$ there is a positive constant $R$ depending only on $n$ and $s$ such that the inequality* (2.7) *holds.*

*Proof.* (i) Let $s > 0$. Then we have immediately,

$$U_l(x) \ll \frac{\|U\|}{T^{|l|}}\sum_\beta \frac{(s|l| + |\beta|)!}{\beta!}\left(\frac{x}{X}\right)^\beta$$

$$= \frac{\|U\|}{T^{|l|}}\frac{(s|l|)!}{(1-|x|/X)^{s|l|+1}} \quad (|x| := x_1 + \cdots + x_q).$$

Here $\|U\| = \|U\|^{(s)}_{T,X}$ and $U(x) \ll V(x)$ means that $U(x)$ is majorized by $V(x)$. This, together with the Stirling formula, implies (2.7).

(ii) Let $s \le 0$ and $U(t, x) \in G^s_n(T, X)$. Then

$$U_l(x) \ll \frac{\|U\|}{T^{|l|}}|l|!^s\sum_\beta \frac{\{n|l| + (n+1)|\beta|\}!}{\{(n-s)|l| + n|\beta|\}!}\frac{|l|!^{-s}}{|\beta|!}\frac{|\beta|!}{\beta!}\left(\frac{x}{X}\right)^\beta.$$

By employing the Stirling formula, we have

$$\frac{\{n|l| + (n+1)|\beta|\}!}{\{(n-s)|l| + n|\beta|\}!}\frac{|l|!^{-s}}{|\beta|!}$$

$$\le C\frac{(2\pi|l|)^{-s/2}}{(2\pi|\beta|)^{1/2}}\left(\frac{n+1}{n}\right)^{n(|l|+|\beta|)}\left(\frac{1}{n-s}\right)^{-s|l|}\left(\frac{n|l| + (n+1)|\beta|}{|\beta|}\right)^{|\beta|},$$

for some positive constant $C$. Since

$$\left(1+\frac{1}{n}\right)^n < e \quad \text{and} \quad \left(\frac{n|l| + (n+1)|\beta|}{|\beta|}\right)^{|\beta|} < (n+1)^{|\beta|}e^{|l|},$$

we get

$$U_l(x) \ll C' \| U \| \left( \frac{e^2(n-s)^s}{T} \right)^{|l|} \frac{|l|!^s}{|l|^{s/2}} \sum_\beta \frac{|\beta|!}{\beta!} \left( \frac{e(n+1)}{X} x \right)^\beta$$

$$= C' \| U \| \left( \frac{e^2(n-s)^s}{T} \right)^{|l|} \frac{|l|!^s}{|l|^{s/2}} \frac{1}{1 - e(n+1)|x|/X},$$

for some non negative constant $C'$. This shows $U_l(x) \in \mathcal{O}(\|x\| < X/e(n+1))$ and hence the inequality (2.7) follows immediately. □

Now we shall prove the following,

LEMMA 2.3. *Let $s \in \mathbf{R}$. Then it holds that:*

(2.8) $$G^s = \bigcup_{T,X>0} G^s(T,X),\ G^{(s)} = \bigcup_{X>0} \bigcap_{T>0} G^s(T,X) \quad \textit{when } s > 0.$$

(2.9) $$G^s = \bigcup_{T,X>0} G^s_n(T,X),\ G^{(s)} = \bigcup_{X>0} \bigcap_{T>0} G^s_n(T,X) \quad \textit{when } s \leq 0.$$

*Proof.* Lemma 2.2, (2.4) and (2.5) imply

$$G^s \supset \bigcup_{T,X>0} G^s(T,X), \quad G^{(s)} \supset \bigcup_{X>0} \bigcap_{T>0} G^s(T,X)\ (s > 0),$$

and the same relations for $G^s_n(T,X)\,(s \leq 0)$.

In order to prove the converse, let $U_l(x) \in \mathcal{O}(\|x\| \leq X)\,(l \in \mathbf{N}^p)$ and assume

(2.10) $$\max_{\|x\| \leq X} |U_l(x)| \leq C \frac{|l|!^s}{T^{|l|}} \quad (l \in \mathbf{N}^p),$$

for a positive constant $T$ and a non negative constant $C$. Let $U_l(x) = \sum U_{l\beta} x^\beta / \beta!$. Then by employing Cauchy's integral formula on a polycircle $\Pi_{i=1}^q \{|x_i| = \xi_i X\}$ $(\xi_i > 0, \xi_1 + \cdots + \xi_q = 1)$, we have

$$|U_{l\beta}| \leq C \frac{|l|!^s}{T^{|l|} X^{|\beta|}} \frac{\beta!}{\xi^\beta} \quad (\xi^\beta := \xi_1^{\beta_1} \cdots \xi_q^{\beta_q}).$$

Since $\xi^\beta$ takes its maximum on the above mentioned domain at a point $\xi = (\beta_1/|\beta|, \cdots, \beta_q/|\beta|)$, we have

$$|U_{l\beta}| \leq C \frac{|l|!^s}{T^{|l|} X^{|\beta|}} \frac{|\beta|^{|\beta|} \beta!}{\beta^\beta}.$$

By the Stirling formula, we have $|\beta|^{|\beta|}\beta!/\beta^{\beta} \leq C'(2\pi|\beta|)^{(q-1)/2}|\beta|!$ for some positive constant $C'$. Hence we have

$$|U_{l\beta}| \leq C''|\beta|^{(q-1)/2}\frac{|l|!^{s}|\beta|!}{T^{|l|}X^{|\beta|}}, \tag{2.11}$$

for some non negative constant $C''$.

(A) Consider the case $s \geq 1$. From (2.11) and the Stirling formula, we get

$$|U_{l\beta}| \leq C|l|^{s/2}|\beta|^{(q-1)/2}\frac{(s|l|+|\beta|)!}{T^{|l|}X^{|\beta|}},$$

for some non negative constant $C$, and hence $U(t,x) = \sum U_l(x)t^l/l! \in G^s(T', X')$ for any $0 < T' < T$ and $0 < X' < X$.

(B) Consider the case $0 < s < 1$. In this case we have

$$|U_{l\beta}| \leq C|l|^{s/2}|\beta|^{(q-1)/2}\frac{(s|l|+|\beta|)!}{(s^sT)^{|l|}X^{|\beta|}},$$

for some non negative constant $C$. Let $E = \exp(-1/e)$. Then $U(t, x) \in G^s(T', X')$ for any $0 < T' < ET$ and $0 < X' < X$, since $E = \min\{s^s; 0 < s < 1\}$.

(C) Consider the case $s \leq 0$. By employing the Stirling formula, we have

$$\begin{aligned}
&|l|!^s|\beta|!\frac{\{(n-s)|l|+n|\beta|\}!}{\{n|l|+(n+1)|\beta|\}!}\\
&\leq C\frac{|\beta|^{1/2}}{|l|^{-s/2}}(n+1)^{-|\beta|}(n-s)^{-s|l|}\left(\frac{n-s}{n}\right)^{n(|l|+|\beta|)}\left(\frac{|l|+|\beta|}{|l|}\right)^{-s|l|}\\
&\leq C'\frac{|\beta|^{1/2}}{|l|^{-s/2}}\{(n-s)^{-s}e^{-s}\}^{|l|}\{(n+1)^{-1}e^{-2s}\}^{|\beta|}.
\end{aligned}$$

Now in view of (2.11), we have

$$|U_{l\beta}| \leq C''\frac{|\beta|^{q/2}}{|l|^{-s/2}}\frac{1}{\{(n-s)^se^sT\}^{|l|}\{(n+1)e^{2s}X\}^{|\beta|}}\frac{\{n|l|+(n+1)|\beta|\}!}{\{(n-s)|l|+n|\beta|\}!}.$$

Hence for any $T'$ and $X'$ and $0 < T' < (n-s)^se^sT$ and $0 < X' < (n+1)e^{2s}X$, we have $U(t, x) \in G_n^s(T', X')$. □

The reader may feel to be curious for the definition of $G_n^s(T, X; k)\,(s \leq 0)$, but its validity will be found by the following example.

