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UNITARY REPRESENTATIONS OF UNIPOTENT GROUPS ASSOCIATED WITH THETA SERIES

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1. Unipotent group of real $(g+2) \times (g+2)$-matrices

$$N_{g+2}(\mathbf{R}) = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} & \cdots & \cdots & \mathbf{R} \\ & 1 & \mathbf{R} & \cdots & \cdots & \mathbf{R} \\ & & & \ddots & & \vdots \\ & & & & 1 & \mathbf{R} \\ & & & & & 1 \end{pmatrix}$$

may be regarded as a split extension of $N_g(\mathbf{R})$ by Heisenberg group of real $(g+2) \times (g+2)$-matrices

$$\mathscr{H}_{g+2}(\mathbf{R}) = \begin{pmatrix} 1 & \mathbf{R} & \mathbf{R} & \cdots & \mathbf{R} & \mathbf{R} \\ & 1 & 0 & \cdots & 0 & \mathbf{R} \\ & & 1 & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & \mathbf{R} \\ & & & & 1 & \mathbf{R} \\ & & & & & 1 \end{pmatrix}.$$

We may choose a coordinate system of $N_{g+2}(\mathbf{R})$

$$(x_0, \hat{x}, x, \xi) \qquad (x_0 \in \mathbf{R}\,;\ \hat{x}, x \in \mathbf{R}^g\,;\ \xi \in N_g(\mathbf{R}))$$

which corresponds to

$$\begin{pmatrix} 1 & \hat{x} & x_0 \\ & \xi & {}^t x \\ & & 1 \end{pmatrix}.$$

From the matrix composition

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$$\begin{pmatrix} 1 & \hat{x} & x_0 \\ & \xi & {}^t x \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \hat{y} & y_0 \\ & \eta & {}^t y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \hat{x}\eta + \hat{y} & x_0 + y_0 + \hat{x}{}^t y \\ & \xi\eta & {}^t(x + y{}^t\xi) \\ & & 1 \end{pmatrix},$$

we obtain law of composition

(1) $(x_0, \hat{x}, x, \xi) \circ (y_0, \hat{y}, y, \eta) = (x_0 + y_0 + \hat{x}\,{}^t y, \hat{x}\eta + \hat{y}, x + y{}^t\xi, \xi\eta).$

We denote discrete subgroups of integral matrices

$$N_{g+2}(\mathbf{Z}) = \{(b_0, \hat{b}, b, \beta) \mid b_0 \in \mathbf{Z}\,;\, \hat{b}, b \in \mathbf{Z}^g, \beta \in N_g(\mathbf{Z})\},$$
$$N_{g+2}(\mathbf{Z}\,;\, n) = \{(b_0, \hat{b}, b, \beta) \in N_{g+2}(\mathbf{Z}) \mid \beta \equiv \mathrm{I} \bmod n\}.$$

By means of right action of $N_{g+2}(\mathbf{R})$ the complex L_2-spaces with normalized Haar measure

$$L_2(N_{g+2}(\mathbf{Z})\backslash N_{g+2}(\mathbf{R})), \quad L_2(N_{g+2}(\mathbf{Z}\,;\, n)\backslash N_{g+2}(\mathbf{R}))$$

are spaces of unitary representations of $N_{g+2}(\mathbf{R})$.

In the present note using the above coordinate system $(x_0, \hat{x}, x, \xi)$ of $N_{g+2}(\mathbf{R})$, we shall construct irreducible invariant spaces in $L_2(N_{g+2}(\mathbf{Z}\,;\, n) \backslash N_{g+2}(\mathbf{R}))$, which are associated with theta series of level n.

