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ON SUBCLASSES OF INFINITELY DIVISIBLE DISTRIBUTIONS ON R RELATED TO HITTING TIME DISTRIBUTIONS OF 1-DIMENSIONAL GENERALIZED DIFFUSION PROCESSES

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1. Introduction

A distribution μ on $\mathbf{R}_+ = [0, \infty)$ is said to be a CME_+^f distribution if there are an increasing (in the strict sense) sequence of positive real numbers $\{a_k\}_{k=1}^{\ell}$ and $0 = b_0 < b_1 < \cdots < b_m < b_{m+1} = \infty$ $(0 \leqq m < \ell < \infty)$ such that, for each $j = 0, \ldots, m$, there is at least one a_k satisfying $b_j < a_k < b_{j+1}$ and the Laplace transform $\mathscr{L}\mu(s) = \int_{\mathbf{R}_+} e^{-sx}\mu(dx)$ of μ is represented as

$$\mathscr{L}\mu(s) = \prod_{i=1}^{\ell} a_i(s + a_i)^{-1} \qquad \text{if } m = 0,$$

$$= \prod_{i=1}^{\ell} a_i(s + a_i)^{-1} / \prod_{j=1}^{\ell} b_j(s + b_j)^{-1} \quad \text{if } m \geqq 1.$$

The author [8] shows that the upward first passage time distributions of birth and death processes belong to the class CME_+^f. He [9] also shows that the class of distributions of hitting times of single points of generalized diffusion processes is a proper subclass of the closure CME_+, in the weak convergence sense, of CME_+^f. Let CME_-^f be the class of distributions on $\mathbf{R}_- = (-\infty, 0]$ whose mirror images belong to CME_+^f. That is, $\mu \in CME_-^f$ if and only if $\bar{\mu}(du) = \mu(-du)$ belongs to CME_+^f. Let CME^f be the class of $\mu = \mu_1 * \mu_2$ with $\mu_1 \in CME_+^f$ and $\mu_2 \in CME_-^f$. Sato [4] shows that the distributions of sojourn times of birth and death processes with weight not necessarily positive belong to CME^f.

We denote the class of infinitely divisible distributions on $\mathbf{R}$ (or $\mathbf{R}_\pm$) by $\mathscr{I}(\mathbf{R})$ (or $\mathscr{I}(\mathbf{R}_\pm)$). The classes CME_+^f and CME_+ are contained in $\mathscr{I}(\mathbf{R}_+)$. The class CME^f is contained in $\mathscr{I}(\mathbf{R})$. Some interesting classes of infinitely divisible distributions on $\mathbf{R}_+$ (for example, BO, CE_+, ME_+, $CME_+, \ldots$) are introduced in [1] and [8] and representations of their Laplace transforms, compactness

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conditions and convergence conditions are investigated. Sato's result [4] suggests that it is natural to extend those classes to classes on $\mathbf{R}$. We denote by B_+ the class BO in this paper.

The main purpose of this paper is to define classes B, CE, ME, CME on $\mathbf{R}$, obtain representations of their characteristic functions or Laplace transforms, and express convergence conditions by their characteristics. This will be done in Sections 2 ~ 5. Thorin [6] extended the notion of generalized Γ-convolutions on the half real line, which is a natural subclass of B_+ and class L containing the class of stable distributions and the class CE_+, to those on the whole real line and gets a convergence condition (parallel to our Theorem 2.1). In Section 6, we define and study a subclass ME_+^d of ME_+ and a subclass CME_+^d of CME_+. It is shown in [9] that hitting time distributions of one dimensional generalized diffusion processes with non-natural boundaries belong to the class CME_+^d.

In the naming of the classes, C, M, and E suggest convolution, mixture, and exponential distributions, respectively. The superscripts f and d suggest finite and discrete, respectively.

Necessary and sufficient condition for strong unimodality for a subclass of CME_+ is given in [7]. An extension of the result to CME will be given in [10].

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2. Class B

For a topological space A, we denote by $\mathscr{P}(A)$ the totality of Borel probability measures on A. For $\mu_1, \mu_2 \in \mathscr{P}(\mathbf{R})$, we denote by $\mu_1 * \mu_2$ the convolution of μ_1 and μ_2. For $A, B \subset \mathscr{P}(\mathbf{R})$, we denote by $A * B$ the totality of $\mu = \mu_1 * \mu_2$ with $\mu_1 \in A$ and $\mu_2 \in B$. The characteristic function of $\mu \in \mathscr{P}(\mathbf{R})$ is denoted by $\mathscr{F}\mu(s)$. We define the bilateral Laplace transform $\mathscr{L}\mu(s) = \int_{\mathbf{R}} e^{-sx}(dx)$ if the integral is finite. A representation of the characteristic functions of infinitely divisible distributions is well known. Namely, a distribution $\mu \in \mathscr{P}(\mathbf{R})$ is infinitely divisible if and only if there are $\gamma \in \mathbf{R}$, $\sigma > 0$ and a measure ν on $\mathbf{R}_0 = \mathbf{R}\backslash\{0\}$ satisfying

$$\int_{\mathbf{R}_0} (x^2 \wedge 1)\nu(dx) < \infty \tag{2.1}$$

such that

(2.2)
$$\mathcal{F}\mu(z) = \exp\{i\gamma z - \sigma^2 z^2/2 + \int_{\mathbf{R}_0} (e^{izx} - 1 - \frac{izx}{1+x^2})\nu(dx)\}.$$

Here, $a \wedge b = \min\{a, b\}$. This representation is unique. We call (2.2) the canonical representation $[\gamma, \sigma^2, \nu]$ of $\mu \in \mathcal{I}(\mathbf{R})$. The measure ν is called Lévy measure of μ. The following theorem is well known.

THEOREM A. *Let $\mu_n \in \mathcal{I}(\mathbf{R})$ with canonical representation $[\gamma_n, \sigma_n, \nu_n]$ and let $\mu \in \mathcal{P}(\mathbf{R})$. Then the following* (i) *and* (ii) *are equivalent:*

(i) *μ_n converges weakly to μ as $n \to \infty$.*

(ii) *μ is infinitely divisible. Let $[\gamma, \sigma, \nu]$ be its canonical representation.*

(a) *For every bounded continuous function f which vanishes near the origin,*

$$\int f(u)\nu_n(du) \to \int f(u)\nu(du) \quad as \quad n \to \infty.$$

(b) *For $\varepsilon > 0$ set*

$$A_{n,\varepsilon} = \sigma_n^2 + \int_{|y|<\varepsilon} y^2 \nu_n(du).$$

Then

$$\lim_{\varepsilon \downarrow 0} \limsup_{n\to\infty} A_{n,\varepsilon} = \lim_{\varepsilon \downarrow 0} \liminf_{n\to\infty} A_{n,\varepsilon} = \sigma^2.$$

(c)
$$\lim_{n\to\infty} \gamma_n = \gamma.$$

We say that a distribution μ on $\mathbf{R}$ is a B distribution if $\mu \in \mathcal{I}(\mathbf{R})$ and its Lévy measure ν is absolutely continuous with density ℓ represented as

$$\ell(y) = \int_{(0,\infty)} e^{-yu} Q(du) \quad \text{for} \quad y > 0,$$
$$= \int_{(-\infty,0)} e^{-yu} Q(du) \quad \text{for} \quad y < 0,$$

where, Q is a measure on $\mathbf{R}_0$ satisfying

(2.3)
$$\int_{\mathbf{R}_0} |u|^{-1} \wedge |u|^{-3} Q(du) < \infty.$$

We denote by B_+ the class of B distributions on R_+. The class B_+ here was denoted by BO in [8] and called g.c.m.e.d. (generalized convolutions of mixtures of exponential distributions) in [1]. The above integrability condition (2.3) for Q is equivalent to the condition (2.1) for ν. We call Q the Q-measure of $\mu \in B$. A B distribution μ is uniquely represented by the triplet (γ, o^2, Q). We describe a necessary and sufficient condition for weak convergence in B in terms of this triplet.