EXAMPLE 2.4. Let $L = I - atD_x^{-1}$, where $t, x \in \mathbf{C}$ and $a \in \mathbf{C}$. In this operator, $k_+ = -1$ and hence $s_+ = 0$. Therefore, the mapping, $L: G^0 \to G^0$, is bijective by Theorem A. This is proved as follows. We consider the equation, $LU(t, x) = F(t, x)$ in $G_1^0(T, X)$. Let $U(t, x) = \sum U_{l\beta} t^l x^\beta / l!\beta!$ and $tD_x^{-1} U(t, x) = \sum V_{l\beta} t^l x^\beta / l!\beta!$. Then we have $V_{l\beta} = lU_{l-1,\beta-1}$. Hence the operator norm of $tD_x^{-1}$ in the space $G_1^0(T, X)$ is estimated by

$$\| tD_x^{-1} \| \leq TX \sup_{l,\beta \geq 1} l \frac{(l + 2\beta - 3)!}{(l + \beta - 2)!} \frac{(l + \beta)!}{(l + 2\beta)!} = TX.$$

By this estimate, we can employ the principle of contraction map in $G_1^0(T, X)$ if $|a| TX < 1$.

On the other hand, suppose we employ the following norm,

$$\| U \|_{T,X}^{(0)} := \inf \{C \,;\, | U_{l\beta} | \leq C\beta! / T^l X^\beta\}.$$

This norm seems to be natural than that of $G_1^0(T, X)$ according to the definition of $G^0$ (see (1.2)), but we can not estimate the operator norm of $tD_x^{-1}$ since

$$| V_{l\beta} | \frac{T^l X^\beta}{\beta!} \leq TX \| U \|_{T,X}^{(0)} \frac{l}{\beta} \to \infty \quad \text{as} \quad \frac{l}{\beta} \to \infty.$$

## 3. Lemmas

Let $a(x) = \sum a_\beta x^\beta / \beta! \in \mathcal{O}(\| x \| \leq \rho X)$ $(\rho > 0)$ and put

$$\| a \|_{\rho X} := \max_{\|x\| \leq \rho X} | a(x) |. \tag{3.1}$$

Then from the proof of (2.11), we have

$$| a_\beta | \leq C(q) \| a \|_{\rho X} \frac{(|\beta| + [q/2])!}{(\rho X)^{|\beta|}}, \tag{3.2}$$

for some positive constant $C(q)$ depending only on the dimension $q$ of $x$. Here $[q/2]$ denotes the integral part of $q/2$.

Now we prove the following,

LEMMA 3.1. *Let $a(x)$ be as above.*

(i) *Let $U(t,x) \in G^s(T,X; k)$ $(s > 0)$. Then for any $\rho > 1$, $a(x)U(t, x) \in G^s(T, X; k)$ and*

$$\|aU\| \leq C(q)\frac{[q/2]!}{(1-1/\rho)^{[q/2]+1}}\|a\|_{\rho X}\|U\|. \tag{3.3}$$

(ii) *Let* $U(t,x) \in G_n^s(T,X;k)$ $(s \leq 0)$ *and* $k \geq n-1$. *Then for any* $\rho > e^{-s}/(n+1)$, $a(x)U(t,x) \in G_n^s(T,X;k)$ *and*

$$\|aU\| \leq C(q)\frac{[q/2]!}{\{1-e^{-s}/(n+1)\rho\}^{[q/2]+1}}\|a\|_{\rho X}\|U\|. \tag{3.4}$$

*Hence, for any holomorphic function* $a(x)$ *in a neighbourhood of* $x=0$ *and* $U(t,x) \in G^s(T,X;k)$ *or* $G_n^s(T,X;k)$ *there is a positive constant* $X_0(\leq X)$ *such that* $a(x)U(t,x) \in G^s(T,Y;k)$ *or* $G_n^s(T,Y;k)$ *for any* $0 < Y \leq X_0$, *and*

$$\|aU\| \leq \{|a(0)| + O(Y)\}\|U\|, \tag{3.5}$$

*where* $O(Y) \geq 0$ *and* $O(Y)/Y$ *is bounded as* $Y \downarrow 0$.

*Proof.* We put $a(x)U(t,x) = \sum V_{l\beta}t^l x^\beta/l!\beta!$. Then

$$V_{l\beta} = \sum_{0\leq\gamma\leq\beta} a_\gamma U_{l,\beta-\gamma}\frac{\beta!}{\gamma!(\beta-\gamma)!}.$$

(i) The case $s > 0$. We have

$$|V_{l\beta}| \leq C(q)\|a\|_{\rho X}\frac{\|U\|}{T^{|l|}X^{|\beta|}}$$

$$\times \sum_{0\leq\gamma\leq\beta}\frac{(|\gamma|+[q/2])!(|\beta|-|\gamma|+s|l|+k)!}{\rho^{|\gamma|}}\frac{\beta!}{\gamma!(\beta-\gamma)!}.$$

Since $\sum_{|\gamma|=i}\beta!/\gamma!(\beta-\gamma)! = |\beta|!/i!(|\beta|-i)!$ and $(|\beta|-i+s|l|+k)!|\beta|!/(|\beta|-i)! \leq (|\beta|+s|l|+k)!$, we have

$$\|aU\| \leq C(q)\|a\|_{\rho X}\|U\|\sum_{i=0}^{\infty}\frac{(i+[q/2])!}{i!}\rho^{-i}$$

$$= C(q)\|a\|_{\rho X}\|U\|\frac{[q/2]!}{(1-1/\rho)^{[q/2]+1}}.$$

(ii) The case $s \leq 0$. We have

$$|V_{l\beta}| \leq C(q)\|a\|_{\rho X}\frac{\|U\|}{T^{|l|}X^{|\beta|}}$$

$$\times \sum_{i=0}^{|\beta|} \frac{(i+[q/2])!}{\rho^i} \frac{\{n|l|+(n+1)(|\beta|-i)+k\}!}{\{(n-s)|l|+n(|\beta|-i)+k\}!} \frac{|\beta|!}{i!(|\beta|-i)!}.$$

We consider the following inequality.

$$\frac{\{n|l|+(n+1)(|\beta|-i)+k\}!}{\{(n-s)|l|+n(|\beta|-i)+k\}!} \frac{|\beta|!}{(|\beta|-i)!} \frac{\{(n-s)|l|+n|\beta|+k\}!}{\{n|l|+(n+1)|\beta|+k\}!}$$

$$= \prod_{j=1}^{i} \frac{|\beta|-j+1}{(n+1)(|\beta|-j+1)+n|l|+k}$$

$$\times \prod_{j=1}^{i} \prod_{p=1}^{n} \frac{n(|\beta|-j+1)+(n-s)|l|+k-p+1}{(n+1)(|\beta|-j+1)+n|l|+k-p}$$

$$\leq (n+1)^{-i} \prod_{j=1}^{i} \prod_{p=1}^{n} \frac{n(|\beta|-j+1)+(n-s)|l|+k-p+1}{(n+1)(|\beta|-j+1)+n|l|+k-p}.$$

Since $1 \leq j \leq i \leq |\beta|$, $1 \leq p \leq n$, $k \geq n-1$ and $s \leq 0$, we have

$$\frac{n(|\beta|-j+1)+(n-s)|l|+k-p+1}{(n+1)(|\beta|-j+1)+n|l|+k-p}$$

$$\leq \frac{n(|\beta|-j+1)+(n-s)|l|+k-p+1}{n(|\beta|-j+1)+n|l|+k-p+1}$$

$$\leq \frac{n(|\beta|-j+1)+(n-s)|l|}{n(|\beta|-j+1)+n|l|} \leq \frac{n-s}{n}.$$

This implies

$$\frac{\{n|l|+(n+1)(|\beta|-i)+k\}!}{\{(n-s)|l|+n(|\beta|-i)+k\}!} \frac{|\beta|!}{(|\beta|-i)!} \frac{\{(n-s)|l|+n|\beta|+k\}!}{\{n|l|+(n+1)|\beta|+k\}!}$$

$$\leq (n+1)^{-i} \left(\frac{n-s}{n}\right)^{ni} < \{(n+1)e^s\}^{-i}.$$

Hence we have

$$|V_{l\beta}| \leq C(q) \|a\|_{\rho X} \frac{\|U\|}{T^{|l|}X^{|\beta|}} \frac{\{n|l|+(n+1)|\beta|+k\}!}{\{(n-s)|l|+n|\beta|+k\}!}$$

$$\times \sum_{i=0}^{|\beta|} \frac{(i+[q/2])!}{i!} \left(\frac{e^{-s}}{(n+1)\rho}\right)^i,$$

and this implies the desired inequality (3.4). □

*Remark* 3.2. When $s \geq 1$, if $a(t, x) \in G^s(\rho T, \rho X)(\rho > 1)$ and $U(t, x) \in G^s(T, X; k)$, then $a(t, x)U(t, x) \in G^s(T, X; k)$ and

$$\| aU \|^{(s)}_{T,X;k} \leq \left(\frac{\rho}{\rho - 1}\right)^2 \| a \|^{(s)}_{\rho T, \rho X} \| U \|^{(s)}_{T,X;k}, \tag{3.6}$$

(see Miyake [7, Lemma 2.4]). By using this inequality and (3.5), the results in Theorems A and B hold by assuming the coefficients belong to $G^s$ or $G^{(s)}$ in the case $s \geq 1$.

