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# RESIDUAL AUTOMORPHIC REPRESENTATIONS OF $Sp_4$

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## Introduction

Let $G = Sp_4$ be the symplectic group of degree two defined over an algebraic number field $F$ and $K$ the standard maximal compact subgroup of the adele group $G(\mathbf{A})$. By the general theory of Eisenstein series ([14]), one knows that the Hilbert space $L^2(G(F)\backslash G(\mathbf{A}))$ has an orthogonal decomposition of the form

$$L^2(G(F)\backslash G(\mathbf{A})) = L^2(G) \oplus L^2(B) \oplus L^2(P_1) \oplus L^2(P_2),$$

where $B$ is a Borel subgroup and $P_i$ are standard maximal parabolic subgroups in $G$ for $i = 1,2$. The purpose of this paper is to study the space $L^2_d(B)$ associated to discrete spectrums in $L^2(B)$.

In order to obtain such discrete spectrums, we follow Langlands' way. To be more precise, let $T = \{t(a, b) = \mathrm{diag}(a, b, a^{-1}, b^{-1})\}$ be a maximal split torus and $B = NT$ a Levi decomposition. For any *quadratic* character $\mu$ of $F^*\backslash \mathbf{A}^*$, the character $\chi(\mu, \mu)$ of $T(\mathbf{A})$ is defined to be $\chi(\mu, \mu)(t(a, b)) = \mu(ab)$ for $t(a, b) \in T(\mathbf{A})$. Let $I(\chi(\mu, \mu)//\mathbf{A})$ be the space of functions $\Phi$ on $G(\mathbf{A})$ satisfying $\Phi(ntg) = \chi(\mu, \mu)(t)\Phi(g)$ for any $n \in N(\mathbf{A})$, $t \in T(\mathbf{A})$ and $g \in G(\mathbf{A})$, and let

$$I(\mu, \beta_1) = \mathrm{Ind}\,(B(\mathbf{A}) \uparrow G(\mathbf{A}); e^{\langle\beta_1, H(\cdot)\rangle}\chi(\mu, \mu))$$

be the normalized induced representation of $G(\mathbf{A})$, where $\beta_1$ is a fundamental weight of $G$. For an admissible vector $\Phi \in I(\chi(\mu, \mu)//\mathbf{A})$, one can define the Eisenstein series $E(g, \Phi, \Lambda)$ and take its iterated residue $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$, $E^1(g, \Phi, \Lambda) = \mathrm{Res}_{\langle\Lambda, \alpha_1^\vee\rangle = 1} E(g, \Phi, \Lambda)$. Through this procedure, one obtains a mapping

$$\mathcal{R}_\mu : \Phi \longmapsto \mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$$

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from the space of admissible vectors in $I(\mu, \beta_1)$ into the space of automorphic forms on $G(\mathbf{A})$. In this situation, the following are conjectured.

CONJECTURE. *For each nontrivial quadratic character $\mu$ of $F^*\backslash \mathbf{A}^*$,*

(1) *$\mathfrak{R}_\mu$ is nontrivial.*

(2) *The image of $\mathfrak{R}_\mu$ is contained in $L^2_d(B)$, so that $\mathfrak{R}_\mu$ is extended to an intertwining operator from $I(\mu, \beta_1)$ into $L^2_d(B)$.*

(3) *$\pi(\mu)$, the image of this intertwining operator, is irreducible and is of multiplicity one in $L^2_d(B)$.*

On condition that these conjectures are true, it seems $L^2_d(B)$ has an irreducible decomposition $L^2_d(B) = \pi(\mu_0) \oplus (\oplus_\mu \pi(\mu))$, where $\mu$ runs over all nontrivial quadratic characters of $F^*\backslash \mathbf{A}^*$ and $\pi(\mu_0)$ denotes the space of constant functions. In this paper, we will show that a part of the conjecture is true for $\mu$ with "square free conductor".

Now we explain contents of this paper. Let $S$ be a finite set of finite places of $F$. For $v \in S$, $\mathbf{k}_v$ denotes the residual field of $F_v$. Let $r_v : K_v \to Sp_4(\mathbf{k}_v)$ be the reduction homomorphism for $v \in S$ and $K_S = \{(k_v) \in K \mid k_v \in \mathrm{Ker}(r_v)$ for $v \in S\}$ a normal subgroup of $K$. We study the subspace $L^2_d(B, K_S)$ consisting of $K_S$-invariant elements of $L^2_d(B)$, which becomes naturally a representation space of the finite group $K/K_S$. Thus we are interested in an irreducible decomposition of $L^2_d(B, K_S)$.