THEOREM 2.1. *Let $\mu_n \in B$ and let $(\gamma_n, \sigma_n^2, Q_n)$ be its triplet. In order that μ_n converges to $\mu \in \mathcal{P}(\mathbf{R})$ as $n \to \infty$, it is necessary and sufficient that $\mu \in B$ with triplet (γ, σ^2, Q) and the following conditions are satisfied.*

(i) *For any function f with compact support in $\mathbf{R}$ such that $|u| f(u)$ is continuous,*

$$\int f(u)\,Q_n(du) \to \int f(u)\,Q(du) \quad \text{as} \quad n \to \infty.$$

(ii) *Let $A_{n,M} = \sigma_n^2 + 2\int_{|u|>M} |u|^{-3} Q_n(du)$. Then*

$$\lim_{M\to\infty} \limsup_{n\to\infty} A_{n,M} = \lim_{M\to\infty} \liminf_{n\to\infty} A_{n,M} = \sigma^2.$$

(iii)
$$\lim_{n\to\infty} \gamma_n = \gamma.$$

Proof. We prove the theorem checking the conditions of Theorem A.

Sufficiency. Assume that $\mu \in B$ and (i) ~ (iii) hold. Let ν_n and ν be the Lévy measures of μ_n and μ, respectively. By (i) and (ii), we have

$$\int f(u)\,(|u|^{-1} \wedge |u|^{-3})\,Q_n(du) \to \int f(u)\,(|u|^{-1} \wedge |u|^{-3})\,Q(du)$$

as $n \to \infty$ for every continuous function f on $\mathbf{R}$ vanishing at infinity. Hence for $0 < a < b$

$$\begin{aligned}\int_a^b \nu_n(dy) &= \int_0^\infty u^{-1}(e^{-au} - e^{-bu})\,Q_n(du) \\ &\to \int_0^\infty u^{-1}(e^{-au} - e^{-bu})\,Q(du) \\ &= \int_a^b \nu(dy) \quad \text{as} \quad n \to \infty.\end{aligned}$$

In the same manner,

$$\int_1^\infty \nu_n(dy) = \int_0^\infty u^{-1} e^{-u} Q_n(du) \to \int_1^\infty \nu(dy),$$

$$\int_{-b}^{-a} \nu_n(dy) \to \int_{-b}^{-a} \nu(dy),$$

$$\int_{-\infty}^{-1} \nu_n(dy) \to \int_{-\infty}^{-1} \nu(dy) \quad \text{as} \quad n \to \infty.$$

Hence we get the condition (a) in Theorem A. Note that

$$\text{(2.4)} \qquad \int_{|y|<\varepsilon} y^2 \nu_n(dy) = \int_{\mathbf{R}_0} \Big(\int_0^{\varepsilon} y^2 e^{-|u|y} dy\Big) Q_n(du)$$
$$= \int_{\mathbf{R}_0} \Big(\int_0^{|u|\varepsilon} y^2 e^{-y} dy\Big) \,|\, u \,|^{-3} Q_n(du)$$
$$= \textstyle\sum_{i=1}^4 F_n^i(\varepsilon)$$

where, for $\varepsilon > 0$,

$$F_n^1(\varepsilon) = \int_{|u| \geqq \varepsilon^{-2}} 2 \,|\, u \,|^{-3} Q_n(du),$$

$$F_n^2(\varepsilon) = -\int_{|u| \geqq \varepsilon^{-2}} \Big(\int_{|u|\varepsilon}^{\infty} y^2 e^{-y} dy\Big) \,|\, u \,|^{-3} Q_n(du),$$

$$F_n^3(\varepsilon) = \int_{|u| \leqq \varepsilon^{-2}} \Big(\int_0^{\varepsilon} y^2 e^{-|u|y} dy\Big) Q_n(du).$$

By (ii), we have that $\{F_n^1(\varepsilon)\}$ is bounded in n and

$$|\, F_n^2(\varepsilon) \,| \leqq \frac{1}{2} F_n^1(\varepsilon) \int_{\varepsilon^{-1}}^{\infty} y^2 e^{-y} dy \to 0 \text{ as } n \to 0 \text{ and } \varepsilon \to 0.$$

In the following, we may assume that ε^{-2} is a continuity point of Q. By (i), we have, for fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} F_n^3(\varepsilon) = \int_{|u| \leqq \varepsilon^{-2}} \Big(\int_0^{\varepsilon} y^2 e^{-|u|y} dy\Big) Q(du).$$

By (2.3) and by bounded convergence theorem,

$$\int_{|u| \leqq \varepsilon^{-2}} \Big(\int_0^{\varepsilon} y^2 e^{-|u|y} dy\Big) Q(du)$$
$$= \int_{|u| \leqq 1} \Big(\int_0^{\varepsilon} y^2 e^{-|u|y} dy\Big) Q(du) + \int_{1 \leqq |u| \leqq \varepsilon^{-2}} \Big(\int_0^{|u|\varepsilon} y^2 e^{-y} dy\Big) \,|\, u \,|^{-3} Q(du) \to 0 \text{ as } \varepsilon \to 0.$$

Thus, we have

$$\lim_{\varepsilon \downarrow 0} \lim_{n\to\infty} F_n^3(\varepsilon) = 0.$$

Hence, we have

$$\text{(2.5)}\qquad \lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\,[\sigma_n^2+\int_{|y|<\varepsilon} y^2\nu_n(dy)]$$

$$=\lim_{M\uparrow\infty}\limsup_{n\to\infty}\,[\sigma_n^2+\int_{|u|>M} 2\,|\,u\,|^{-3}Q(du)]$$

and

$$\text{(2.6)}\qquad \lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\,[\sigma_n^2+\int_{|y|<\varepsilon} y^2\nu_n(dy)]$$

$$=\lim_{M\uparrow\infty}\liminf_{n\to\infty}\,[\sigma_n^2+\int_{|u|>M} 2\,|\,u\,|^{-3}Q(du)].$$

Thus the condition (b) of Theorem A holds. The condition (c) is trivial.

Necessity. Let $\mu_n\to\mu$. Then $\mu\in\mathscr{I}(\mathbf{R})$ by Theorem A. By Theorem A(a), we have, for any continuity point $a>0$ of ν.

$$\int_a^\infty \nu_n(du)\to\int_a^\infty \nu(du)\ \text{ as } n\to\infty.$$

Hence we have, for a.e. $a>0$,

$$\text{(2.7)}\qquad \int_0^\infty u^{-1}e^{-au}Q_n(du)\to\int_a^\infty \nu(du)\ \text{ as } n\to\infty.$$

Similarly we have, for a.e. $a<0$,

$$\text{(2.8)}\qquad \int_{-\infty}^0 |\,u\,|^{-1}e^{-au}Q_n(du)\to\int_{-\infty}^a \nu(du)\ \text{ as } n\to\infty.$$

By (2.4) we have,

$$\int_{|y|<\varepsilon} y^2\nu_n(dy)\geqq F_n^1(\varepsilon)(1-2^{-1}\int_{\varepsilon^{-1}}^\infty y^2e^{-y}dy).$$

Thus $\{F_n^1(\varepsilon)\}$ is bounded in n. Then we see, by (2.7) and (2.8), that there is a finite measure $\tilde{Q}$ on $\mathbf{R}$ such that for $a>0$

$$\int_0^\infty u^{-1}e^{-au}Q_n(du)\to\int_0^1 e^{-au}\tilde{Q}(du)+\int_1^\infty u^2e^{-au}\tilde{Q}(du),$$

$$\int_{-\infty}^0 |\,u\,|^{-1}e^{-a|u|}Q_n(du)\to\int_{-1}^0 e^{-a|u|}\tilde{Q}(du)+\int_{-\infty}^{-1} u^2e^{-a|u|}\tilde{Q}(du)$$

as $n\to\infty$. Note that $\tilde{Q}$ does not have a point mass at $\{0\}$ since $\lim_{a\to\infty}\int_{|y|>a}\nu(dy)=0$. Set $Q(du)=(|\,u\,|\vee|\,u\,|^3)\tilde{Q}(du)$. Then, Q is a measure on $\mathbf{R}_0$ satisfying

(2.3). We have

$$\int_I |u|^{-1} Q_n(du) \to \int_I |u|^{-1} Q(du) \text{ as } n \to \infty, \tag{2.9}$$

for every finite interval I in $\mathbf{R}$ both end points of which are continuity points of Q. Thus, (i) holds. We have, by (i),

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} F_n^3(\varepsilon) = 0.$$

Since $\{F_n^1\{\varepsilon)\}$ is bounded in n,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} | F_n^2(\varepsilon) | = 0.$$

We have (2.5) and (2.6). Hence (ii) holds. The proof is complete.