Next, we shall estimate the operator norm of an integro-differential operator $t^\sigma D_t^j D_x^\alpha$ ($\sigma \in \mathbf{N}^p$, $j \in \mathbf{Z}^p$, $\alpha \in \mathbf{Z}^q$) acting on $G^s(T, X; k)$ $(s > 0)$ or $G^s_n(T, X; k)$ $(s \leq 0)$.

LEMMA 3.3. (i) *Let $s \in \mathbf{R}$ and $(\sigma, j, \alpha) \in \mathbf{N}^p \times \mathbf{Z}^p \times \mathbf{Z}^q$ satisfy*

$$s|j| + (1 - s)|\sigma| + |\alpha| = s(|j| - |\sigma|) + |\sigma| + |\alpha| = -\delta \leq 0. \tag{3.7}$$

*Then the mapping, $t^\sigma D_t^j D_x^\alpha : G^s(T, X; k) \to G^s(T, X; k)$ $(s > 0)$ or $t^\sigma D_t^j D_x^\alpha : G^s_n(T, X; k) \to G^s_n(T, X; k)$ $(s \leq 0)$, is bounded and its operator norm is estimated by*

$$\| t^\sigma D_t^j D_x^\alpha \| \leq C(\sigma, j, \alpha, s, n) T^{|\sigma| - |j|} X^{-|\alpha|} k^{-\delta}. \tag{3.8}$$

*Here $C(\sigma, j, \alpha, s, n)$ is a positive constant depending only on $\sigma$, $j$, $\alpha$, $s$ and $n$.*

(ii) Let $(o, j, \alpha)$ *satisfy* (3.7) *with $\delta = 0$. Then the operator norm of $D_t^j D_x^\alpha$ is estimated as follows*:

(A) *If $s > 0$, then*

$$\| D_t^j D_x^\alpha \| \leq T^{-|j|} X^{-|\alpha|}. \tag{3.9$_s$}$$

(B) *If $s \leq 0$, then*

$$\| D_t^j D_x^\alpha \| \leq \begin{cases} (1 + \varepsilon(n)) e^{|j| + |\alpha|} T^{-|j|} X^{-|\alpha|} & \text{if } |j| + |\alpha| \geq 0, \\ (1 + \varepsilon(n)) e^{s(|j| + |\alpha|)} T^{-|j|} X^{-|\alpha|} & \text{if } |j| + |\alpha| < 0, \end{cases} \tag{3.9$_s$}$$

*where $0 \leq \varepsilon(n) \to 0$ as $n \to +\infty$.*

*Proof.* Let $U(t, x) = \sum U_{l\beta} t^l x^\beta / l! \beta!$ and $t^\sigma D_t^j D_x^\alpha U(t, x) = \sum V_{l\beta} t^l x^\beta / l! \beta!$. Then

$$V_{l\beta} = \frac{l!}{(l - \sigma)!} U_{l + j - \sigma, \beta + \alpha} \quad (l - \sigma,\ l + j - \sigma \in \mathbf{N}^p;\ \beta + \alpha \in \mathbf{N}^q).$$

(i) First, we consider the case $s > 0$. In this case we have

$$| V_{l\beta} | \frac{T^{|l|} X^{|\beta|}}{(s|l| + |\beta| + k)!}$$

$$\leq \frac{\| U \|}{T^{|j|-|\sigma|} X^{|\alpha|}} \frac{l!}{(l-\sigma)!} \frac{\{s(|l| + |j| - |\sigma|) + |\beta| + |\alpha| + k\}!}{(s|l| + |\beta| + k)!}$$

$$\leq \frac{\| U \|}{T^{|j|-|\sigma|} X^{|\alpha|}} \frac{|l|!}{(|l| - |\sigma|)!} \frac{(s|l| + |\beta| + k - |\sigma| - \delta)!}{(s|l| + |\beta| + k)!}.$$

In the case $s \geq 1$, this implies immediately

$$\| t^{\sigma} D_t^j D_x^{\alpha} \| \leq C T^{|\sigma|-|j|} X^{-|\alpha|} k^{-\delta}, \tag{3.10}$$

for some positive constant $C$. In the case $0 < s < 1$, there is a positive constant $C$ such that

$$\| t^{\sigma} D_t^j D_x^{\alpha} \| \leq C s^{-|\sigma|} T^{|\sigma|-|j|} X^{-|\alpha|} k^{-\delta}. \tag{3.11}$$

Next, we consider the case $s \leq 0$. In this case we have

$$| V_{l\beta} | T^{|l|} X^{|\beta|}$$

$$\leq \frac{\| U \|}{T^{|j|-|\sigma|} X^{|\alpha|}} \frac{l!}{(l-\sigma)!} \frac{\{n(|l| + |j| - |\sigma|) + (n+1)(|\beta| + |\alpha|) + k\}!}{\{(n-s)(|l| + |j| - |\sigma|) + n(|\beta| + |\alpha|) + k\}!}$$

$$\leq \frac{C \| U \|}{T^{|j|-|\sigma|} X^{|\alpha|}}$$

$$\times \frac{\{n|l| + (n+1)|\beta| + n(|j| - |\sigma|) + (n+1)|\alpha| + |\sigma| + k\}!}{\{(n-s)|l| + n|\beta| + n(|j| - |\sigma|) + (n+1)|\alpha| + |\sigma| + \delta + k\}!}.$$

By the Stirling formula we have

$$\frac{\{n|l| + (n+1)|\beta| + n(|j| - |\sigma|) + (n+1)|\alpha| + |\sigma| + k\}!}{\{(n-s)|l| + n|\beta| + n(|j| - |\sigma|) + (n+1)|\alpha| + |\sigma| + \delta + k\}!}$$

$$\leq C k^{-\delta} \frac{\{n|l| + (n+1)|\beta| + n(|j| - |\sigma|) + (n+1)|\alpha| + |\sigma| + k\}!}{\{(n-s)|l| + n|\beta| + n(|j| - |\sigma|) + (n+1)|\alpha| + |\sigma| + k\}!}$$

$$\leq C(\sigma, j, \alpha, s, n) k^{-\delta} \frac{\{n|l| + (n+1)|\beta| + k\}!}{\{(n-s)|l| + n|\beta| + k\}!},$$

and hence we obtain (3.8) for $s \leq 0$.