First, it is not hard to show that there is a decomposition

$$L^2_d(B, K_S) = \bigoplus_{\mu \in A_S(F)} L^2_d(B, K_S, X(\mu))$$

(corollary to Proposition 1). Here, $A_S(F)$ is the set of characters $\mu$ of $F^*\backslash \mathbf{A}^1$ such that the corresponding $\chi(\mu, \mu)$ are characters of $T(F) \backslash T^1/(K_S \cap T(A))$ of order at most 2 and $L^2_d(B, K_S, X(\mu))$ is the space generated by residues of $E^1(g, \Phi, \Lambda)$ for $K_S$-invariant elements $\Phi \in I(\chi(\mu, \mu)//\mathbf{A})$.

Next, by further calculations of residues of Eisenstein series, it is shown that $\mu = \mu_0$ is the trivial character then $L^2_d(B, K_S, X(\mu_0))$ consists of constant functions (Theorem 2) and if $\mu$ is nontrivial then one has an irreducible decomposition

$$L^2_d(B, K_S, X(\mu)) = \bigoplus_{\lambda \in \Gamma(S,\mu)} L^2_d(B, K_S, X(\mu))_\lambda \cong \bigoplus_{\lambda \in \Gamma(S,\mu)} \lambda_S(\mu)$$

(Theorem 1). Each $\lambda_S(\mu)$ is a $K/K_S$-irreducible subspace in $I(\chi(\mu, \mu)//\mathbf{A})$ and $\Gamma(S, \mu)$ is a certain subset of the set of all maps from $S$ to the two points set $\{0,1\}$. Isomorphisms $L^2_d(B, K_S, X(\mu))_\lambda \cong \lambda_S(\mu)$ are derived from the constant term map. Irreducible representations of $Sp_4(\mathbf{k}_v)$, $v \in S$ occurring in a tensor product decomposition of $\lambda_S(\mu)$ are described by using the labelling of Enomoto ([7]) and Srinivasan ([20]) (Lemma 7).

Using these results, we can classify those automorphic representations realized in $L_d^2(B)$ which have (non-zero) vectors fixed by $K_S$ for some $S$. Let $A_\infty(F)$ be the union of $A_S(F)$ for all finite sets $S$ of finite places and $I^1(\mu, \beta_1)$ for $\mu \in A_\infty(F)$ the $G(\mathbf{A})$ module generated by all vectors in $I(\mu, \beta_1)$ fixed by $K_S$ for some $S$. Then Theorem 1 implies that there is a nontrivial intertwining operator $\mathcal{R}_\mu$ from $I^1(\mu, \beta_1)$ into $L_d^2(B)$ for each nontrivial $\mu \in A_\infty(F)$. Further, it is known by [3. Corollary 3.3.7] and [6. Proposition 3, 4° (b)] that $I^1(\mu, \beta_1)$ coincides with $I(\mu, \beta_1)$ if $F$ is totally imaginary. Unfortunately, we have been unable to prove the irreduciblity of $\pi(\mu)$, the image of $\mathcal{R}_\mu$. However, we can show that the number of irreducible constituents of $\pi(\mu)$ is at most $2^{|S_r(\mu)|}$ (Theorem 3), where $S_r(\mu)$ is the set of finite places $v$ such that $\mu_v$ is ramified, and each irreducible constituent of $\pi(\mu)$ is of multiplicity one in $L_d^2(B)$ (Theorem 4). The last assertion is derived from the fact that each $K$-type $\lambda_S(\mu)$ is of multiplicity one in $L_d^2(B, K_S)$.

The major part of this paper will be devoted to calculations of residues of Eisenstein series. It will be carried out in Sections 2, 3, 5 and 7. Main results are stated in Sections 6, 7 and 8. In Section 4, we will recall the representation theory of the finite group $Sp_4(\mathbf{F}_q)$, which we need for precise expressions of intertwining operators occuring in the constant terms of Eisenstein series.

## 1. Preliminaries

Let $F$ be an algebraic number field of finite degree over $\mathbf{Q}$. For each place $v$ of $F$, let $F_v$ be the completion of $F$ at $v$ and $|\cdot|_v$ the normalized absolute value on $F_v$. $V_f$ denotes the set of all finite places of $F$. For $v \in V_f$ let $\mathcal{O}_v$ be the valuation ring of $F_v$, $\mathcal{P}_v = p_v\mathcal{O}_v$ the maximal ideal of $\mathcal{O}_v$ and $\mathbf{k}_v = \mathcal{O}_v/\mathcal{P}_v$ the residual field. Let $\mathbf{A}$ be the adele ring of $F$, $|\cdot|_\mathbf{A} = \prod_v |\cdot|_v$ the idele norm and $\mathbf{A}^1$ the group consisting of ideles with idele norm one. The infinite part of $\mathbf{A}$ is denoted by $\mathbf{A}_\infty$.