COROLLARY. *The class B is closed under convolution and weak convergence.*

THEOREM 2.2. *The class B coincides with the closure of $B_+ * B_-$.*

Proof. Since the class B is closed, it is enough to show that the normal distributions and B distributions without Gaussian components are approximated by $B_+ * B_-$ distributions. For $\sigma^2 > 0$, set $\alpha_n = (2n/\sigma^2)^{1/2}$ and let

$$\begin{aligned} q_n(x) &= 0 \quad \text{for } |x| < \alpha_n, \\ &= n \quad \text{for } a_n \leqq |x|. \end{aligned}$$

Then $\mu_n = (0, 0, q_n(x)dx) \in B_+ * B_-$. We have, for $M < \alpha_n$

$$2\int_{|u|>M} |u|^{-3} q_n(u)du = \sigma^2$$

and for every finite interval I in $\mathbf{R}$,

$$\int_I |u|^{-1} q_n(u)du \to 0$$

as $n \to \infty$. Hence $\mu_n \to (0, \sigma^2, 0)$ as $n \to \infty$ by Theorem 2.1. Now, let $(0,0,Q) \in B$. Define Q_n by $Q_n = Q|_{[-n,n]}$.Then $(0, 0, Q_n) \in B_+ * B_-$.
Since

$$\int_{|u|>M} |u|^{-3} Q_n(du) \to \int_{|u|>M} |u|^{-3} Q(du)$$

as $n \to \infty$ and

$$\int_{|u|>M} |u|^{-3} Q(du) \to 0$$

as $M \to \infty$, $(0, 0, Q_n) \to (0, 0, Q)$. The proof is complete.

3. Class *ME*

We say that a probability distribution μ on $\mathbf{R}_+$ is an ME_+ distribution if there is a probability measure G on $(0, \infty]$ such that

$$\begin{aligned} \mu[0, x] &= G(\{\infty\}) && \text{if} \quad x = 0, \\ &= \int_{(0,\infty]} (1 - e^{-xu}) G(du) && \text{if} \quad x > 0, \end{aligned}$$

where the value of the integrand $1 - e^{-xu}$ at infinity for $x > 0$ is defined by its limit 1 as $u \to \infty$. We call G the mixing distribution of μ. We denote by ME_+ the class of ME_+ distributions. It is easy to see that the Laplace transform of $\mu \in ME_+$ is represented by its mixing distribution G as:

$$\begin{aligned} \mathscr{L}\mu(s) &= G(\{\infty\}) + \int_{(0,\infty)} e^{-sx} dx \int_{(0,\infty)} u e^{-xu} G(du) \\ &= \int_{(0,\infty]} \frac{u}{s+u} G(du). \end{aligned} \tag{3.1}$$

Define ME_- by the mirror image of ME_+. That is, ME_- if and only if $\mu \in \mathscr{P}(\mathbf{R}_-)$ and

$$\begin{aligned} \mu[x, 0] &= G(\{-\infty\}) && \text{if} \quad x = 0 \\ &= \int_{[-\infty,0)} (1 - e^{-xu}) G(du) && \text{if} \quad x < 0 \end{aligned}$$

with $G \in \mathscr{P}([-\infty, 0))$. Let $ME = ME_+ * ME_-$. A representation of the Laplace transform of $\mu \in ME_+$ is obtained by Steutel [5]. We state here his representation.

THEOREM B. *A probability measure μ on $\mathbf{R}_+$ is an ME_+ distribution if and only if there is a nonnegative and absolutely continuous measure Q on R_+ with density bounded by* 1 *a.e. satisfying $\int_0^1 u^{-1} Q(du) < \infty$ such that, for $z \in \mathbf{R}$,*

$$\mathscr{F}\mu(z) = \exp\left[\int_{\mathbf{R}_+} (e^{izx} - 1) \left(\int_{\mathbf{R}_+} e^{-xu} Q(du)\right) dx\right].$$

By this theorem, we easily get the representation of the characteristic function of $\mu \in ME$:

$$\mathscr{F}\mu(z) = \exp\left[\int_{\mathbf{R}_0} (e^{izx} - 1)\ell(x)\,dx\right] \tag{3.2}$$

where

$$\begin{aligned} \ell(x) &= \int_{\mathbf{R}_+} e^{-xu} Q(du) \quad \text{for } x > 0, \\ &= \int_{\mathbf{R}_-} e^{-xu} Q(du) \quad \text{for } x < 0 \end{aligned}$$

and Q is an absolutely continuous measure on $\mathbf{R}$ with density bounded by 1 a.e. satisfying

$$\int_{|u|<1} |u|^{-1} Q(du) < \infty.$$

Hence $ME \subset B$ and the above Q is the Q-measure of μ.

Remark 3.1. Let $\mu \in ME_+$ and let G be its mixing distribution. Let ℓ be the density of the Lévy measure of μ and let Q be the Q-measure of μ. Then

$$\begin{aligned} G(\{\infty\}) &= \exp\{-\int_0^\infty \ell(x)\,dx\} \\ &= \exp\{-\int_0^\infty \frac{1}{u} Q(du)\}. \end{aligned}$$

Proof. It is easy to see that

$$G(\{\infty\}) = \lim_{s\to\infty} \mathscr{L}\mu(s) = \exp\{-\int_0^\infty \ell(x)\,dx\}.$$

Since

$$\begin{aligned} \int_0^\infty \ell(x)\,dx &= \int_0^\infty \left(\int_0^\infty e^{-ux} dx\right) Q(du) \\ &= \int_0^\infty \frac{1}{u} Q(dx), \end{aligned}$$

we get the conclusion.

THEOREM 3.1. *Let $\mu_n \in ME_+$ and $\mu \in \mathscr{P}(\mathbf{R}_+)$. Let G_n be the mixing distribution of μ_n. Then μ_n converges weakly to μ if and only if $\mu \in ME_+$ and G_n converges weakly to G, the mixing distribution of μ, as a sequence of distributions on $(0, \infty]$ as $n \to \infty$.*

Proof. Let F_n and F be the distribution functions of μ_n and μ, respectiveiy. Assume that $\mu \in ME_+$ and $G_n \to G$ weakly on $(0, \infty]$ as $n \to \infty$. Then, obviously we have, for $x > 0$,

$$F_n(x) = \int_{(0,\infty]} (1 - e^{-xu}) G_n(du)$$

$$\to F(x) = \int_{(0,\infty]} (1 - e^{-xu}) G(du) \quad \text{as } n \to \infty.$$

This shows that $\mu_n \to \mu$. Conversely, we assume that $\mu_n \to \mu$ weakly as $n \to \infty$. Then we have $F_n(x) \to F(x)$ as $n \to \infty$ for all continuity point $x > 0$. For $\varepsilon > 0$, we can choose $x > 0$ sufficiently large so that $1 - F_n(x) < \varepsilon$ for all n. Hence,

$$e^{-x\delta} G_n(0, \delta) \leqq \int_{(0,\delta)} e^{-xu} G_n(du) < \varepsilon,$$

i.e.

$$G_n(0, \delta) < \varepsilon e^{x\delta}.$$

This means that $\{G_n\}$ is a conditionally compact sequence as measures on $(0, \infty]$. Choosing subsequence $\{n'\}$ of $\{n\}$ so that $G_{n'}$ converges to a distribution G on $(0, \infty]$, we have

$$F_{n'}(x) = \int_{(0,\infty]} (1 - e^{-xu}) G_{n'}(du)$$

$$\to \int_{(0,\infty]} (1 - e^{-xu}) G(du) \quad \text{as } n' \to \infty$$

for $x > 0$. Hence

$$F(x) = \int_{(0,\infty]} (1 - e^{-xu}) G(du)$$

for continuity point x of F. Since the right hand side is continuous for $x > 0$ and since F is right continuous, the equality holds for all $x > 0$. Letting $x \to 0$, we get $F(0) = G(\{\infty\})$. Hence

$$F(x) = 1 - \int_{(0,\infty]} e^{-xu} G(du).$$

By the uniqueness for Laplace transforms, G_n converges weakly to G on $(0, \infty]$ as $n \to \infty$. The proof is complete.