(ii) The inequality $(3.9)_s$ $(s > 0)$ is obvious from the above proof. To prove

$(3.9)_s$ $(s \leq 0)$, we first consider the case $n|j| + (n+1)|\alpha| = n(|j| + |\alpha|) + |\alpha| \geq 0$.

$$\frac{\{n(|l|+|j|)+(n+1)(|\beta|+|\alpha|)+k\}!}{\{(n-s)(|l|+|j|)+n(|\beta|+|\alpha|)+k\}!}\frac{\{(n-s)|l|+n|\beta|+k\}!}{\{n|l|+(n+1)|\beta|+k\}!}$$

$$=\prod_{i=1}^{n|j|+(n+1)|\alpha|}\frac{n|l|+(n+1)|\beta|+k+i}{(n-s)|l|+n|\beta|+k+i}\leq\left(\frac{n+1}{n}\right)^{n(|j|+|\alpha|)+|\alpha|}$$

$$\leq\begin{cases}e^{|j|+|\alpha|}(1+1/n)^{|\alpha|} & \text{if } |j|+|\alpha|>0,\\ (1+1/n)^{|\alpha|} & \text{if } |j|+|\alpha|=0.\end{cases}$$

Next, in the case $n|j| + (n+1)|\alpha| < 0$ we have

$$\frac{\{n(|l|+|j|)+(n+1)(|\beta|+|\alpha|)+k\}!}{\{(n-s)(|l|+|j|)+n(|\beta|+|\alpha|)+k\}!}\frac{\{(n-s)|l|+n|\beta|+k\}!}{\{n|l|+(n+1)|\beta|+k\}!}$$

$$=\prod_{i=1}^{-n|j|-(n+1)|\alpha|}\frac{(n-s)(|l|+|j|)+n(|\beta|+|\alpha|)+k+i}{n(|l|+|j|)+(n+1)(|\beta|+|\alpha|)+k+i}$$

$$\leq\left(\frac{n-s}{n}\right)^{-n(|j|+|\alpha|)-|\alpha|}$$

$$\leq\begin{cases}e^{s(|j|+|\alpha|)}(1-s/n)^{-|\alpha|} & \text{if } |j|+|\alpha|<0,\\ (1-s/n)^{-|\alpha|} & \text{if } |j|+|\alpha|=0.\end{cases}$$

These imply the inequality $(3.9)_s$ $(s \leq 0)$. □

## 4. Proofs of Theorems A and B

*Proof of Theorem* A. In the case $s \leq 0$ we fix $n = 1$, and we write $G_1^s(T, X; k)$ by $G^s(T, X; k)$. Let

$$A=\sum_{\sigma,j,\alpha}^{\text{finite}}a_{\sigma j\alpha}(x)t^{\sigma}D_t^jD_x^{\alpha}.$$

Then Theorem A is obtained by showing that $A$ defines a contraction map in $G^s(T, X; k)$ by a suitable choice of $T$, $X$ and $k$.

By Proposition 1.4, we know that the condition $s_+ \leq s \leq s_-$ is equivalent to

(4.1) $\quad s|j| + (1-s)|\sigma| + |\alpha| = s(|j| - |\sigma|) + |\sigma| + |\alpha| = -\delta \leq 0,$

for $(\sigma, j, \alpha) \in \mathbf{N}^p \times \mathbf{Z}^p \times \mathbf{Z}^q$ such that $a_{\sigma j\alpha}(x) \not\equiv 0$. We also notice the following facts.

(i) If $s_+ < s < s_-$, then $\delta = 0$ only if $|j| = |\sigma|$.

(ii) If $s = s_+$, then $\delta = 0$ only if $|\sigma| \geq |j|$.

(iii) If $s = s_-$, then $\delta = 0$ only if $|\sigma| \leq |j|$.

First, we estimate the operator norm of $a_{\sigma j\alpha}(x)t^\sigma D_t^j D_x^\alpha$ acting on $G^s(T, X; k)$ in the case $\delta = 0$.

(A) The case $|\sigma| = |j| = |\alpha| = 0$.

$$\| a_{oj\alpha}(x)\, D_t^j D_x^\alpha \|_{T,X;k}^{(s)} \leq | a_{oj\alpha}(0) | + O(X). \tag{4.2}$$

(B) The case $|\sigma| = |j| > 0$ and $|j| + |\alpha| = 0$.

$$\| a_{\sigma j\alpha}(x)t^\sigma D_t^j D_x^\alpha \|_{T,X;k}^{(s)} \leq A_{\sigma j\alpha s} X^{-|\alpha|} \to 0 \quad \text{as} \quad X \downarrow 0. \tag{4.3}$$

Here and in what follows $A_{\sigma j\alpha s}$ denote various positive constants independent of $T$, $X$ and $k$.

(C) The case $s = s_+ \in \mathbf{R}$ and $|\sigma| > |j|$.

$$\| a_{\sigma j\alpha}(x)t^\sigma D_t^j D_x^\alpha \|_{T,X;k}^{(s)} \leq A_{\sigma j\alpha s} T^{|\sigma|-|j|} X^{-|\alpha|} \to 0 \quad \text{as} \quad T \downarrow 0 \tag{4.4}$$

for any fixed $X$.

(D) The case $s = s_- \in \mathbf{R}$ and $|\sigma| < |j|$.

$$\| a_{\sigma j\alpha}(x)t^\sigma D_t^j D_x^\alpha \|_{T,X;k}^{(s)} \leq A_{\sigma j\alpha s} T^{|\sigma|-|j|} X^{-|\alpha|} \to 0 \quad \text{as} \quad T \uparrow +\infty \tag{4.5}$$

for any fixed $X$.

Next, in the case $\delta > 0$ we have

$$\| a_{\sigma j\alpha}(x)t^\sigma D_t^j D_x^\alpha \|_{T,X;k}^{(s)} \leq A_{\sigma j\alpha s} T^{|\sigma|-|j|} X^{-|\alpha|} k^{-\delta} \to 0 \quad \text{as} \quad k \to +\infty \tag{4.6}$$

for any fixed $X$ and $T$.

Now by using the assumption (1.6), $\sum_{|j|=|\alpha|=0} | a_{oj\alpha}(0) | < 1$, we can prove (i), (ii) and (iii) in Theorem A, except the cases $s_+ = -\infty$ in (ii) and $s_- = +\infty$ in (iii).

Consider the case $s_+ = -\infty$ in (ii). Since the Newton polygon $\mathrm{N}(L)$ lies in the lower half plane, we can take $\tilde{s}$ such that $\delta < 0$ in (4.1) for any $s < \tilde{s}$ except the cases $(|\sigma| - |j|, |j| + |\alpha|) = (0,0)$. Then $A$ becomes a contraction map in $G^s(T, X; k)$ for any fixed $T > 0$ by taking small $X$ depending on $s$ and next taking large $k$ depending on $s$, $X$ and $T$. This implies the bijectivity of the mapping $(L)_{-\infty}$.

In the case $s_- = +\infty$ in (iii), the proof of the bijectivity of the mapping $(L)_{(+\infty)}$ is essentially the same as the above, so we omit it.

(iv) The proof is somewhat different from the above. Since $|\sigma| \geq |j|$ in this case, we set

$$A = \Big( \sum_{|\sigma|=|j|=|\alpha|=0} + \sum_{\substack{|\sigma|=|j|\geq 0\\ |\alpha|<0}} + \sum_{|\sigma|>|j|} \Big) a_{\sigma j \alpha}(x) t^{\sigma} D_t^j D_x^{\alpha}$$

$$\underset{\text{put}}{=} A_1 + A_2 + A_3.$$

Let $U(t, x) = \sum U_l(x) t^l / l!$ and $F(t, x) = \sum F_l(x) t^l / l!$. Then the equation, $LU(t, x) = F(t, x)$, implies the following relations.

$$U_l(x) - \sum_{|j|=|\alpha|=0} a_{oj\alpha}(x)\, D_x^{\alpha}\, U_{l+j}(x) - \sum_{\substack{|\sigma|=|j|\geq 0\\ |\alpha|<0}} \frac{l!}{(l-\sigma)!} a_{\sigma j \alpha}(x) D_x^{\alpha}\, U_{l+j-\alpha}(x)$$

$$= F_l(x) + R^{(l)}(x, D_x^{\alpha}\, U_{\mu}(x)\ ; |\mu| < |l|, \text{ finite number of } \alpha).$$

Let $\mathcal{U}^{(N)}(x) = {}^t(U_l(x); |l| = N)$ be a column vector with length $d(N) := (p+N-1)!/(p-1)!N!(= \#\ \{l \in \mathbf{N}^p\ ; |l| = N\})$. Then the above relations imply a sequence of systems of integro-differential equations of the form,

$$\{I - \mathcal{A}^{(N)}(x, D_x^{\alpha}\ ; |\alpha| \leq 0)\}\mathcal{U}^{(N)}(x) \tag{4.7}$$

$$= \mathcal{F}^{(N)}(x) + \mathcal{R}^{(N)}(x, D_x^{\alpha}\, U_l(x); |l| < N)\ \ (N \in \mathbf{N}).$$

Let $G(X; k)$ $(X > 0,\ k \in \mathbf{N})$ be a Banach space of holomorphic functions with norm

$$\| U \|_{X;k} := \sup_{\beta} |U_{\beta}| \frac{X^{|\beta|}}{(|\beta| + k)!} < +\infty \text{ for } U(x) = \sum_{\beta} U_{\beta} \frac{x^{\beta}}{\beta!}. \tag{4.8}$$

Then $D_x^{\alpha}$ $(|\alpha| \leq 0)$ defines a bounded operator in $G(X; k)$ with norm estimated by

$$\| D_x^{\alpha} \| \leq \Big(\frac{X}{k}\Big)^{-|\alpha|}.$$

Let $\mathcal{G}^{(N)}(X; k) = \Pi^{d(N)}\, G(X; k)$ be a Banach space with norm

$$\| \mathcal{U}^{(N)} \|_{X;k} := \max_{|l|=N} \| U_l(x) \|_{X;k},\ \mathcal{U}(x) = {}^t(U_l(x)) \in \mathcal{G}^{(N)}(X; k). \tag{4.9}$$

The following lemma is a special case of Lemma 4.4 in Miyake [7].