2. We use the following notations freely:

$\mathbf{Z}_{\geq 0} = \{$non-negative integer$\}$

$\mathbf{Z}^g_{\geq 0} = \{j = (j_1, \ldots, j_g) \mid j_i \in \mathbf{Z}_{\geq 0}\}$

$|j| = j_1 + j_2 + \cdots + j_g$

$\varepsilon_i = (\overbrace{0, \ldots, 0}^{i-1}, 1, \overbrace{0, \ldots, 0}^{g-i})$

τ = a symmetric complex $g \times g$-matrix with positive definite imaginary part,

$z = \hat{x} + x\tau, \quad \bar{z} = \hat{x} + x\bar{\tau}.$

The vector space of theta series of level n has a basis

(2) $$\bullet^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid z) = \sum_{\ell \in \mathbf{Z}^g} \exp[\pi n \sqrt{-1}\{(\ell + \tfrac{a}{n})\tau{}^t(\ell + \tfrac{a}{n}) + 2z{}^t(\ell + \tfrac{a}{n})\}]$$
$$(a \in \mathbf{Z}^g/n\mathbf{Z}^g).$$

To each theta series we associate a family of real analytic functions on $N_{g+2}(\mathbf{R})$:

(3) $$\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$= \exp[-2\pi n\sqrt{-1}x_0] \sum_{\ell\in\mathbf{Z}^g} ((x+\ell+\frac{a}{n})\,{}^t\xi^{-1})^j$$

$$\exp[\pi n\sqrt{-1}\{(x+\ell+\frac{a}{n})\,{}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}) + 2\hat{x}\xi^{-1}(x+\ell+\frac{a}{n})\}]$$

$$(a \in \mathbf{Z}^g/n\mathbf{Z}^g,\ j \in \mathbf{Z}^g_{\geq 0},\ n \geq 1)$$

PROPOSITION 1.

$$(4)\quad \Phi_j^{(n)}\begin{bmatrix}\frac{a}{n}\\ 0\end{bmatrix}(\tau \mid (b_0, \hat{b}, b, \beta)\circ(x_0, \hat{x}, x, \xi))$$

$$= \Phi_j^{(n)}\begin{bmatrix}\frac{a}{n}\,{}^t\beta^{-1}\\ 0\end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi)\ \ ((b_0, \hat{b}, b, \beta) \in N_{g+2}(\mathbf{Z})).$$

Proof. For each $(b_0, \hat{b}, b, \beta) \in N_{g+2}(\mathbf{Z})$, we have

$$\Phi_j^{(n)}\begin{bmatrix}\frac{a}{n}\\ 0\end{bmatrix}(\tau \mid (b_0, \hat{b}, b, \beta)\circ(x_0, \hat{x}, x, \xi))$$

$$= \Phi_j^{(n)}\begin{bmatrix}\frac{a}{n}\\ 0\end{bmatrix}(\tau \mid (b_0+x_0+\hat{b}\,{}^tx,\ \hat{b}\xi+\hat{x},\ b+x\,{}^t\beta,\ \beta\xi))$$

$$= \exp[-2\pi n\sqrt{-1}(b_0+x_0+\hat{b}\,{}^tx)] \sum_{\ell\in\mathbf{Z}^g} ((x\,{}^t\beta+b+\ell+\frac{a}{n})\,{}^t(\beta\xi)^{-1})^j$$

$$\exp[\pi n\sqrt{-1}\{(x\,{}^t\beta+b+\ell+\frac{a}{n})\,{}^t(\beta\xi)^{-1}\tau(\beta\xi)^{-1}\,{}^t(x\,{}^t\beta+b+\ell+\frac{a}{n}) + 2(\hat{x}+\hat{b}\xi)\xi^{-1}\,{}^t(x\,{}^t\beta+b+\ell+\frac{a}{n})\}]$$

$$= \exp[-2\pi n\sqrt{-1}x_0]\exp[-2\pi n\sqrt{-1}\,{}^tx] \sum_{\ell\in\mathbf{Z}^g} ((x+\ell+\frac{a}{n}\,{}^t\beta^{-1})\,{}^t\xi^{-1})^j$$

$$\exp[\pi n\sqrt{-1}(x+\ell+\frac{a}{n}\,{}^t\beta^{-1})\,{}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}\,{}^t\beta^{-1})]$$

$$\exp[2\pi n\sqrt{-1}\hat{x}\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}\,{}^t\beta^{-1}\exp[2\pi n\sqrt{-1}\hat{b}\,{}^tx]$$

$$= \Phi_j^{(n)}\begin{bmatrix}\frac{a}{n}\,{}^t\beta^{-1}\\ 0\end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi).$$