THEOREM 3.2. *Let $\mu_+ \in ME_+$, $\mu_- \in ME_-$ and let $\mu = \mu_+ * \mu_- \in ME$. Then μ is absolutely continuous on $\mathbf{R}_0$ and has a point mass $\mu_+(\{0\})\mu_-(\{0\})$ at the origin.*

Let h be the density of μ on $\mathbf{R}_0$. Let G_+ and G_- be mixing distributions of μ_+ and μ_-, respectively. Denote $\psi_+(s) = \mathscr{L}\mu_+(s)$ and $\psi_-(s) = \mathscr{L}\mu_-(s)$. Then the following hold:

(i)
$$h(x) = (h_+ * h_-)(x) + \mu_-(\{0\})h_+(x) = \int_{(0,\infty)} \psi_-(-u)ue^{-ux}G_+(du) \quad \text{for} \quad x > 0,$$

and

$$h(x) = (h_+ * h_-)(x) + \mu_+(\{0\})h_-(x) = \int_{(-\infty,0)} \psi_+(-v)\,|\,v\,|\,e^{-vx}G_-(dv) \quad \text{for} \quad x < 0,$$

where h_+ and h_- are densities of μ_+ and μ_- on $(0, \infty)$ and $(-\infty, 0)$, respectively.
(ii) *Denote $d_- = \sup\{v < 0\,;\, G_-([v, 0)) > 0\}$ and $d_+ = \inf\{v > 0\,;\, G_+((0, v]) > 0\}$. If $d_- < d_+$, then the Laplace transform $\mathscr{L}\mu(s)$ of μ exists for $-d_+ < s < -d_-$ and is represented as*

$$\mathscr{L}\mu(s) = \int_{(-\infty,0)} \psi_+(-v)\frac{v}{s+v}G_-(dv) + \int_{(0,\infty)} \psi_-(-u)\frac{u}{s+u}G_+(du) + G_+(\{\infty\})G_-(\{-\infty\}). \tag{3.3}$$

Proof. (i) Let F, F_+ and F_- be the distribution functions of μ, μ_+ and μ_-, respectively. Let $x > 0$. Then,

$$\begin{aligned} &F(x) \\ &= \int_{(-\infty,0)} F_+(x-y)F_-(dy) + F_+(x)\mu_-(\{0\}) \\ &= \int_{(-\infty,0)} h_-(y)dy\int_0^{x-y} h_+(z)dz + \mu_-(\{0\})\Big(\int_0^x h_+(z)dz + \mu_+(\{0\})\Big). \end{aligned}$$

By this we get

$$h(x) = \int_{(-\infty,0)} h_+(x-y)h_-(y)dy + \mu_-(\{0\})h_+(x) \quad \text{for} \quad x > 0.$$

By the definition of the classes ME_+ and ME_- we have

$$\int_{(-\infty,0)} h_+(x-y)h_-(y)dy$$

$$= \int_{(-\infty,0)} (\int_{(0,\infty)} ue^{-u(x-y)} G_+(du)) (\int_{(-\infty,0)} |v| e^{-vy} G_-(dv)) dy$$

$$= \int_{(0,\infty)} G_+(du) \int_{(-\infty,0)} \frac{vu}{v-u} e^{-ux} G_-(dv)$$

$$= \int_{(0,\infty)} (\int_{(-\infty,0)} \frac{v}{v-u} G_-(dv)) ue^{-ux} G_+(du).$$

Thus,

$$h(x) = \int_{(0,\infty)} \phi_-(-u) ue^{-ux} G_+(du) < \infty.$$

In the same way we get the representation for $x < 0$.
(ii) If $-d_+ < s < -d_-$, the right hand side of (3.3) is well defined. Denote by $A(s)$ the right hand side of (3.3). Set

$$\tilde{\phi}_-(s) = \int_{(-\infty,0)} \frac{v}{s+v} G_-(dv)$$

and

$$\tilde{\phi}_+(s) = \int_{(0,\infty)} \frac{u}{s+u} G_+(du).$$

Note that, by (3.1),

$$\phi_-(s) = \tilde{\phi}_-(s) + G_-(\{-\infty\})$$

and

$$\phi_+(s) = \tilde{\phi}_+(s) + G_+(\{-\infty\}).$$

We have

$$\begin{aligned} &A(s) \\ &= A_1(s) + \tilde{\phi}_-(s) G_+(\{\infty\}) + \tilde{\phi}_+(s) G_-(\{-\infty\}) + G_+(\{\infty\}) G_-(\{-\infty\}), \end{aligned}$$

where

$$A_1(s) = \int_{(-\infty,0)} \tilde{\phi}_+(-v) \frac{v}{s+v} G_-(dv) + \int_{(0,\infty)} \tilde{\phi}_-(-u) \frac{u}{s+u} G_+(du).$$

The function $A_1(s)$ is written as

$$\begin{aligned} &A_1(s) \\ &= \int_{(0,\infty)} \int_{(-\infty,0)} \frac{uv}{u-v} (\frac{1}{s+v} - \frac{1}{s+u}) G_-(dv) G_+(du) \end{aligned}$$

$$= \int_{(0,\infty)} \int_{(-\infty,0)} \frac{uv}{(s+u)(s+v)} G_-(dv) G_+(du)$$

$$= \tilde{\phi}_+(s)\, \tilde{\phi}_-(s).$$

Hence we have $A(s) = \phi_+(s)\phi_-(s) = \mathscr{L}\mu(s)$. The proof is complete.

THEOREM 3.3. *A sequence in ME is shift compact if and only if it is conditionally compact.*

Proof. Let $\{\mu_n\} \subset ME$ be a shift compact sequence. That is, there is a sequence $\{\gamma_n\} \subset \mathbf{R}$ such that $\{\mu_n * \delta_{\gamma_n}\}$ is conditionally compact, where δ_{γ_n} is the Dirac measure concentrated at γ_n. Let $\ell_n(y)$ be that density of the Lévy measure of μ_n. Note that since

$$\ell_n(y) \leqq \int_0^\infty e^{-|y|u} du = |y|^{-1} \quad \text{for } y \neq 0,$$

the sequence $\{\int_{\mathbf{R}_0} \frac{y}{1+y^2} \ell_n(y) dy\}$ is bounded. We have

$$\mathscr{L}(\mu_n * \delta_{\gamma_n})(z) = \exp[i\gamma_n z + \int_{\mathbf{R}_0} (e^{izy} - 1)\ell_n(y) dy]$$

$$= \exp[iz\{\gamma_n + \int_{\mathbf{R}_0} \frac{y}{1+y^2} \ell_n(y) dy\} + \int_{\mathbf{R}_0} (e^{izy} - 1 - \frac{izy}{1+y^2} \ell_n(y) dy].$$

Hence $\{\gamma_n\}$ must be bounded. It follows that $\{\mu_n\}$ is conditionally compact. The converse is obvious.

4. Class *CE*

Let CE_+^f be the class of $\mu \in \mathscr{P}(\mathbf{R}_+)$ such that $\mathscr{L}\mu(s) = \prod_{k=1}^m a_k(s + a_k)^{-1}$ with $1 \leqq m < \infty$ and $0 < a_1 < a_2 < \cdots < a_m$ and let CE_-^f be the mirror image of CE_+^f. Let $CE^f = CE_+^f * CE_-^f$. We denote by CE the closure of CE^f. Let Z be the set of integers and set $Z_0 = Z \backslash \{0\}$.

THEOREM 4.1. *Let $\mu \in \mathscr{P}(\mathbf{R})$. Then, μ is a CE distribution if and only if $\mu \in \mathscr{I}(\mathbf{R})$ and there is an $\mathbf{R}_0$-valued non-decreasing sequence $\{a_k\}_{k \in Z_0 \cap I}$ for an interval I containing 0 such that*

(4.1) $$a_k > 0 \quad \textit{for } k > 0,$$
$$< 0 \quad \textit{for } k < 0,$$

(4.2) $$\sum a_k^{-2} < \infty$$

and the Lévy measure ν *of* μ *is represented as*

(4.3) $$\nu(dx) = (x^{-1}\textstyle\sum_{k>0} e^{-a_k x})dx \quad \text{for } x > 0,$$
$$= (|x|^{-1}\textstyle\sum_{k<0} e^{-a_k x})dx \quad \text{for } x < 0.$$

We call $\{a_k\}$ the parameter sequence of μ.