We fix, once and for all, a finite subset $S$ of $V_f$. Let $U_S$ be the compact subgroup consisting of ideles $(a_v) \in \mathbf{A}^1$ such that $|a_v|_v = 1$ for all places $v$ and $a_v \in 1 + \mathcal{P}_v$ for any $v \in S$. $\Omega_S$ denotes the Pontrjagin dual of the compact group $F^*\backslash \mathbf{A}^1/U_S$.

Take an element $\mu \in \Omega_S$. If we decompose $\mu$ to the product of characters $\mu_v$ of $F_v^*$, then, by definition, $\mu_v$ is trivial on $\mathcal{O}_v^*$ or $1 + \mathcal{P}_v$ according as $v \notin S$ or $v \in S$. Hence, for each $v \in S$, the restriction of $\mu_v$ to $\mathcal{O}_v^*$ induces the character ${}^*\mu_v$ of $\mathbf{k}_v^*$. We denote by $\xi(z, \mu)$ the Hecke $L$-function of $\mu$ with the ordinary $\Gamma$ factor so that it satisfies the functional equation $\xi(z, \mu) = \varepsilon(\mu)\xi(1 - z, \mu^{-1})$. If $\mu$ is the trivial character $\mu_0$, then we write simply $\xi(z)$ for $\xi(z, \mu_0)$. The residue of $\xi(z)$ at $z = 1$ is denoted by $c(F)$.

For an algebraic group $\underline{G}$ defined over $F$ and an $F$-algebra $A$, $\underline{G}(A)$ denotes the group of $A$-rational points of $\underline{G}$.

Let $G = Sp_4$ be the symplectic group of degree two, that is

$$G(F) = \{g \in GL_4(F) \mid g\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}{}^t g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $T$ and $N$ be a maximal split torus and a maximal unipotent subgroup of $G$, respectively, as follows:

$$T(F) = \{t(a, b) = \mathrm{diag}(a, b, a^{-1}, b^{-1}) \in G(F)\}$$

$$N(F) = \left\{\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} I & S \\ 0 & I \end{pmatrix} \in G(F) \mid A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, S = {}^tS\right\}$$

Then $B = TN$ is a Borel subgroup in $G$.

Let $X(T)$ (resp. $X^*(T)$) be the character (resp. cocharacter) group of $T$. There is a natural pairing $\langle\, , \rangle : X(T) \times X^*(T) \to \mathbf{Z}$. We take $\alpha_1, \alpha_2 \in X(T)$ such that $\alpha_1(t(a, b)) = ab^{-1}$ and $\alpha_2(t(a, b)) = b^2$, Then $\{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2\}$ (resp. $\{\alpha_1, \alpha_2\}$) is the set of positive roots (resp. simple roots) of $G$ with respect to $(B, T)$. Further, $\beta_1 = \alpha_1 + \alpha_2/2$ and $\beta_2 = \alpha_1 + \alpha_2$ are the fundamental weights of $G$ with respect to $(B, T)$. The coroot corresponding to $\alpha_i$ is denoted by $\alpha_i^\vee$ for $1 \leq i \leq 4$. Since $G$ is simply connected, one has $X(T) = \mathbf{Z}\beta_1 + \mathbf{Z}\beta_2$ and $X^*(T) = \mathbf{Z}\alpha_1^\vee + \mathbf{Z}\alpha_2^\vee$. Set $\mathfrak{a} = X(T) \otimes \mathbf{R}, \mathfrak{a}_\mathbf{C} = \mathfrak{a} + \sqrt{-1}\ \mathfrak{a}$, $\mathfrak{a}^* = X^*(T) \otimes \mathbf{R}$ and $\mathfrak{a}_\mathbf{C}^* = \mathfrak{a}^* + \sqrt{-1}\,\mathfrak{a}^*$. Then $\{\beta_1, \beta_2\}$ and $\{\alpha_1^\vee, \alpha_2^\vee\}$ is the pair of dual bases for $\mathfrak{a}$ and $\mathfrak{a}^*$. The set $C^+ = \{a\beta_1 + b\beta_2 \mid a, b \in \mathbf{R}_+\}$ is the Weyl chamber in $\mathfrak{a}$ corresponding to $(B, T)$. Let $\sigma$ (resp. $\tau$) be the reflection with respect to the line $\mathbf{R}\beta_2$ (resp. $\mathbf{R}\beta_1$) in $\mathfrak{a}$. Then the Weyl group $W$ of $G$ is generated by $\sigma, \tau$.