3. Right invariant vector fields

$$D_0 = -\frac{\partial}{\partial x_0},\quad \hat{D}_i = \frac{\partial}{\partial \hat{x}_i},\quad D_i = \hat{x}_i\frac{\partial}{\partial x_0} + \frac{\partial}{\partial \hat{x}_i}\quad (1 \leq i \leq g)$$

on $\mathscr{H}_{g+2}(\mathbf{R})$ are naturally extended to right invariant vector fields on $N_{g+2}(\mathbf{R})$ as follows,

(5) $$D_0 = -\frac{\partial}{\partial x_0}, \quad \hat{D}_i = \frac{\partial}{\partial \hat{x}_i}, \quad D_i = \hat{x}_i \frac{\partial}{\partial x_0} + \sum_{p=1}^{g} \xi_{pi} \frac{\partial}{\partial x_p} \quad (1 \le i \le g),$$

because

$$-D_0 f(x_0, \hat{x}, x, \xi) = \frac{f((x_0, \hat{x}, x, \xi) \circ (s, 0, 0, I)) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0}$$

$$= \frac{f(x_0 + s, \hat{x}, x, \xi) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0} = \frac{\partial}{\partial x_0} f(x_0, \hat{x}, x, \xi),$$

$$\hat{D}_i f(x_0, \hat{x}, x, \xi) = \frac{f((x_0, \hat{x}, x, \xi) \circ (0, s\varepsilon_i, 0, I)) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0}$$

$$= \frac{f(x_0, \hat{x} + s\varepsilon_i, x, \xi) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0} = \frac{\partial}{\partial \hat{x}_i} f(x_0, \hat{x}, x, \xi),$$

$$D_i f(x_0, \hat{x}, x, \xi) = \frac{f((x_0, \hat{x}, x, \xi) \circ (0, 0, s\varepsilon_i, I)) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0}$$

$$= \frac{f(x_0 + s\hat{x}_i, \hat{x}, x + s\varepsilon_i {}^t\xi, \xi) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0}$$

$$= \Big(\hat{x}_i \frac{\partial}{\partial x_0} + \sum_{q=1}^{g} \xi_{qi} \frac{\partial}{\partial x_p}\Big) f(x_0, \hat{x}, x, \xi).$$

LEMMA 1.

(6) $$D_i(x + \ell + \frac{a}{n})^t\xi^{-1} = \varepsilon_i \qquad (1 \le i \le g).$$

Proof. Denoting $\xi^{-1} = (\xi^*_{ik})$ we have

$$D_i(x + \ell + \frac{a}{n})^t\xi^{-1} = \Big(\sum_{p=1}^{g} \xi_{pi} \frac{\partial}{\partial x_p}\Big) x^t\xi^{-1}$$

$$= \Big(\sum_{p=1}^{g} \xi_{pi}\xi^*_{1p}, \cdots, \sum_{p=1}^{g} \xi_{pi}\xi^*_{gp}\Big) = \varepsilon_i.$$

PROPOSITION 2.

(7) $$D_0 \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) = 2\pi n\sqrt{-1}\, \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi),$$

(8) $$\hat{D}_i \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) = 2\pi n\sqrt{-1}\, \Phi_{j+\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi),$$

(9) $$D_i \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) = j_i \Phi_{j-\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi),$$

$$+ 2\pi n\sqrt{-1} \sum_{p=1}^{g} \tau_{ip} \Phi_{j+\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi),$$

$$(a \in \mathbf{Z}^g/n\mathbf{Z}^g, \quad j \in \mathbf{Z}^g_{\geq 0}, \quad n \geq 1, \quad 1 \leq i \leq g).$$