LEMMA 4.1. *Let the condition* (1.6) *be satisfied. Then there is a positive constant* $X_0$ *such that* $\mathcal{A}^{(N)}(x, D_x^{\alpha}; |\alpha| \leq 0)$ *becomes a contraction map in* $\mathcal{G}^{(N)}(X; k)$ *for any* $X \leq X_0$ *and any* $N \in \mathbf{N}$ *by taking large* $k$ *depending only on* $X_0$ *and* $N$.

The bijectivity of the mapping $(L)_{+\infty}$ is now easily proved.

This completes the proof of Theorem A. □

*Proof of Theorem* B. (i) Let $s_0 > 0$. By Proposition 1.4,

$$s_0|j| + (1-s_0)|\sigma| + |\alpha| = s_0(|j| - |\sigma|) + |\sigma| + |\alpha| = -\delta \leq 0 \tag{4.10}$$

for any $(\sigma, j, \alpha)$ with $a_{\sigma j\alpha}(x) \not\equiv 0$. We set

$$A = \Big(\sum_{\substack{s_0|j|+|\alpha|=0 \\ \sigma=0}} + \sum_{\delta=0,|\sigma|>0} + \sum_{\delta>0}\Big) a_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha \tag{4.11}$$

$$\underset{\text{put}}{=} A_1 + A_2 + A_3.$$

We shall estimate the operator norm of the mapping,

$$A: G^{s_0}(X^{s_0}, X) \to G^{s_0}(X^{s_0}, X). \tag{4.12}$$

Since $X^{s_0(|\sigma|-|j|)}X^{-|\alpha|} = X^{|\sigma|+\delta}$, we have:

$$\|A_1\| \leq \sum_{s_0|j|+|\alpha|=0} |a_{oj\alpha}(0)| + O(X), \tag{4.13}$$

$$\|A_i\| \to 0 \quad \text{as} \quad X \downarrow 0 \ (i = 2,3). \tag{4.14}$$

Hence the assumption (1.8) implies the existence of a positive constant $X_0$ such that $A$ becomes a contraction map in $G^{s_0}(X^{s_0}, X)$ for any $0 < X \leq X_0$. This proves the bijectivity of $(L)_{s_0}$. Indeed, it is sufficient to notice that $G^{s_0}(T, X) \subset G^{s_0}(X^{s_0}, X)$ for any $T$ with $0 < T < X^{s_0}$ and conversely $G^{s_0}(T, X) \supset G^{s_0}(Y^{s_0}, Y)$ if we choose sufficiently small $Y$.

If we employ the space $G^{s_0}(X^{s_0}, X; k)$ instead of $G^{s_0}(X^{s_0}, X)$, we can see that the existence domain of solutions in $(T, X)$-plane depends only on operators $a_{\sigma j\alpha}(x)t^\sigma D_t^j D_x^\alpha$ such that $\delta = 0$ by letting $k \to +\infty$ as in the proof of Theorem A.

(ii) Let $s_0 \leq 0$. From $(3.9)_s$ $(s \leq 0)$ and the assumption (1.9), the operator norm of the mapping

$$A: G_n^{s_0}(X^{s_0}, X; n-1) \to G_n^{s_0}(X^{s_0}, X; n-1) \tag{4.15}$$

is estimated by

$$\|A\| \leq \kappa + \varepsilon(n) + \varepsilon(n, X), \tag{4.16}$$

where $\kappa < 1$, $\varepsilon(n) \to 0$ as $n \to \infty$, and $\varepsilon(n, X) \to 0$ as $X \downarrow 0$ for any fixed $n$. Hence the mapping (4.15) becomes a contraction map by a suitable choice of $n$ and

$X$. This implies the injectivity of the mapping $(L)_{(s_0)}$. The above proof shows that an equation,

$$L(t, x; D_t, D_x)\, U(t, x) = F(t, x) \in G^{(s_0)},$$

has a unique solution $U(t, x) \in G_n^{s_0}(X^{s_0}, X; n-1)$ for sufficiently small $X$ and large $n$. This proves the surjectivity of the mapping $L: G^{s_0} \to G^{(s_0)}$, but not the surjectivity of $(L)_{(s_0)}$, because $s_0 \leq 0$. □

## Chapter 2. Operators with Euler type principal part

### 5. Statement of results

The reasonings in the preceding sections go well for more general operators of the following form which we called of Cauchy-Goursat-Fuchs type in Miyake [7]:

$$L_m = P_m(\delta_t) + \sum_{\sigma,j,\alpha}^{\text{finite}} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha, \tag{5.1}$$

where $m \geq 1$, $\delta_t = (t_1 D_{t_1}, \cdots, t_p D_{t_p})$ and

$$P_m(\delta_t) = \sum_{0 \leq |j| \leq m} a_j \delta_t^j \quad (a_j \in \mathbf{C},\ j \in \mathbf{N}^p), \tag{5.2}$$

where $\delta_t^j = (t_1 D_{t_1})^{j_1} \cdots (t_p D_{t_p})^{j_p}$ for $j \in \mathbf{N}^p$.

Let $\mathrm{N}(L_m)$ be the Newton polygon of $L_m$. We assume the following condition.

(B.1) A point $(m, 0)$ is a vertex of $\mathrm{N}(L_m)$.

Let $k_+$ (resp. $k_-$) be the slope of a side of $\mathrm{N}(L_m)$ with an end point $(m, 0)$ which is in the upper (resp. lower) half plane. Let $s_\pm$ be the numbers defined by (1.5).

We shall study the bijectivity of the mappings,

$(L_m)_s$ $\qquad L_m: G^s \to G^s$,

$(L_m)_{(s)}$ $\qquad L_m: G^{(s)} \to G^{(s)}$,

for $s_+ \leq s \leq s_-$.

We assume the following additional condition.

(B.2) If $|\sigma| + m(|\sigma| - |j| - 1) < 0$, then $|\alpha| + (m+s)(|j| - |\sigma|) \leq 0$ for $(\sigma, j, \alpha)$ with $b_{\sigma j\alpha}(x) \not\equiv 0$.

Now we can prove the following theorem corresponding to Theorem A.

THEOREM C. *Let* (B.1) *and* (B.2) *be satisfied, and further assume that there is a positive constant* $\varepsilon_0$ *such that*

(5.3) $$|P_m(l)| > \varepsilon_0(|l|+1)^m \text{ (Poincaré condition)},$$

*holds for any* $l \in \mathbf{N}^p$, *and*

(5.4) $$\sum_{\substack{|\sigma|=|j|\leq m \\ |\alpha|=0}} |b_{\sigma j\alpha}(0)| \tau^{j-\sigma}\xi^\alpha < \varepsilon_0 \text{ (Spectral condition)},$$

*holds for some* $\tau \in \mathbf{R}_+^p$ *and* $\xi \in \mathbf{R}_+^q$. *Then we have*:

(i) *Let* $s_+ < s < s_-$. *Then* $(L_m)_{(s)}$ *is bijective. Furthermore if* $m+s>0$, *then* $(L_m)_s$ *is bijective.*

(ii) $(L_m)_{(s_-)}$ *is bijective.*

(iii) *Let* $s=s_+$ *and* $m+s_+>0$. *If one of the following conditions is satisfied, then* $(L_m)_{s_+}$ *is bijective*:

(5.5) $b_{oja}(0) = 0$ *for* $|j| = -1$ *and* $s_+|j|+|\alpha| = m$.