Let $K_\infty$ be the standard maximal compact subgroup in $G(\mathbf{A}_\infty)$ and $K_v = G(\mathcal{O}_v)$ for $v \in V_f$. The product $K = K_\infty \times \prod K_v$ is a maximal compact subgroup in $G(\mathbf{A})$. For $v \in S$, let $r_v : K_v \to G(\mathbf{k}_v)$ be the reduction homomorphism. The compact group $K_S = K_\infty \times \prod_{v \in V_f - S} K_v \times \prod_{v \in S} \mathrm{Ker}(r_v)$ is a normal subgroup of $K$.

The homomorphism $H$ from $T(\mathbf{A})$ onto $\mathfrak{a}^*$ is defined to be $H(t(a, b)) = \log|a|_\mathbf{A}\alpha_1^\vee + \log|ab|_\mathbf{A}\alpha_2^\vee$. We set $T^1 = \mathrm{Ker}(H)$ and $T(\mathbf{R})_+ = \{t(a, b) \mid a, b \in \mathbf{R}_+\}$. Then $T(\mathbf{R})_+$ is diagonally embedded in $T(\mathbf{A}_\infty)$ and $T(\mathbf{A})$ has a direct product decomsosition $T(\mathbf{A}) = T(\mathbf{R})_+T^1$. Thus the map $\mathfrak{a}_\mathbf{C} \to$ Hom

$(T(\mathbf{A})/T^1, \mathbf{C}^*) : \Lambda \mapsto e^{<\Lambda, H(\cdot)>}$ becomes an isomorphism.

If we define the character $\chi(\mu, \nu)$ of $T(\mathbf{A})$ by $\chi(\mu, \nu)(t(a, b)) = \mu(a)\nu(b)$ for $\mu, \nu \in \Omega_S$, $t(a, b) \in T(\mathbf{A})$, then the correspondence $(\mu, \nu) \mapsto \chi(\mu, \nu)$ gives a bijection from $\Omega_S \times \Omega_S$ to $\Omega_S(T) = \mathrm{Hom}(T(F) \setminus T^1/(K_S \cap T(\mathbf{A})), \mathbf{C}^*)$. For $\chi \in \Omega_S(T)$ and a place $v$, $v$-component of $\chi$ is denoted by $\chi_v$. If $v \in S$, the restriction of $\chi_v$ to $T(\mathcal{O}_v)$ induces the character $^*\chi_v$ of $T(\mathbf{k}_v)$. The Weyl group $W$ naturally acts on $\Omega_S(T)$. The set of $W$-orbits in $\Omega_S(T)$ is denoted by $\Omega_S(T)/W$.

The space of square integrable functions on $N(\mathbf{A})T(\mathbf{R})_+T(F)\setminus G(\mathbf{A})/K_S$ has an orthogonal decomposition

$$L^2(N(\mathbf{A})T(\mathbf{R})_+T(\mathrm{F})\setminus G(\mathbf{A})/K_S) = \bigoplus_{\chi \in \Omega_S(T)} I(\chi//\mathbf{A})_S.$$

Here, for $\chi \in \Omega_S(T)$, let $I(\chi//\mathbf{A})$ be the space of all functions $\Phi$ on $G(\mathbf{A})$ satisfying $\Phi(ntg) = \chi(t)\Phi(g)$ for any $n \in N(\mathbf{A})$, $t \in T(\mathbf{A})$ and $g \in G(\mathbf{A})$ and $I(\chi//\mathbf{A})_S$ the subspace of right $K_S$ invariant elements in $I(\chi//\mathbf{A})$.

For a moment, we fix a $\chi \in \Omega_S(T)$. Let $I(\chi_v//F_v)$ for each $v$ denote the space of functions $\Phi_v$ on $G(F_v)$ satisfying $\Phi_v(n_v t_v g_v) = \chi_v(t_v)\Phi_v(g_v)$ for any $n_v \in N(F_v)$, $t_v \in T(F_v)$ and $g_v \in G(F_v)$. If $v \notin S$, $I(\chi_v//F_v)$ contains a unique spherical function $e(\chi_v)$ such that $e(\chi_v)(1_4) = 1$. Then one has a restricted tensor product decomposition