Proof. Denote by CE^d the subclass of $\mathscr{I}(\mathbf{R})$ consisting of distributions whose Lévy measure is of the form (4.3) satisfying conditions (4.1) and (4.2). The assertion of the theorem is that $CE = CE^d$. Let $\mu \in CE^d$ and let $\{a_k\}$ be its parameter sequence. Set

(4.4) $$q(x) = \sum_{k>0} 1_{(a_k,\infty)}(x) + \sum_{k<0} 1_{(-\infty,a_k)}(x),$$

where $1_A(x)$ is the indicator function of a set A. Noting that $\{a_k\}$ is a monotone sequence, we have by (4.2) that $\int_{\mathbf{R}} |u|^{-3}q(u)du < \infty$. It is easy to see that the Lévy measure ν of μ is written as

$$\nu(dx) = (\int_{\mathbf{R}_+} e^{-xu}q(u)du)dx \quad \text{for } x > 0,$$
$$= (\int_{\mathbf{R}_-} e^{-xu}q(u)du)dx \quad \text{for } x < 0.$$

Hence, μ is a B distribution with triplet $(\gamma, \sigma^2, q(x)dx)$ with some γ and σ^2. Now we show that μ is approximated by CE^f-distributions. Let

$$q_{1,n}(x) = \sum_{0<k\leqq n} 1_{(a_k,\infty)}(x) + \sum_{0>k\geqq -n} 1_{(-\infty,a_k)}(x)q(x)$$

and

$$\gamma_{1,n} = \int_{\mathbf{R}_+} \frac{x}{1+x^2}\{\int_{\mathbf{R}_+} e^{-xu}q_{1,n}(u)du\}dx + \int_{\mathbf{R}_-} \frac{x}{1+x^2}\{\int_{\mathbf{R}_-} e^{-xu}q_{1,n}(u)du\}dx.$$

In case $\sigma^2 > 0$, set $\alpha_n = (2n/\sigma^2)^{1/2}$ and let

$$q_{2,n}(x) = 0 \quad \text{for } |x| < \alpha_n,$$
$$= n \quad \text{for } |x| \geqq \alpha_n,$$

and choose $\beta_n > 0$ so that

(4.5) $$(\gamma - \gamma_{1,n})/\beta_n \to 0 \quad \text{as } n \to \infty$$

and $\beta_n > \alpha_n$. In case $\sigma^2 = 0$, let

$$q_{2,n}(x) \equiv 0$$

and choose $\beta_n > 0$ so to satisfy (4.5). Let δ_n be the integral part of $\{|\gamma - \gamma_{1,n}| / \int_{\mathbf{R}_+} \frac{1}{1+x^2} e^{-\beta_n x} dx\}$ and let

$$\begin{aligned} \tilde{q}_n(x) &= 0 \qquad \text{for } x < \beta_n, \\ &= \delta_n \qquad \text{for } x \geqq \beta_n. \end{aligned}$$

Define

$$\begin{aligned} q_{3,n}(x) &= \tilde{q}_n(x) \qquad \text{if } \gamma > \gamma_{1,n} \\ &= \tilde{q}_n(-x) \quad \text{if } \gamma \leqq \gamma_{1,n}. \end{aligned}$$

Let

$$\gamma_n = \gamma_{1,n} + \operatorname{sign}(\gamma - \gamma_{1,n}) \delta_n \int_{\mathbf{R}_+} \frac{1}{1+x^2} e^{-\beta_n x} dx. \tag{4.6}$$

Then, $Q_n(dx) = \{\sum_{j=1}^{3} q_{j,n}(x)\} dx$ satisfies (2.3). Let $\mu_n = (\gamma_n, 0, Q_n) \in B$. Since

$$\gamma_n = \int_{\mathbf{R}_+} \frac{x}{1+x^2} dx \int_{(0,\infty)} e^{-xu} Q_n(du) + \int_{\mathbf{R}_-} \frac{x}{1+x^2} dx \int_{(-\infty,0)} e^{-xu} Q_n(du),$$

μ_n is approximated by CE^f-distributions. It is easy to see that $Q_n(I) \to \int_I q(x) dx$ for every bounded interval I in $\mathbf{R}$. We have by (4.2) that

$$\begin{aligned} &\lim_{M\to\infty} \limsup_{n\to\infty} 2\int_{|u|>M} u^{-3} q_{1,n}(u) du \\ &= \lim_{M\to\infty} \sum_k \frac{1}{(|a_k| \vee M)^2} = 0. \end{aligned}$$

We see by (4.6) that, for every M,

$$\begin{aligned} &\lim_{n\to\infty} 2\int_{|u|>M} |u|^{-3} \{q_{2,n}(u) + q_{3,n}(u)\} du \\ &= \lim_{n\to\infty} \{\sigma^2 + \delta_n/\beta_n^2\} \to \sigma^2. \end{aligned}$$

We have by (4.6) that

$$|\gamma_n - \gamma| \leqq \int_{\mathbf{R}_+} \frac{1}{1+x^2} e^{-\beta_n x} dx \to 0 \quad \text{as } n \to \infty.$$

Thus by Theorem 2.1, $\mu_n \to \mu$ as $n \to \infty$. Hence, CE^d-distributions can be approximated by CE^f-distributions. Now we show that the class CE^d is closed under weak convergence. Let $\mu_n \in CE^d$ and let $\mu_n \to \mu \in \mathcal{P}(\mathbf{R})$. Then, by Theorem 2.1, $\mu \in B$. Let q_n be the density of Q-measure of μ_n. Consider the convergence of the Q-measures on $(0, \infty)$. Since q_n is a nondecreasing function, Q-measure of μ is absolutely continuous, its density q is nondecreasing. Moreover, $q_n(x)$ converges to $q(x)$ at every continuity point of q. Noting that q_n is a step function of step size 1, we have that q is also a step function with step size being positive integers. The same argument yields that the Q-measure of μ has a density q also on $(-\infty, 0)$ and that q is a nonincreasing step function on $(-\infty, 0)$ with step size being negative integers. By (2.3), $q(x) = 0$ near $x = 0$. Hence the class CE^d is closed. Hence $CE^d = CE$.

Remark 4.1. The condition (4.2) for the parameter sequence $\{a_n\}$ of $\mu \in CE$ is equivalent to

$$\int_{|x|<1} x^2 \nu(dx) < \infty$$

for the Lêvy measure ν of μ.

Remark 4.2. (i) A measure ν of the form (4.3) with subsidiary conditions (4.1) and (4.2) satisfies $\int_{|x|>1} |x| \nu(dx) < \infty$. Hence, for a CE distribution, instead of (2.2) we can use another representation of its characteristic function. Let $\mu \in CE$ with canonical representation $[\gamma, \sigma^2, \nu]$. Then its characteristic function is represented as

$$\text{(4.7)} \qquad \mathcal{F}\mu(z) = \exp\{i\gamma' z - \sigma^2 z^2/2 + \int_{\mathbf{R}_0} (e^{ixz} - 1 - izx)\nu(dx)\}.$$

Here

$$\text{(4.8)} \qquad \gamma' = \gamma + \int_{\mathbf{R}_0} x^3(1 + x^2)^{-1}\nu(dx).$$

We call (4.7) the modified representation of $\mu \in CE$. We denote the modified representation of μ by $\{\gamma', \sigma^2, \nu\}$ or $\{\gamma', \sigma^2, \{a_j\}\}$, where $\{a_j\}$ is the parameter sequence of μ. Using this representation, as is shown in the next theorem, we can represent the Laplace transforms of CE distributions as rather simple products.

(ii) Let $\mu_n \in CE$ and let $[\gamma_n, \sigma_n^2, \nu_n]$ and $\{\gamma_n', \sigma_n^2, \nu_n\}$ be the canonical and the modified representations of μ_n, respectively. If $[\gamma_n, \sigma_n^2, \nu_n]$ satisfies the condition of Theorem A with $\mu = [\gamma, \sigma^2, \nu] = \{\gamma', \sigma^2, \nu\}$, then

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{|x|<\varepsilon}|x|^3\nu_n(dx)=0. \tag{4.9}$$

Hence by Theorem A, $\gamma_n'\to\gamma'$ as $n\to\infty$. The converse is also valid. Hence for CE distributions, the condition (iii) of Theorem 2.1 can be replaced $\lim_{n\to\infty}\gamma_n'=\gamma'$.