Proof. From the definition of $\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$ and Lemma 1 we have

$$D_0\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) = \frac{\partial}{\partial x_0} \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$= 2\pi n\sqrt{-1}\,\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi),$$

$$\hat{D}_i\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) = \frac{\partial}{\partial \hat{x}_i} \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$= \exp[-2\pi n\sqrt{-1}x_0] \sum_{\ell \in \mathbf{Z}^g} ((x + \ell + \frac{a}{n})\,{}^t\xi^{-1})^j \exp[\pi n$$

$$\sqrt{-1}(x + \ell + \frac{a}{n})\,{}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x + \ell + \frac{a}{n})\, \frac{\partial}{\partial \hat{x}_i} \exp[2\pi n\sqrt{-1}\hat{x}\xi^{-1}\,{}^t(x + \ell + \frac{a}{n})]$$

$$= 2\pi n\sqrt{-1} \exp[-2\pi n\sqrt{-1}x_0] \sum_{\ell \in \mathbf{Z}^g} ((x + \ell + \frac{a}{n})\,{}^t\xi^{-1})^{j+\varepsilon_i}$$

$$\exp[\pi n\sqrt{-1}\{(x + \ell + \frac{a}{n})\,{}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x + \ell + \frac{a}{n})\}]$$

$$= 2\pi n\sqrt{-1}\,\Phi_{j+\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$D_i\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) = (\hat{x}_i \frac{\partial}{\partial x_0} + \sum_{p=1}^{g} \xi_{pi} \frac{\partial}{\partial x_p})\, \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$= -2\pi n\sqrt{-1}\hat{x}_i\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) + \exp[-2\pi n\sqrt{-1}x_0]$$

$$\sum_{\ell \in \mathbf{Z}^g} j_i((x + \ell + \frac{a}{n})\,{}^t\xi^{-1})^{j-\varepsilon_i} \exp[\pi n\sqrt{-1}\{(x + \ell + \frac{a}{n})\,{}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x + \ell + \frac{a}{n})$$

$$+ 2\hat{x}\xi^{-1}\,{}^t(x + \ell + \frac{a}{n})\}]$$

$$+ \sum_{p=1}^{g} 2\pi n\sqrt{-1}\,\tau_{ip} \exp[-2\pi n\sqrt{-1}x_0] \sum_{\ell \in \mathbf{Z}^g} ((x + \ell + \frac{a}{n})\,{}^t\xi^{-1})^{j+\varepsilon_p}$$

$$\exp[\pi n\sqrt{-1}\{(x + \ell + \frac{a}{n})\,{}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x + \ell + \frac{a}{n}) + 2\hat{x}\xi^{-1}\,{}^t(x + \ell + \frac{a}{n})\}]$$

$$+ 2\pi n\sqrt{-1}\hat{x}_i\Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$= j_i\Phi_{j-\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) + 2\pi n\sqrt{-1} \sum_{p=1}^{g} \tau_{ip}\Phi_{j+\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi).$$

PROPOSITION 3.

$$(10)\quad (D_i - \sum_{p=1}^{g} \tau_{ip}\hat{D}_i)\Phi_j^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi) = j_i\Phi_{j-\varepsilon_i}^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi).$$

This is a direct consequence of (8) and (9).
Since $N_{g+2}(\mathbf{Z}\,;\,n)\backslash N_{g+2}(\mathbf{R})$ is compact, Proposition 1 means

$$\Phi_j^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi) \in L_2(N_{g+2}(\mathbf{Z} : n)\backslash N_{g+2}(\mathbf{R}))$$
$$(a \in \mathbf{Z}^g/n\mathbf{Z}^g\,;\, j \in \mathbf{Z}^g_{\geq 0}).$$

THEOREM. *We denote by* $K^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}$ *the completion of the vector spanned by*

$$\Phi_j^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi) \qquad (j \in \mathbf{Z}^g_{\geq 0})$$

in $L_2(N_{g+2}(\mathbf{Z}\,;\,n)\backslash N_{g+2}(\mathbf{R}))$. *Then* $K^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}$ *is an irreducible invariant subspace wtih respect to the right action of* $N_{g+2}(\mathbf{R})$ *on* $N_{g+2}(\mathbf{Z}\,;\,n)\backslash N_{g+2}(\mathbf{R})$.