$$I(\chi//\mathbf{A}) = \bigotimes_v I(\chi_v//F_v)$$

with respect to $e(\chi_v)$, $v \notin S$. Particularly, each element $\Phi \in I(\chi//\mathbf{A})_S$ can be written as $(\otimes_{v\in S}\Phi_v) \otimes (\otimes_{v\notin S} e(\chi_v))$, where $\Phi_v \in I(\chi_v//F_v)$ is right $\mathrm{Ker}(r_v)$ invariant for each $v \in S$. Further, let $I(^*\chi_v//\mathbf{k}_v)$ for each $v \in S$ be the representation space of $G(\mathbf{k}_v)$ induced by the character trivially extended to $B(\mathbf{k}_v)$ from $^*\chi_v$. Then, for a right $\mathrm{Ker}(r_v)$ invariant element $\Phi_v \in I(\chi_v//F_v)$, there is a unique $^*\Phi_v \in I(^*\chi_v//\mathbf{k}_v)$ such that $\Phi_v(k_v) = {}^*\Phi_v(r_v(k_v))$ for any $k_v \in K_v$. By the Iwasawa decomposition $G(F_v) = B(F_v)K_v$, the correspondence $\Phi_v \mapsto {}^*\Phi_v$ gives a bijection from the subspace of right $\mathrm{Ker}(r_v)$ invariant elements in $I(\chi_v//F_v)$ to $I(^*\chi_v//\mathbf{k}_v)$. Hence one has the bijection

$$\text{(1.1)} \qquad I(\chi_v//\mathbf{A})_S \to \bigotimes_{v\in S} I(^*\chi_v//\mathbf{k}_v) : \Phi = (\otimes_{v\in S}\Phi_v) \otimes (\otimes_{v\notin S}\ e(\chi_v)) \mapsto {}^*\Phi = \otimes_{v\in S}{}^*\Phi_v.$$

For $\varphi = \sum_i f_i \otimes \Phi_i \in C_c^\infty(\mathfrak{a}^*) \otimes L^2(N(\mathbf{A})T(\mathbf{R})_+T(F)\setminus G(\mathbf{A})/K_S)$, the incomplete theta series

$$\varphi^\wedge(g) = \sum_i \sum_{\gamma \in B(F)\setminus G(F)} f_i(H(\gamma g))\Phi_i(\gamma g)$$

converges and belongs to the Hilbert space $L^2(G(F)\backslash G(\mathbf{A})/K_S)$. Let $L^2(B, K_S)$ be the closure of $\{\varphi^\wedge \mid \varphi \in C_c^\infty(\mathfrak{a}^*) \otimes L^2(N(\mathbf{A})T(\mathbf{R})_+T(F) \backslash G(\mathbf{A})/K_S)\}$ in $L^2(G(F)\backslash G(\mathbf{A})/K_S)$.

Take an orbit $X \in \Omega_S(T)/W$ and define the space $L^2(B, K_S, X)$ by the closed linear span of $\{\varphi^\wedge \mid \varphi \in C_c^\infty(\mathfrak{a}^*) \otimes I(\chi/\!/\mathbf{A})_S, \chi \in X\}$. If $L^2_d(B, K_S)$ and $L^2_d(B, K_S, X)$ denote the space associated to discrete spectrums in $L^2(B, K_S)$ and $L^2(B, K_S, X)$, respectively, there is an orthogonal decomposition

$$L^2_d(B, K_S) = \bigoplus_{X \in \Omega_S(T)/W} L^2_d(B, K_S, X).$$

A theorem of Harish-Chandra says the space $L^2_d(B, K_S, X)$ is of finite dimension. Further, $K$ acts on this space by right translation. Since the action of $K_S$ is trivial, $L^2_d(B, K_S, X)$ becomes a representation space of the finite group $K/K_S$. In Section 6, we will give completely an irreducible decomposition of $L^2_d(B, K_S, X)$.

Let $X \in \Omega_S(T)/W$ and $\chi \in X$. For $\Phi \in I(\chi/\!/\mathbf{A})$ and $\Lambda \in \mathfrak{a}_\mathbf{C}$, the Eisenstein series

$$E(g, \Phi, \Lambda) = \sum_{\gamma \in B(F)\backslash G(F)} e^{\langle \Lambda+\delta, H(\gamma g)\rangle} \Phi(\gamma g)$$

defines an automorphic form on $G(F)\backslash G(\mathbf{A})$ provided $\mathrm{Re}\,\Lambda \in C^+ + \delta$, where let $\delta = \beta_1 + \beta_2$. The constant term of $E(g, \Phi, \Lambda)$ is given by

$$E_0(g, \Phi, \Lambda) = \sum_{w \in W} e^{\langle w\Lambda+\delta, H(g)\rangle} M(w, \Lambda, \chi)\, \Phi(g).$$