(5.6) $b_{\sigma ja}(0) = 0$ *for* $(\sigma, j, \alpha)$ *such that* $|\sigma| < |j|$,
$|\sigma| + m(|\sigma| - |j| - 1) < 0$ *and* $|\alpha| + (m+s_+)(|j| - |\sigma|) = 0$.

We remark that we may assume $\tau = (1,\cdots,1)$ and $\xi = (1,\cdots,1)$ in (5.4) without loss of generality (see Remark 1.2).

*Remark* 5.1. (i) From the proof, we see that if $b_{\sigma j\alpha}(x) \equiv 0$ when $|\sigma| + m(|\sigma| - |j| - 1) < 0$, then the same results as in Theorem A hold.

(ii) In the above theorem, some conditions can be weakend. For example, in (i) if $b_{\sigma ja}(0) = 0$ as in (5.6), then the condition $m+s>0$ can be replaced by another one (see also an example in §7). But we do not discuss such a problem in this paper.

Combining the arguments in Miyake [7] with the proof of Theorem C, we can prove the following,

THEOREM D. *Let* $k_- = 0$ *and assume that there is a positive constant* $\varepsilon_0$ *such that*

(5.7) $$\inf_{\tau\in\bar{\mathbf{R}}_+^p, |\tau|=1} \Big|\sum_{|j|=m} a_j\tau^j\Big| > \varepsilon_0 \quad (\bar{\mathbf{R}}_+ := [0, \infty)),$$

(5.8)
$$\sum_{\substack{|\sigma|=|j|=m\\|\alpha|=0}} |b_{\sigma j\alpha}(0)|\,\tau^{j-\sigma}\xi^{\alpha} < \varepsilon_0,$$

*for some* $\tau \in \mathbf{R}^p_+$ *and* $\xi \in \mathbf{R}^q_+$. *Then the mapping,*

$$L_m : G^{+\infty}/G^s \to G^{+\infty}/G^s,$$

*is bijective for every* $s$ *with* $s_+ \leq s < +\infty$.

We omit the proof of this theorem, since it is the same as [7, Theorem 1.1] which studied the case $k_+ > 0$.

## 6. Proof of Theorem C

The proof is essentially the same as that of Theorem A, so we omit the detail except different points.

We introduce a Banach space $G^s(T, X; k; m)$ instead of $G^s(T, X; k)$ $(s > 0)$ or $G^s_1(T, X; k)$ $(s \leq 0)$ as follows.

Let $U(t, x) = \sum U_{l\beta}t^l x^\beta / l!\beta! \in G^{+\infty}$. Then $U(t, x) \in G^s(T, X; k; m)$ if $\| U\|^{(s)}_{T,X;k;m} < +\infty$ which is defined below.

(6.1)
$$\| U\|^{(s)}_{T,X;k;m} := \sup_{l,\beta} |U_{l,\beta}| \frac{T^{|l|}X^{|\beta|}\,|l|!^m}{\{(s+m)|l| + |\beta| + k\}!} \quad (s > 0),$$

(6.2)
$$\| U\|^{(s)}_{T,X;k;m} := \sup_{l,\beta} |U_{l,\beta}|\, T^l X^\beta \frac{\{(1-s)|l| + |\beta| + k\}!\,|l|!^m}{\{(1+m)|l| + 2|\beta| + k\}!} \quad (s \leq 0).$$

Then it can be proved that

(6.3)
$$G^s = \bigcup_{T,X>0} G^s(T, X; k; m), \quad G^{(s)} = \bigcup_{X>0}\bigcap_{T>0} G^s(T, X; k; m),$$

for any fixed $k$ and $m$ (see Lemma 2.3 and Miyake [7, §2]).

Corresponding to Lemma 3.1 we can prove the following,

LEMMA 6.1. *Let* $U(t, x) \in G^s(T, X; k; m)$ *and* $a(x) \in \mathcal{O}(\|x\| \leq \rho X)$ $(\rho > 0)$. *Then we have*

(i) *If* $s > 0$, *then* $a(x)U(t, x) \in G^s(T, X; k; m)$ *for any* $\rho > 1$ *and it holds that*

(6.4)
$$\| aU\| \leq C(q) \frac{[q/2]!}{(1-1/\rho)^{[q/2]+1}} \| a\|_{\rho X} \| U\|.$$

*Here* $C(q)$ *is the same constant appeared in Lemma* 3.1.

(ii) *If* $s \le 0$, *then* $a(x)U(t, x) \in G^s(T, X; k; m)$ *for any* $\rho > (1-s)/2$ *and it holds that*

$$\| aU \| \le C(q) \frac{[q/2]!}{(1-(1-s)/2\rho)^{[q/2]+1}} \| a \|_{\rho X} \| U \|. \tag{6.5}$$

*Hence for any holomorphic function* $a(x)$ *in a neighbourhood of the origin, there is a positive constant* $X_0$ *such that* $a(x)$ *defines a bounded operator on* $G^s(T, X; k; m)$ *for any* $0 < X \le X_0$ *and we have*

$$\| aU \|^{(s)}_{T,X;k;m} \le \{|a(0)| + O(X)\} \| U \|^{(s)}_{T,X;k;m}, \tag{6.6}$$

*where* $O(X)/X$ *is bounded as* $X \downarrow 0$.

The proof is similar to that of Lemma 3.1, and it is sufficient to notice the following inequality. Let $s \le 0$. Then for any $i$ with $0 \le i \le |\beta|$ we have

$$\frac{\{(1-s)|l| + |\beta| + k\}!}{\{(1+m)|l| + 2|\beta| + k\}!} \frac{|\beta|!}{(|\beta| - i)!} \frac{\{(1+m)|l| + 2(|\beta| - i) + k\}!}{\{(1-s)|l| + |\beta| - i + k\}!}$$

$$= \prod_{j=1}^{i} \frac{|\beta| + 1 - j}{(1+m)|l| + 2(|\beta| + 1 - j) + k}$$

$$\times \prod_{j=1}^{i} \frac{(1-s)|l| + |\beta| - j + 1 + k}{(1+m)|l| + 2(|\beta| - j) + 1 + k}$$

$$\le \left(\frac{1-s}{2}\right)^i.$$

We remark that $P_m(\delta_t)$ defines an invertible operator in both spaces $G^s$ and $G^{(s)}$ under the assumption (5.3) and its inverse operator $P_m^{-1}(\delta_t)$ is given by

$$P_m^{-1}(\delta_t) U(t, x) = \sum_l U_l(x) \frac{t^l}{P_m(l) l!}. \tag{6.7}$$

Therefore our problem is reduced to prove the bijectivity of the mapping,

$$(\tilde{L}_m)_s \qquad L_m P_m^{-1}(\delta_t) : G^s \to G^s \quad \text{or}$$

$$(\tilde{L}_m)_{(s)} \qquad L_m P_m^{-1}(\delta_t) : G^{(s)} \to G^{(s)}.$$

Since $L_m P_m^{-1}(\delta_t) = I - \sum b_{\sigma j \alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t) \underset{\text{put}}{=} I - B$, it is sufficient to prove that $B$ becomes a contraction map in a suitable space $G^s(T, X; k; m)$ under the assumptions of the theorem as in the proof of Theorem A. For that purpose, we have to estimate the operator norm of the mapping, (6.8) $$t^{\sigma} D_t^j D_x^{\alpha} P_m^{-1}(\delta_t) : G^s(T, X; k; m) \to G^s(T, X; k; m).$$

We note that the assumption (B.1) and the condition that $s_+ \le s \le s_-$ imply that

(6.9) $$s|j| + (1-s)|\sigma| + |\alpha| - m \underset{\text{put}}{=} -\delta \le 0,$$

for any $(\sigma, j, \alpha)$ with $b_{\sigma j\alpha}(x) \not\equiv 0$. We note also,

(6.10) $$\begin{cases} \text{If } s_+ < s < s_-, \text{ then } \delta = 0 \text{ only if } |\sigma| = |j|. \\ \text{If } s = s_+, \text{ then } \delta = 0 \text{ only if } |\sigma| \ge |j|. \\ \text{If } s = s_-, \text{ then } \delta = 0 \text{ only if } |\sigma| \le |j|. \end{cases}$$