THEOREM 4.2. *A distribution μ is a CE distribution if and only if there are $\gamma'\in\mathbf{R}$, $\sigma^2\geqq 0$ and an $\mathbf{R}_0$-valued non-decreasing sequence $\{a_n\}_{n\in Z_0\cap I}$ for an interval I containing* 0 *such that* (4.1) *and* (4.2) *are satisfied and the Laplace transform of μ is represented as*

$$\mathscr{L}\mu(s)=\exp(-\gamma's+\sigma^2s^2/2)\textstyle\prod_n a_n(s+a_n)^{-1}e^{a_n^{-1}s} \tag{4.10}$$

for $-a_1<\operatorname{Re}s<-a_{-1}$.

Proof. Let $s=x+iy$. Note that

$$\begin{aligned}&\log(|(1+s/a_n)e^{-a_n^{-1}s}-1|+1)\\ &\leqq|(1+s/a_n)e^{-a_n^{-1}s}-1|\\ &\leqq|e^{-a_n^{-1}s}-1+s/a_n|+|s/a_n||e^{-a_n^{-1}s}-1|\\ &\leqq|s/a_n|^2R^{-2}(1+R)e^R\quad\text{for }|s/a_n|<R.\end{aligned}$$

Hence by (4.2), it is easy to see that the right hand side of (4.10) is convergent for $-a_1<\operatorname{Re}s<-a_{-1}$. For $s=-iz$, $z\in\mathbf{R}$, it is equal to

$$\exp(i\gamma'z-\sigma^2z^2/2)\textstyle\prod_n a_n(-iz+a_n)^{-1}e^{-ia_n^{-1}z}.$$

We can rewrite the above formula as

$$\begin{aligned}&\exp\{i\gamma'z-\sigma^2z^2/2+\textstyle\sum[\log\{a_n(-iz+a_n)^{-1}\}-ia_n^{-1}z]\}\\ &=\exp\{i\gamma'z-\sigma^2z^2/2+\textstyle\sum_{n>0}\int_0^\infty(e^{izx}-1-izx)x^{-1}e^{-a_nx}dx+\\ &\qquad+\textstyle\sum_{n<0}\int_{-\infty}^0(e^{izx}-1-izx)|x|^{-1}e^{-a_nx}dx\}\\ &=\exp\{i\gamma'z-\sigma^2z^2/2+\int_0^\infty(e^{izx}-1-izx)[\textstyle\sum_{n>0}x^{-1}e^{-a_nx}]dx+\\ &\qquad+\int_{-\infty}^0(e^{izx}-1-izx)[\textstyle\sum_{n<0}|x|^{-1}e^{-a_nx}]dx\}.\end{aligned}$$

Here we choose the branch of the logarithm so the argument is between $-\pi$ and π. On the other hand, $\int e^{sx}\mu(dx)$ is finite if $-a_1 < \operatorname{Re} s < -a_{-1}$. This shows the validity of Theorem 4.2.

The above representation shows that the class of densities of CE distributions coincides with the class of PF densities defined in Karlin [2] p. 335.

The quantities γ' appearing in (4.10) and (4.8) are identical. Write the closures of CE_+^f and CE_-^f as CE_+ and CE_-, respectively. It is easy to show that the class CE_+ coincides with the class CE_+ defined in [8] and the class CE_+ (resp. CE_-) coincides with the class of CE distributions with supports in $\mathbf{R}_+$ (resp. $\mathbf{R}_-$).

5. Class *CME*

In [8], the class CME_+ is defined by $CME_+ = ME_+ * CE_+$ and it is proved that the class CME_+ is the closure of CME_+^f. Let $CME_- = ME_- * CE_-$. Then, the class CME_- is the closure of CME_-^f. We denote by CME the closure of CME^f. This class contains both CME_+ and CME_-. Define ME_+^f as follows: $\mu \in ME_+^f$ if and only if $\mu \in ME_+$ and the mixing distribution G of μ is supported on a finite number of points in $(0, \infty]$. Let ME_-^f be the mirror image of ME_+^f and let $ME^f = ME_+^f * ME_-^f$.

THEOREM 5.1. $CME = CE * ME$.

Proof. By definition CE is the closure of CE^f. It is easy to see that ME is the closure of ME^f. Hence we have $CME^f \subset CE * ME \subset CME$. Now we show that $CE * ME$ is closed, which will prove the theorem. Let $\{\mu_n\}$ be a sequence in $CE * ME$ converging to a distribution μ. Let $\mu_n^1 \in CE$ and $\mu_n^2 \in ME$ be such that $\mu_n = \mu_n^1 * \mu_n^1$, for $n = 1, 2, \ldots$. Since the components $\{\mu_n^1\}$ and $\{\mu_n^2\}$ are both shift compact, $\{\mu_n^2\}$ is conditionally compact by Theorem 3.3. Hence $\{\mu_n^1\}$ is also conditionally compact. Now we can choose a subsequence n' so that $\mu_{n'}^1 \to \mu^1 \in CE$ and $\mu_{n'}^2 \to \mu^2 \in ME$ as $n' \to \infty$ and we have

$$\mu = \mu^1 * \mu^2.$$

Hence, $CE * ME$ is closed.

Remark 5.1. A distribution $\mu \in CME$ is determined by the modified representation $\{\gamma, \sigma^2, \boldsymbol{a} = \{a_j\}\}$ of its CE component and the Q-measure Q of its ME

component. Let us call $(\gamma, \sigma^2, \boldsymbol{a}, Q)$ the quadruplet of the CME distribution μ. Since there are many ways of decomposing μ as $\mu = \mu_1 * \mu_2$ with $\mu_1 \in CE$ and $\mu_2 \in ME$, there are many quadruplets that determine μ. But, among them, there is a unique decomposition which maximizes the density of the Q-measure of μ_1. Choosing $\mu_1 \in CE$ and $\mu_2 \in ME$ in this way, the quadruplet $(\gamma', \sigma^2, \boldsymbol{a}, Q)$ is uniquely determined by μ. In the following, by qudruplet of μ, we always mean this quadruplet.

The parameter sequence $\boldsymbol{a} = \{a_j\}_{j \in Z_0 \cap I}$ may possibly be empty. In case a_j is not defined, we regard $a_j = \infty$ if $j > 0$ and $a_j = -\infty$ if $j < 0$.

6. Representation of Laplace transforms of distributions of classes ME_+^d and CME_+^d

We say that a distribution on $(0, \infty]$ is discrete if its support is a finite or countably infinite set which has no accumulation point in $[0, \infty)$. A distribution μ on $\mathbf{R}_+$ is said to belong to class ME_+^d if μ belongs to ME_+ and its mixing distribution is discrete.

THEOREM 6.1. *Let $\{\alpha_j\}$ and $\{\beta_j\}$ be sequences of positive real numbers such that $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$ and $\alpha_j, \beta_j \to \infty$ as $j \to \infty$. Then the infinite product*

$$f(s) = \prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}) \tag{6.1}$$

absolutely and uniformly converges on each compact set in $\mathbf{C} \setminus \{-\alpha_1, -\alpha_2, \ldots\}$ and there is $\mu \in ME_+^d$ such that

$$\mathcal{L}\mu(s) = f(s) \quad \textit{for } s > 0.$$

Moreover, $\mathcal{L}\mu(s)$ is written as

$$\mathcal{L}\mu(s) = \exp\int_0^\infty (e^{-sx} - 1)\{\int_0^\infty e^{-xu} q(u) du\} dx, \tag{6.2}$$

where

$$\begin{aligned} q(u) &= 0 \qquad 0 < u < \alpha_1, \\ &= 1 \qquad \alpha_j < u < \beta_j, \quad j = 1,2,\ldots. \\ &= 0 \qquad \beta_j < u < \alpha_{j+1}, \; j = 1,2,\ldots. \end{aligned}$$