$M(w, \Lambda, \chi)\Phi(g)$ equals

$$e^{\langle -w\Lambda-\delta, H(g)\rangle} \int_{wN(\mathbf{A})w^{-1}\cap N(\mathbf{A})\backslash N(\mathbf{A})} e^{\langle \Lambda+\delta, H(w^{-1}ng)\rangle} \Phi(w^{-1}ng)\, dn. \tag{1.2}$$

Here, for any closed connected subgroup $N'$ of the unipotent group $N$, we use the Haar measure $dn$ on $N'(\mathbf{A})$ such that the volume of $N'(F)\backslash N'(\mathbf{A})$ equals one. This $M(w, \Lambda, \chi)$ defines a linear map from $I(\chi/\!/\mathbf{A})_S$ to $I(w\chi/\!/\mathbf{A})_S$. It is known by general theory that both $E(g, \Phi, \Lambda)$ and $M(w, \Lambda, \chi)$ have meromorphic continuation on $\mathfrak{a}_\mathbf{C}$ as functions of $\Lambda$ and $M(w, \Lambda, \chi)$ satisfies the functional equation of the form

$$M(w_1w_2, \Lambda, \chi) = M(w_1, w_2\Lambda, w_2\chi)\, M(w_2, \Lambda, \chi) \tag{1.3}$$

for any $w_1, w_2 \in W$.

## 2. Calculation of $M(w, \Lambda, \chi)$

Throughout this section, we fix a $\chi \in \Omega_S(T)$ and $\Phi = \otimes_v \Phi_v \in I(\chi//\mathbf{A})_S$, $\Phi_v = e_v(\chi)$ for $v \notin S$. First we calculate the integral (1.2) for the generator $\sigma$, $\tau$ of $W$. Let

$$w_\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad w_\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

be representatives in $K$ of $\sigma$ and $\tau$, respectively. Further, let

$$n_\sigma(x) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix}, \quad n_\tau(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be the one parameter subgroups associated to the simple roots $\alpha_1$ and $\alpha_2$, respectively.

In what follows we identify $\Phi$ (resp. $\Phi_v$) with $^*\Phi$ (resp. $^*\Phi_v$) by the isomorphism (1.1), hence we will often neglect the symbol $*$. To mention a statement of results, we need a few more notations. Take an $r \in \mathbf{C}$ with absolute value one and an arbitrary $z \in \mathbf{C}$. Define, for each $v \in S$, two operators $\mathscr{A}_v(\sigma, {}^*\chi_v)$ and $\mathscr{A}_v(\sigma, z, r)$ on $I({}^*\chi_v//\mathbf{k}_v)$ by

$$\mathscr{A}_v(\sigma, {}^*\chi_v)\Phi_v(k_v) = q_v^{-\frac{1}{2}} \sum_{x \in \mathbf{k}_v} \Phi_v(w_\sigma^{-1} n_\sigma(x) r_v(k_v))$$

$$\mathscr{A}_v(\sigma, z, r)\Phi_v(k_v) = \left(\frac{1 - rq_v^{-z}}{1 - rq_v^{-z-1}}\right) q_v^{-1}$$
$$\times \left\{\sum_{x \in \mathbf{k}_v} \Phi_v(w_\sigma^{-1} n_\sigma(x) r_v(k_v)) + \left(\frac{rq_v^{-z}}{1 - rq_v^{-z}}\right)(q_v - 1)\Phi_v(r_v(k_v))\right\}$$

for $k_v \in K_v$. Replacing $\sigma$ by $\tau$, $\mathscr{A}_v(\tau, {}^*\chi_v)$ and $\mathscr{A}_v(\tau, z, r)$ are similarly defined. Let $S_i(\chi)$ for $1 \leq i \leq 4$ be the set of $v \in S$ such that ${}^*\chi_v \circ \alpha_i^\vee$ is trivial. The following is clear by definition.