Proof. From Proposition 2 $K^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}$ is an invariant subspace with respect to subgroup $\mathscr{H}_{g+2}(\mathbf{R})$. From Proposition 3 $K^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}$ is generated by a primitive element

$$\Phi_0^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}(\tau \mid x_0, \hat{x}, x, \xi),$$

hence $K^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}$ is irreducible with respect to $\mathscr{H}_{g+2}(\mathbf{R})$. It is sufficient to prove that $K^{(n)}\begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}$ is invariant with respect to subgroup $N_g(\mathbf{R})$. For each element A of Lie algebra of $N_g(\mathbf{R})$ we denote

$$D_A f(x_0, \hat{x}, x, \xi) = \frac{f((x_0, \hat{x}, x, \xi)\circ(0,0,0,\exp(sA))) - f(x_0, \hat{x}, x, \xi)}{s}\Big|_{s=0}$$

$$= \frac{f(x_0, \hat{x}\exp(sA), x, \xi\exp(sA)) - f(x_0, \hat{x}, x, \xi)}{s}\Bigg|_{s=0}.$$

It is enough to show

$$D_A \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) \in K^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix}.$$

Since ther exist constants λ_{jk}, μ_{jh} depending on A and τ such that

$$((x+\ell+\frac{a}{n}){}^t\xi^{-1}(-A))^j = \sum_{|k|=|j|} \lambda_{jk}((x+\ell+\frac{a}{n}){}^t\xi^{-1})^k,$$

$$2\pi n\sqrt{-1}((x+\ell+\frac{a}{n}){}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}))\ ((x+\ell+\frac{a}{n}){}^t\xi^{-1})^j$$

$$= \sum_{|h|=|j|+2} \mu_{jh}((x+\ell+\frac{a}{n}){}^t\xi^{-1})^h,$$

$$D_A \Phi_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi)$$

$$= \frac{1}{s}\Big[\{\exp[-2\pi n\sqrt{-1}x_0] \sum_{\ell\in\mathbf{Z}^g} ((x+\ell+\frac{a}{n}){}^t\xi^{-1}\exp(-s{}^tA))^j$$

$$\exp[\pi n\sqrt{-1}(x+\ell+\frac{a}{n}){}^t\xi^{-1}$$

$$\exp(-s{}^tA)\tau\exp(-sA)\xi^{-1}\,{}^t(x+\ell+\frac{a}{n})]\ \exp[2\pi n\sqrt{-1}\hat{x}\xi^{-1}\,{}^t(x+\ell+\frac{a}{n})]$$

$$-\exp[-2\pi n\sqrt{-1}x_0] \sum_{\ell\in\mathbf{Z}^g} ((x+\ell+\frac{a}{n}){}^t\xi^{-1})^j \exp[\pi n\sqrt{-1}$$

$$\{(x+\ell+\frac{a}{n}){}^t\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}) + 2\hat{x}\xi^{-1}\,{}^t(x+\ell+\frac{a}{n})\}]\Big]\Big|_{s=0}$$

$$= \sum_{|k|=|j|} \lambda_{jk}\exp[-2\pi n x_0] \sum_{\ell\in\mathbf{Z}^g} ((x+\ell+\frac{a}{n})\ {}^t\xi^{-1})^k$$

$$\exp[\pi n\sqrt{-1}\{(x+\ell+\frac{a}{n}){}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}) + 2\hat{x}\xi^{-1}\,{}^t(x+\ell+\frac{a}{n})\}]$$

$$+ \sum_{|h|=|j|+2} \mu_{jk}\exp[-2\pi n x_0] \sum_{\ell\in\mathbf{Z}^g} ((x+\ell+\frac{a}{n})\ {}^t\xi^{-1})^h$$

$$\exp[\pi n\sqrt{-1}\{(x+\ell+\frac{a}{n}){}^t\xi^{-1}\tau\xi^{-1}\,{}^t(x+\ell+\frac{a}{n}) + 2\hat{x}\xi^{-1}\,{}^t(x+\ell+\frac{a}{n})\}]$$

$$= \sum_{|k|=|j|} \lambda_{jk}\Phi_k^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi) + \sum_{|h|=|j|+2} \mu_{jh}\Phi_h^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid x_0, \hat{x}, x, \xi).$$

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