Proof. First step. We show the absolute and uniform convergence of f on each compact set in $\mathbf{C} \setminus \{-\alpha_1, -\alpha_2, \cdots\}$. Set

$$a_j(s) = (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})$$

and

$$b_j(s) = 1 - a_j(s).$$

Then we have

(6.3) $$b_j(s) = s(1 + s/\alpha_j)^{-1}\{(\alpha_j)^{-1} - (\beta_j)^{-1}\}$$

and the inside of the braces in (6.3) is positive. Let $D_T = \{s ; |s| < T\}$. If there is i such that $\alpha_i \leqq T < \alpha_{i+1}$, then choose M so that $1/M < 1/T - 1/\alpha_{i+1}$. Then we get that, for $s \in D_T$ and for all $j \geqq i+1$,

$$|1/s + 1/\alpha_j| \geqq |1/s| - |1/\alpha_j|$$

$$> 1/T - 1/\alpha_j \geqq 1/T - 1/\alpha_{i+1} > 1/M.$$

That is,

(6.4) $$|s/(1 + s/\alpha_j)| < M.$$

Moreover, $|b_j(s)| < 1$ for large j, since $\alpha_j, \beta_j \to \infty$ as $j \to \infty$. We denote by $U_{T,\delta}$ the set D_T with the δ-neighborhoods of $-\alpha_1, \ldots, -\alpha_i$ excluded. Since $s/(1 + s/\alpha_j)$ is bounded in j and $s \in U_{T,\delta}$, there is $M > 0$ such that

$$\sum_{j=1}^{\infty} |b_j(s)| \leqq \sum_{j=1}^{\infty} M(1/\alpha_j - 1/\alpha_{+1}) \leqq M/\alpha_1 < \infty$$

for $s \in U_{T,\delta}$. By this we have that $\sum_{j=1}^{\infty} b_j(s)$ converges absolutely and uniformly on any compact set in $\mathbf{C}\setminus\{-\alpha_1, -\alpha_2, \ldots\}$. Hence the infinite product $f(s)$ converges absolutely and uniformly on any compact set in $\mathbf{C}\setminus\{-\alpha_1, -\alpha_2, \ldots\}$.

Second step. We show that f is the Laplace transform of the ME_+^d distribution μ defined by (6.2). Note that

$$f_n(s) = \prod_{j=1}^{n} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})$$

is the Laplace transform of an ME_+^f distribution μ_n (Steutel [5]). Moreover, f_n is written as

$$f_n(s) = \exp \int_0^{\infty} (e^{-sx} - 1)\{\int_0^{\infty} e^{-xu} q_n(u)\,du\}\,dx,$$

where

$$\begin{aligned} q_n(u) &= 0 \qquad u < \alpha_1 \\ &= 1 \qquad a_j < u < \beta_j \qquad j = 1,2,\ldots, n, \\ &= 0 \qquad \beta_j < u < \alpha_{j+1} \quad j = 1,2,\ldots, n, \end{aligned}$$

Here we understand $\alpha_{n+1} = \infty$. We have

$$q_n(u)\,du \longrightarrow q(u)\,du \text{ as } n \longrightarrow \infty$$

and

$$\int_M^\infty \frac{q_n(u)}{u^2}\,du \leqq 1/M \longrightarrow 0 \text{ as } M \longrightarrow \infty.$$

By the continuity theorem for B_+ (Bondesson [1]), letting μ be the distribution with Laplace transform of the form (6.2), we have $\mu_n \longrightarrow \mu$ as $n \longrightarrow \infty$. Hence $f_n(s) \longrightarrow \mathcal{L}\mu(s)$ for $s > 0$ as $n \longrightarrow \infty$. On the other hand, $f_n(s)$ converges to $f(s)$ as $n \longrightarrow \infty$ absolutely and uniformly on any compact set in $\mathbf{C} \setminus \{-\alpha_1, -\alpha_2, \ldots\}$, $\mathcal{L}\mu(s) = f(s)$ should hold for $s > 0$. By Theorem 3.1, the mixing distribution G_n of μ_n converges weakly to the mixing distribution G of μ as a distribution on $(0, \infty]$. Since the support of G_n is contained in $\{\alpha_j\}_{j=1}^n \cup \{\infty\}$, the support of G is contained in $\{\alpha_j\}_{j=1}^\infty \cup \{\infty\}$. Hence $\mu \in ME_+^d$. The proof is complete.

THEOREM 6.2. *Let $\{\alpha_j\}$ and $\{\beta_j\}$ be non-decreasing infinite sequences of positive real numbers satisfying $\alpha_i \neq \beta_j$ for all i, j. Let $\mu \in ME_+$ such that*

$$\mathcal{L}\mu(s) = \textstyle\prod_{j=1}^\infty (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}) \tag{6.5}$$

for $s \geqq 0$. Then

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots. \tag{6.6}$$

Moreover, if α_j, $\beta_j \longrightarrow \infty$ as $j \longrightarrow \infty$, then $\mu \in ME_+^d$.

Proof. By the assumption,

$$\begin{aligned}\mathcal{L}\mu(s) &= \exp\left[\textstyle\sum_{j=1}^\infty \{\log \frac{\alpha_j}{s+\alpha_j} - \log \frac{\beta_j}{s+\beta_j}\}\right] \\ &= \exp\left[\textstyle\sum_{j=1}^\infty \int_{\alpha_j}^{\beta_j} \frac{s}{u(s+u)}\,du\right], \text{ for } s \geqq 0.\end{aligned}$$

Thus the density $q(u)$ of the Q-measure of μ is written as $q(u) = \sum_{j=1}^\infty 1_{[\alpha_j, \beta_j]}(u)$. We show (6.6) by induction. Remind that $q(u)$ is nonnegative and bounded by 1 a.e. Hence $\alpha_1 < \beta_1$. Assume that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n$$

holds for $n \geqq 1$. If $\alpha_n \leqq \alpha_{n+1} < \beta_{n+1}$, then since $\beta_n \leqq \beta_{n+1}$, $q(u) = 2$ on (α_{n+1}, β_n). This can not occur. Hence $\alpha_n < \beta_n < \alpha_{n+1}$. Since q is nonnegative, $\beta_n < \beta_{n+1} < \alpha_{n+1}$ can not occur. Hence $\alpha_n < \beta_n < \alpha_{n+1} < \beta_{n+1}$. The proof is complete.

THEOREM 6.3. *Let $\mu \in ME_+^d$ and let G be its mixing distribution. Suppose that $\{\alpha_j\}_{1\leq j<\infty} = (\mathrm{supp}\ G)\setminus\{\infty\}$, where $\{\alpha_j\}$ is an infinite sequence increasing to ∞. Then there is a sequence of real numbers $\{\beta_k\}_{k=1}^{\infty}$ such that*

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$$

and

$$\mathscr{L}\mu(s) = \prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}),\ s > 0.$$

Proof. Let $p_j = G(\{\alpha_j\})$ and $p_\infty = 1 - \sum_{j=1}^{\infty} p_j$. We have

$$\mathscr{L}\mu(s) = p_\infty + \sum_{j=1}^{\infty} \frac{\alpha_j}{s + \alpha_j} p_j \quad \text{for } s > 0. \tag{6.7}$$

Denote by $f(s)$ the right hand side of (6.7). Set $P = \{-\alpha_j\}_{j=1}^{\infty}$. Then the analytic continuation of f to $\mathbf{C}\setminus P$ is unique and f is a meromorphic function. Every pole of f has degree 1 and the set of poles coincides with P. The function f is term-wise differentiable in $\mathbf{C}\setminus P$ and

$$f'(s) = -\sum_{j=1}^{\infty} \frac{\alpha_j}{(s + \alpha_j)^2} p_j.$$

This shows that f is decreasing in every interval in $\mathbf{R}\setminus P$ and the set of zeros $Z = \{-\beta_j\}_{j\geq 1}$ of f in $\mathbf{R}\setminus P$ satisfies

$$\cdots < -\beta_2 < -\alpha_2 < -\beta_1 < -\alpha_1 < 0.$$

Set $s = a + bi$. Since

$$f(s) = p_\infty + \sum_{j=1}^{\infty} \frac{\alpha_j(a + \alpha_j)}{(a + \alpha_j)^2 + b^2} p_j + i \sum_{j=1}^{\infty} \frac{-\alpha_j b}{(a + \alpha_j)^2 + b^2} p_j,$$

the imaginary part of $f(s)$ vanishes if and only if $b = 0$. Hence f does not have zero points outside $\mathbf{R}$. Set

$$\begin{aligned} E(u, n) &= 1 - u & &\text{for } n = 0, \\ &= (1 - u) \exp\{\textstyle\sum_{k=1}^{n} \frac{u^k}{k}\} & &\text{for } n = 1, 2, \ldots. \end{aligned}$$