LEMMA 1. *Let $v$ be in $S$ and $r$ a complex number with absolute value one.*

(1) *Both $\mathscr{A}_v(\sigma, z, r)$ and $\mathscr{A}_v(\tau, z, r)$ are rational functions of $z$ and are holomorphic on $\{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$.*

(2) *For $v \in S$, $\mathscr{A}_v(\sigma, {}^*\chi_v)$ (resp. $\mathscr{A}_v(\tau, {}^*\chi_v)$) is an intertwining operator from*

*$I(^*\chi_v//\mathbf{k}_v)$ to $I(\sigma^*\chi_v//\mathbf{k}_v)$ (resp. $I(\tau^*\chi_v//\mathbf{k}_v)$).*

(3) *If $v \in S_1(\chi)$ (resp. $v \in S_2(\chi)$), then $\mathscr{A}_v(\sigma, z, r)$ (resp. $\mathscr{A}_v(\tau, z, r)$) is a self-intertwining operator of $I(^*\chi_v//\mathbf{k}_v)$.*

Now we can state an explicit formula of $M(\sigma, \Lambda, \chi)$ and $M(\tau, \Lambda, \chi)$.

Lemma 2. *Let $\chi \in \Omega_S(T)$ and $\Phi = \otimes_v \Phi_v \in I(\chi//\mathbf{A})_S$. Then, one has*

$$M(\sigma, \Lambda, \chi)\Phi(k) = \frac{\xi(\langle\Lambda, \alpha_1^\vee\rangle, \chi \circ \alpha_1^\vee)}{\xi(\langle\Lambda, \alpha_1^\vee\rangle + 1, \chi \circ \alpha_1^\vee)} \prod_{v \in S - S_1(\chi)} \mathscr{A}_v(\sigma, {}^*\chi_v)\Phi_v(k_v)$$

$$\times \prod_{v \in S_1(\chi)} \mathscr{A}_v(\sigma, \langle\Lambda, \alpha_1^\vee\rangle, \chi \circ \alpha_1^\vee(p_v))\Phi_v(k_v)$$

$$M(\tau, \Lambda, \chi)\Phi(k) = \frac{\xi(\langle\Lambda, \alpha_2^\vee\rangle, \chi \circ \alpha_2^\vee)}{\xi(\langle\Lambda, \alpha_2^\vee\rangle + 1, \chi \circ \alpha_2^\vee)} \prod_{v \in S - S_2(\chi)} \mathscr{A}_v(\tau, {}^*\chi_v)\Phi_v(k_v)$$

$$\times \prod_{v \in S_2(\chi)} \mathscr{A}_v(\tau, \langle\Lambda, \alpha_2^\vee\rangle, \chi \circ \alpha_2^\vee(p_v))\Phi_v(k_v)$$

*for any $\Lambda \in \mathfrak{a}_{\mathbf{C}}$ and $k \in K$.*

*Proof.* We prove only the first equality since the second is obtained by similar calculation. For each $v \in V_f$ and $x_v \in F_v$, elements $c(x_v)$ and $u(x_v)$ in $\mathcal{O}_v$ are defined as follows:

$$c(x_v) = \begin{cases} p_v^{-\mathrm{ord}(x_v)} & \text{if } x_v \in F_v - \mathcal{O}_v \\ 1 & \text{if } x_v \in \mathcal{O}_v. \end{cases},$$

$$u(x_v) = \begin{cases} (x_v c(x_v))^{-1} & \text{if } x_v \in F_v - \mathcal{O}_v \\ x_v & \text{if } x_v \in \mathcal{O}_v \end{cases}.$$

Further, for each infinite place $v$ and $x_v \in F_v$ we set

$$c(x_v) = \begin{cases} (x_v^2 + 1)^{-\frac{1}{2}} & \text{if } v \text{ is real} \\ (x_v \bar{x}_v + 1)^{-\frac{1}{2}} & \text{if } v \text{ is imaginary} \end{cases},$$

$$u(x_v) = \begin{cases} x_v c(x_v) & \text{if } v \text{ is real} \\ \bar{x}_v c(x_v) & \text{if } v \text{ is imaginary} \end{cases},$$

where let $x_v \mapsto \bar{x}_v$ be the complex conjugate for imaginary place $v$. Obviously, for any $x = (x_v) \in \mathbf{A}$, both $c(x) = (c(x_v))$ and $u(x) = (u(x_v))$ are contained in $\mathbf{A}$.

For $x = (x_v) \in \mathbf{A}$, set $k_\sigma(x) = (k_\sigma(x_v))$, where

$$k_\sigma(x_v) = \begin{pmatrix} u(x_v) & u(x_v)x_v - c(x_v)^{-1} & 0 & 0 \\ c(x_v) & c(x_v)x_v & 0 & 0 \\ 0 & 0 & c(x_v)x_v & -c(x_v) \\ 0 & 0 & -u(x_v)x_v + c(x_v)^{-1} & u(x_v) \end{pmatrix}.$$

Then $k_\sigma(x)$ is contained in $K$ and one has

$$w_\sigma^{-1}n_\sigma(x) = n_\sigma(-u(x)c(x))\alpha_1^\vee(c(x))k_\sigma(x) \tag{2.1}$$

for $x \in \mathbf{A}$.