Now we can prove the following,

LEMMA 6.2. *Let $P_m(\delta_t)$ be as above and $(\sigma, j, \alpha) \in \mathbf{N}^p \times \mathbf{Z}^p \times \mathbf{Z}^q$ satisfy* (6.9) *and* (B.2). *Then the operator norm of $t^{\sigma} D_t^j D_x^{\alpha} P_m^{-1}(\delta_t)$ of the mapping* (6.8) *is estimated as follows:*

(i) *If* $|\sigma| + m(|\sigma| - |j| - 1) \ge 0$, *then*

(6.11) $$\| t^{\sigma} D_t^j D_x^{\alpha} P_m^{-1}(\delta_t) \| \le C(m, \varepsilon_0, s, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|} k^{-\delta}.$$

(ii) *If* $|\sigma| + m(|\sigma| - |j| - 1) < 0$, *then*

(6.12) $$\| t^{\sigma} D_t^j D_x^{\alpha} P_m^{-1}(\delta_t) \| \le C(m, \varepsilon_0, s, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|}.$$

(iii) *If* $|\sigma| = |j| \le m$ *and* $|\alpha| = 0$, *then*

(6.13) $$\| t^{\sigma} D_t^j D_x^{\alpha} P_m^{-1}(\delta_t) \| \le \varepsilon_0^{-1}.$$

*Proof.* Put $t^{\sigma} D_t^j D_x^{\alpha} P_m^{-1}(\delta_t) U(t, x) = \sum V_{l\beta} t^l x^{\beta}/l!\beta!$. Then we have

(6.14) $$V_{l\beta} = \frac{l!}{(l-\sigma)!} \frac{1}{P_m(l+j-\sigma)} U_{l+j-\sigma, \beta+\alpha}.$$

From this expression, the inequality (6.13) is obvious.

(A) The case $s > 0$. In this case, we have

$$\frac{|V_{l\beta}| T^{|l|} X^{|\beta|}}{\{(s+m)|l| + |\beta| + k\}!} |l|!^m \le C(m, \varepsilon_0, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|} \| U \|$$

$$\times |l|^{|\sigma| + m(|\sigma|-|j|-1)} \frac{\{(s+m)|l| + |\beta| + k + (s+m)(|j| - |\sigma|) + |\alpha|\}!}{\{(s+m)|l| + |\beta| + k\}!}.$$

In the case $|\sigma| + m(|\sigma| - |j| - 1) \ge 0$, we easily obtain (6.11) since

$$(s+m)(|j| - |\sigma|) + |\alpha| = m(|j| - |\sigma| + 1) - |\sigma| - \delta.$$

In the case $|\sigma| + m(|\sigma| - |j| - 1) < 0$, the condition (B.2) implies (6.12) immediately.

(B) The case $s \leq 0$. In this case, we have

$$|V_{l\beta}| T^{|l|} X^{|\beta|} |l|!^m \leq C(m, \varepsilon_0, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|} \|U\|$$

$$\times |l|^{|\sigma|+m(|\sigma|-|j|-1)} \frac{\{(1+m)|l| + 2|\beta| + k + (1+m)(|j| - |\sigma|) + 2|\alpha|\}!}{\{(1-s)|l| + |\beta| + k + (1-s)(|j| - |\sigma|) + |\alpha|\}!}.$$

If $|\sigma| + m(|\sigma| - |j| - 1) \geq 0$, then

$$|V_{l\beta}| T^{|l|} X^{|\beta|} |l|!^m \leq C(m, \varepsilon_0, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|} \|U\|$$

$$\times \frac{\{(1+m)|l| + 2|\beta| + k + |j| - m + 2|\alpha|\}!}{\{(1-s)|l| + |\beta| + k + (1-s)(|j| - |\sigma|) + |\alpha|\}!}.$$

Now by the relation (6.9), we have

$$|V_{l\beta}| T^{|l|} X^{|\beta|} |l|!^m \leq C(m, \varepsilon_0, s, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|} \|U\|$$

$$\times k^{-\delta} \frac{\{(1+m)|l| + 2|\beta| + k + |j| - m + 2|\alpha|\}!}{\{(1-s)|l| + |\beta| + k + |j| - m + 2|\alpha|\}!},$$

which implies (6.11) immediately.
If $|\sigma| + m(|\sigma| - |j| - 1) < 0$, then by (B.2) we have

$$|V_{l\beta}| T^{|l|} X^{|\beta|} |l|!^m \leq C(m, \varepsilon_0, \sigma, j, \alpha) T^{|\sigma|-|j|} X^{-|\alpha|} \|U\|$$

$$\times \frac{\{(1+m)|l| + 2|\beta| + k + (1-s)(|j| - |\sigma|) + |\alpha|\}!}{\{(1-s)|l| + |\beta| + k + (1-s)(|j| - |\sigma|) + |\alpha|\}!},$$

and this implies (6.12). □

*Proof of Theorem* C. First of all, we have to make clear the meaning of assumption (B.2). It is a condition for $(\sigma, j, \alpha)$ with $|\sigma| \leq |j|$. Let $|\sigma| = |j|$. Then $|\sigma| < m$ implies $|\alpha| \leq 0$, and also $|\sigma| \geq m$ implies $|\sigma| + |\alpha| \leq m$ by (B.1). Hence if $|\sigma| = |j| > m$, we have $|\alpha| \leq m - |\sigma| < 0$.

(i) Let us consider the case $s_+ < s < s_-$. In this case, $\delta = 0$ only if $|\sigma| = |j|$. We rewrite the operator $\tilde{L}_m := L_m P_m^{-1}$ as follows.

$$\tilde{L}_m = I - \sum_{i=1}^{4} B_i(t, x; D_t, D_x). \tag{6.15}$$

Here,

$$B_1 = \sum_{\substack{|\sigma|=|j|\leq m \\ |\alpha|=0}} b_{\sigma j \alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t).$$

$$B_2 = \sum_{\substack{|\sigma|=|j| \\ |\alpha|<0}} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t).$$

$$B_3 = \sum_{\sigma,j,\alpha}^{(1)} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t),$$

where the summation is taken over $(\sigma, j, \alpha)$ such that (B.2) is satisfied with $|j| > |\sigma|$.

$$B_4 = \sum_{\sigma,j,\alpha}^{(2)} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t),$$

where the summation is taken over $(\sigma, j, \alpha)$ which is excluded in $B_i$ $(i = 1,2,3)$, and hence in this summation $\delta > 0$, because $s_+ < s < s_-$.

Let us estimate the operator norm of each $B_i$ acting on $G^s(T, X; k; m)$.

(6.16) $$\| B_1 \| \le \varepsilon_0^{-1} \{ \sum_{\substack{|\sigma|=|j|\le m \\ |\alpha|=0}} | b_{\sigma j\alpha}(0) | + O(X) \}.$$

(6.17) $$\| B_2 \| \le CX^a \quad \text{for some } a > 0.$$

(6.18) $$\| B_3 \| \le C(X) T^{-b} \quad \text{for some } b > 0.$$

(6.19) $$\| B_4 \| \le C(T, X) k^{-c} \quad \text{for some } c > 0.$$

By the condition (5.4), we can take small positive constant $X$ so that $\| B_1 \| + \| B_2 \| < 1$. Next we take large $T$ so that $\| B_1 \| + \| B_2 \| + \| B_3 \| < 1$, and finally we take large $k$ so that $\sum_{i=1}^4 \| B_i \| < 1$. This proves the bijectivity of $(L_m)_{(s)}$.

In order to prove the bijectivity of $(L_m)_s$, we need more careful estimate for $B_3$. Let $(\sigma, j, \alpha)$ satisfy the condition (B.2) with $|j| > |\sigma|$ and let $0 < d < m + s$. Then the operator norm of the mapping,

$$t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t) : G^s(X^d, X; k; m) \to G^s(X^d, X; k; m)$$

is estimated by

$$\| t^\sigma D_t^j D_x^\alpha P_m^{-1} \| \le CX^{d(|\sigma|-|j|)-|\alpha|}.$$

Here $d(|\sigma| - |j|) - |\alpha| > 0$ by the above choice of $d$. Hence the operator norm of

$$B_3 : G^s(X^d, X; k; m) \to G^s(X^d, X; k; m)$$

tends to 0 as $X \downarrow 0$. This implies the bijectivity of $(L_m)_s$.