Define a function φ by

$$\varphi(s) = \prod_{j=1}^{\infty} E(-\frac{s}{\alpha_j}, j).$$

Then, since $\sum_{j=1}^{\infty} (\frac{T}{\alpha_j})^j < \infty$ for arbitrary $T > 0$, φ is an entire function and the set of zero points of φ coincides with P ([3] p. 233). Let

$$\varphi_0(s) = \varphi(s) f(s).$$

Then φ_0 is an entire function with the set of zero points coinciding with Z. By Weierstrass's Factorization Theorem ([3] p. 234), there is an entire function g_0 such that φ_0 can be written as

$$\varphi_0(s) = e^{g_0(s)} \prod_{j=1}^{\infty} E(-\frac{s}{\beta_j}, j).$$

Hence,

$$f(s) = e^{g_0(s)} \prod_{j=1}^{\infty} E(-\frac{s}{\beta_j}, j) / \prod_{j=1}^{\infty} E(-\frac{s}{\alpha_j}, j).$$

This yields

$$f(s) = e^{g_0(s)} \prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}) \exp\Big[\sum_{k=1}^{j} \frac{(-s)^k}{k} \{(\beta_j)^{-k} - (\alpha_j)^{-k}\}\Big].$$

We have, for any positive integer M,

$$\prod_{j=1}^{M} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}) \exp\Big[\sum_{k=1}^{j} \frac{(-s)^k}{k} \{(\beta_j)^{-k} - (\alpha_j)^{-k}\}\Big]$$

$$= \{\prod_{j=1}^{M} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})\} \exp\Big[\sum_{j=1}^{M} \sum_{k=1}^{j} \frac{(-s)^k}{k} \{(\beta_j)^{-k} - (\alpha_j)^{-k}\}\Big].$$

If $|s| < \alpha_N$ and $M > N$, then

$$\sum_{j=N+1}^{M} \sum_{k=1}^{j} \frac{|s|^k}{k} \{(\alpha_j)^{-k} - (\beta_j)^{-k}\}$$

$$\leqq \sum_{k=1}^{\infty} \sum_{j=N+1}^{M} \frac{|s|^k}{k} \{(\alpha_j)^{-k} - (\alpha_{j+1})^{-k}\}$$

$$\leqq \sum_{k=1}^{\infty} \frac{1}{k} (|s|/\alpha_N)^k < \infty.$$

It follows that

$$g_1(s) = \sum_{j=1}^{\infty} \sum_{k=1}^{j} \frac{(-s)^k}{k} \{(\beta_j)^{-k} - (\alpha_j)^{-k}\}$$

is an entire function. By Theorem 6.2,

$$\prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})$$

is a meromorphic function. Hence $f(s)$ is written as

$$f(s) = e^{g(s)} \prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}),$$

where $g(s) = g_0(s) + g_1(s)$ is an entire function. For $s > 0$ let $A(s) = \log f(s)$ and $B(s) = \log \prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})$. Since $f(s) = \mathcal{L}\mu(s)$ for $s > 0$, we have

$$A(s) = \int_0^{\infty} (e^{-sx} - 1) \{\int_0^{\infty} e^{-xu} q(u) du\} dx,$$

for $s > 0$ where $0 \leqq q(u) \leqq 1$ a.e. and $\int_0^1 u^{-1} q(u) du < \infty$. Let

$$
\begin{aligned}
q_1(u) &= 0 \quad \text{for } 0 < \alpha_1, \\
&= 1 \quad \text{for } \alpha_j < u < \beta_j \quad j = 1,2,\ldots, \\
&= 0 \quad \text{for } \beta_j < u < \alpha_{j+1} \quad j = 1,2,\ldots.
\end{aligned}
$$

Since, by Theorem 6.1,

$$B(s) = \int_0^\infty (e^{-sx} - 1) \{\int_0^\infty e^{-xu} q_1(u) du\} dx,$$

for $s > 0$ we have

$$g(s) = A(s) - B(s) + C$$

and

$$A(s) - B(s) = \int_0^\infty \frac{s}{(s+u)u} (q_1(u) - q(u)) du,$$

where C is a constant satisfying $e^c = 1$. Since $(A(s) - B(s))/s$ is the Stieltjes transform of $(q_1(x) - q(x))x^{-1}dx$, $(q_1(x) - q(x))x^{-1}$ is obtained by the inversion formula for Stieltjes transform. Since $g(s)$ is an entire function, $(q_1(x) - q(x))x^{-1}dx$ can not have a mass in $(0, \infty)$. Hence

$$q_1(x) - q(x) = 0 \text{ a.e.}$$

and $g(s)$ is a constant C. Hence, we have

$$\mathscr{L}\mu(s) = \prod_{j=1}^\infty (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j}).$$

The proof is complete.

Remark 6.1. Let $\mu \in ME_+^d$ and let G be its mixing distribution. Let $\mathscr{L}\mu(s) = \prod_{j=1}^\infty (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})$. Then

$$G(\{\infty\}) = \prod_{j=1}^\infty \alpha_j/\beta_j$$

Proof. Let Q be the Q-measure of μ. Since, by Remark 3.1,

$$G(\{\infty\}) = \exp\{-\int_0^\infty \frac{1}{u} Q(du)\},$$

and since $-\int_0^\infty \frac{1}{u} Q(du) = \sum_{j=1}^\infty \log(\alpha_j/\beta_j)$, we get the conclusion.

Denote $CME_+^d = CE_+ * ME_+^d$.

THEOREM 6.4. *Let $\mu \in CME_+$. Suppose that its Laplace transform is represented as*

$$\mathscr{L}\mu(s) = \prod_{j=1}^{\infty} (1 + \frac{s}{\beta_j})/(1 + \frac{s}{\alpha_j})$$

where $\{\alpha_j\}$, $(\beta_j\}$ are disjoint divergent non-decreasing sequences of positive reals satisfying $\alpha_j \neq \beta_j$ for all i, j. Then,

(i) *there is a subsequence $\{\alpha_{n_j}\}$ of $\{\alpha_j\}$ such that*

$$0 < \alpha_{n_1} < \beta_1 < \alpha_{n_2} < \beta_2 < \cdots$$

and

(ii) $\sum_{\gamma \in \Gamma} \gamma^{-1} < \infty$ *for* $\Gamma = \{\alpha_j\}_{j=1}^{\infty} \setminus \{\alpha_{n_j}\}_{j=1}^{\infty}$.

Hence $\mu \in CME_+^d$.

Proof. If $\mu \in CME_+$, then there is $\mu_1 \in CE_+$, $\mu_2 \in ME_+$ such that $\mu = \mu_1 * \mu_2$ and there is a finite or infinite sequence $0 < \gamma_1 \leqq \gamma_2 \leqq \cdots$ s

$$\mathscr{L}\mu_1(s) = \prod_{j=1}^{\infty} 1/(1 + \frac{s}{\gamma_j}), \tag{6.8}$$

$$\sum 1/\gamma_n < \infty.$$

See [8]. Hence,

$$\mathscr{L}\mu_2(s) = \prod_{j=1}^{\infty} (1 + \frac{s}{\delta_j})/(1 + \frac{s}{\tau_j})$$

where

$$\{\delta_j\} = \{\beta_j\} \cup (\{\gamma_j\} \setminus \{\alpha_j\}),\ \{\tau_j\} = \{\alpha_j\} \setminus \{\gamma_j\},$$

$$0 < \tau_1 \leqq \tau_2 \leqq \cdots,$$

$$0 < \delta_1 \leqq \delta_2 \leqq \cdots.$$

We may assume that $\{\tau_j\}$ is an infinite sequence. Then δ_j, $\tau_j \to \infty$ as $j \to \infty$. By Theorems 6.1 and 2, we have

$$0 < \tau_1 < \delta_1 < \tau_2 < \delta_2 < \cdots$$

and $\mu_2 \in ME_+^d$. Hence $\mathscr{L}\mu(s)$ can be analytically continued to $\mathbf{C} \setminus \{-\alpha_1, -\alpha_2, \ldots\}$ and zero points of analytic continuation of $\mathscr{L}\mu(s)$ are contained in $\{\beta_j\}$. We have $\{\gamma_j\} \subset \{\alpha_j\}$, $\{\delta_j\} = \{\beta_j\}$ and we have (i) and (ii).

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