(ii) The bijectivity of $(L_m)_{(s_-)}$ is obvious from the above proof, since in this case $\delta > 0$ for any $(\sigma, j, \alpha)$ with $|\sigma| > |j|$.

(iii) Let us consider the case $s = s_+$. We rewrite the operator $\tilde{L}_m$ as follows.

$$\tilde{L}_m = I - \sum_{i=1}^{5} C_i(t, x; D_t, D_x). \tag{6.20}$$

Here,

$$C_1 = \sum_{\substack{|\sigma|=|j|\leq m \\ |\alpha|=0}} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t) (= B_1).$$

$$C_2 = \sum_{\substack{|\sigma|=|j| \\ |\alpha|<0}} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t) (= B_2).$$

$$C_3 = \sum_{\sigma,j,\alpha}^{(1)} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t),$$

where the summation is taken over $(\sigma, j, \alpha)$ such that $|\sigma| + m(|\sigma| - |j| - 1) \geq 0$ and $\delta = 0$.

$$C_4 = \sum_{\sigma,j,\alpha}^{(2)} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t) (= B_3).$$

$$C_5 = \sum_{\sigma,j,\alpha}^{(3)} b_{\sigma j\alpha}(x) t^\sigma D_t^j D_x^\alpha P_m^{-1}(\delta_t),$$

where the summation is taken over $(\sigma, j, \alpha)$ which is excluded in $C_i$ $(i = 1,2,3,4)$, and hence $\delta > 0$ in this summation.

Let $\| C_i \|$ denote the operator norm of $C_i$ acting on $G^s(T, X; k; m)$. As in (i) we can choose small positive constant $X_0$ such that $\| C_1 \| + \| C_2 \| < 1$ for any $0 < X \leq X_0$ and any $T > 0$. Since $\| C_5 \| \leq C(T, X) k^{-c}$ for some $c > 0$, this term does not play any role in the proof of the bijectivity by letting $k \longrightarrow \infty$.

First, consider the case where the condition (5.5) is satisfied, that is,

$$b_{oj\alpha}(0) = 0 \text{ for } |j| = -1 \text{ and } s_+|j| + |\alpha| = m. \tag{5.5}$$

This implies $\| b_{oj\alpha} \|_{\rho X} = O(X)$ as $X \downarrow 0$. Therefore the operator norm of $C_3 : G^{s_+}(X^d, X; k; m) \rightarrow G^{s_+}(X^d, X; k; m)$ is estimated by

$$\begin{aligned} \| C_3 \| &\leq C(q) \textstyle\sum^{(1)} C_{\sigma j\alpha} \| b_{\sigma j\alpha} \|_{\rho X} X^{d(|\sigma|-|j|)-|\alpha|} \\ &\leq C(q) \textstyle\sum^{(1,1)} C'_{oj\alpha} X^{d-s_+-m+1} \\ &\quad + C(q) \textstyle\sum^{(1,2)} C'_{\sigma j\alpha} X^{(d-s_+)(|\sigma|-|j|)-m}, \end{aligned}$$

where the summation $\sum^{(1,1)}$ is taken over $(o, j, \alpha)$ which satisfies (5.5), and $\sum^{(1,2)}$ is taken over $(\sigma, j, \alpha)$ such that $|\sigma| > |j|$ except $|\sigma| = 0$ and $|j| = -1$. Here we used the relation, $-|\alpha| = s_+(|j| - |\sigma|) + |\sigma| - m$. In the summation $\sum^{(1,2)}$, if $|\sigma| - |j| = 1$, then $|\sigma| \geq 1$. Therefore,

$$\mu := \max\left\{\frac{m-|\sigma|}{|\sigma|-|j|}\,;\ (\sigma, j, \alpha) \text{ in the summation } \Sigma^{(1,2)}\right\} < m.$$

Hence for any $d$ with $s_+ + \max\{m-1, \mu\} < d < s_+ + m$, the operator norm of $C_3$ tends to zero as $X \downarrow 0$. As in (i), the operator norm of $C_4 : G^{s_+}(X^d, X; k; m) \rightarrow G^{s_+}(X^d, X; k; m)$ tends to zero as $X \downarrow 0$, since $d < s_+ + m$. This proves the bijectivity of $(L_m)_{s_+}$.

Next, we consider the case where the condition (5.6) is satisfied. Note that the operator norm of

$$C_3 : G^{s_+}(X^d, X; k; m) \rightarrow G^{s_+}(X^d, X; k; m)$$

tends to zero as $X \downarrow 0$ if $d > s_+ + m$. The condition (5.6) assures that we can choose $d > s_+ + m$ so that the operator norm of $C_4 : G^{s_+}(X^d, X; k; m) \rightarrow G^{s_+}(X^d, X; k; m)$ tends to zero as $X \downarrow 0$, as the above.

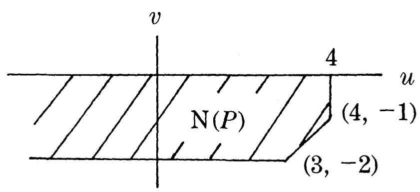
This completes the proof. □

## 7. Example

We consider the following partial differential operator,

$$P = \{tD_t + 1\}D_t^2 + a\{tD_t + 1\}D_t D_x^2 + bD_x^4 \quad (a, b \in \mathbf{C}\backslash\{0\}). \tag{7.1}$$

The Newton polygon $\mathrm{N}(P)$ is given as follows.




(i) By Theorem C, the Cauchy problem,

$$\begin{cases} Pu(t, x) = f(t, x) \in G^s, \\ u(t, x) - w(t, x) = O(t^2) \ (w \in G^s), \end{cases} \tag{7.2}$$

is uniquely solvable in $G^s$ for any $s \geq 2$.

(ii) Theorem A implies that the Cauchy problem,

$$\begin{cases} Pu(t, x) = f(t, x) \in G^s, \\ u(t, x) - w(t, x) = O(x^4) \ (w \in G^s), \end{cases} \tag{7.3}$$

is uniquely solvable in $G^s$ for any $s < 1$.

(iii) Looking at the vertex $(4, -1)$ of $N(P)$, we consider the Goursat problem,

$$\begin{cases} Pu(t, x) = f(t, x) \in G^s, \\ u(t, x) - w(t, x) = O(tx^2) \ (w \in G^s). \end{cases} \tag{7.4}$$

Let

$$L_1 = (\delta_t + 1) + a^{-1}\{tD_t^2 D_x^{-2} + D_t D_x^{-2}\} + a^{-1}bD_t^{-1} D_x^2$$
$$\underset{\text{put}}{=} (\delta_t + 1) - A.$$

Then the unique solvability of the Goursat problem (7.4) is equivalent to the bijectivity of the mapping

$$L_1 : G^s \to G^s. \tag{7.5}$$

Since $s_+ = 1$ and $s_- = 2$, let $1 \leq s \leq 2$. The condition (B.2) is satisfied only if $s = 1$. In this case, the conditions (5.5) and (5.6) are not satisfied, since $ab \neq 0$. So we have to take care to estimate the operator norms for

$$tD_t^2 D_x^{-2}(\delta_t + 1)^{-1},\ D_t D_x^{-2}(\delta_t + 1)^{-1} \text{ and } D_t^{-1} D_x^2(\delta_t + 1)^{-1}$$

acting on the space $G^1(T, X)$, and we have:

$$\| tD_t^2 D_x^{-2}(\delta_t + 1)^{-1} \| \leq T^{-1} X^2,$$
$$\| D_t D_x^{-2}(\delta_t + 1)^{-1} \| \leq T^{-1} X^2/2,$$
$$\| D_t^{-1} D_x^2(\delta_t + 1)^{-1} \| \leq TX^{-2}.$$

Now the operator norm of $A(\delta_t + 1)^{-1} : G^1(X^2, X) \to G^1(X^2, X)$ is estimated by $\| A(\delta_t + 1)^{-1} \| \leq |a|^{-1}(3/2 + |b|)$. Therefore the problem (7.4) is uniquely solvable in $G^1$ if $|a| > |b| + (3/2)$.

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