By definition, for $\Lambda \in (C^+ + \delta) + \sqrt{-1}\,\mathfrak{a}$ and $k \in K$, $M(\sigma, \Lambda, \chi)\Phi(k)$ equals

$$\int_{\mathbf{A}} e^{\langle \Lambda+\delta, H(w_\sigma^{-1}n_\sigma(x))\rangle}\Phi(w_\sigma^{-1}n_\sigma(x)k)\,dx.$$

The measure $dx$ is normalized by $\mathrm{vol}(\mathbf{A}/F) = 1$. By (2.1), this equals

$$\int_{\mathbf{A}} |\,c(x)\,|_{\mathbf{A}}^{\langle \Lambda+\delta,\alpha_1^\vee\rangle}\chi(\alpha_1^\vee(c(x)))\Phi(k_\sigma(x)k)\,dx = \prod_v I_v,$$

where

$$I_v = \int_{F_v} |\,c(x_v)\,|_v^{\langle \Lambda+\delta,\alpha_1^\vee\rangle}\chi_v(\alpha_1^\vee(c(x_v))\,\Phi_v(k_\sigma(x_v)k_v)\,dx_v$$

for each $v$. If $v \notin S$, $I_v$ is easily calculated because $\chi_v$ is unramified and $\Phi_v(k_\sigma(x_v)k_v) = 1$. We obtain

$$I_v = \begin{cases} \pi^{\frac{1}{2}}\dfrac{\Gamma((z+d_v\sqrt{-1})/2)}{\Gamma((z+1+d_v\sqrt{-1})/2)} & \text{if } v \text{ is real} \\ 2\pi\dfrac{\Gamma(z+d_v\sqrt{-1})}{\Gamma(z+1+d_v\sqrt{-1})} & \text{if } v \text{ is imaginary,} \\ \mathrm{vol}(\mathcal{O}_v)\left\{\dfrac{1-\chi_v\circ\alpha_1^\vee(p_v)q_v^{-z-1}}{1-\chi_v\circ\alpha_1^\vee(p_v)q_v^{-z}}\right\} & \text{if } v \in V_f - S \end{cases}$$

where let $z = \langle \Lambda, \alpha_1^\vee \rangle$ and $d_v$ be the real number given by $x_v \circ \alpha_1^\vee = |\cdot|_v^{d_v\sqrt{-1}}$ for each infinite place $v$.

If $v \in S$, then one has

$$\begin{aligned} I_v &= \int_{\mathcal{O}_v} \Phi(k_\sigma(x_v)k_v)\,dx_v \\ &\quad + \int_{F_v-\mathcal{O}_v} q_v^{\mathrm{ord}(x_v)\langle \Lambda,\alpha_1^\vee\rangle}\,|\,x_v\,|_v^{-1}\chi_v\circ\alpha_1^\vee(p_v^{-\mathrm{ord}(x_v)})\,\Phi_v(k_\sigma(x_v)k_v)\,dx_v \\ &= q_v^{-1}\mathrm{vol}(\mathcal{O}_v)\sum_{x\in\mathbf{k}_v}\Phi_v(w_\sigma^{-1}n_\sigma(x)r_v(k_v)) \\ &\quad + \sum_{n=1}^{\infty} q_v^{-n\langle \Lambda,\alpha_1^\vee\rangle}\chi_v\circ\alpha_1^\vee(p_v)^n \end{aligned}$$

$$\times \int_{\mathcal{O}_v^*} \Phi_v\left(\begin{pmatrix} u & 0 & 0 & 0 \\ p_v^n & u^{-1} & 0 & 0 \\ 0 & 0 & u^{-1} & -p_v^n \\ 0 & 0 & 0 & u \end{pmatrix} k_v\right) du$$

$$= q_v^{-\frac{1}{2}} \mathrm{vol}(\mathcal{O}_v)\, \mathcal{A}_v(\sigma, {}^*\chi_v)\, \Phi_v(k_v)$$

$$+ q_v^{-1} \mathrm{vol}(\mathcal{O}_v) \left\{\frac{\chi_v \circ \alpha_1^\vee(p_v) q_v^{-\langle \Lambda, \alpha_1^\vee \rangle}}{1 - \chi_v \circ \alpha_1^\vee(p_v) q_v^{-\langle \Lambda, \alpha_1^\vee \rangle}} \sum_{x \in \mathbf{k}_v^*} {}^*\chi_v \circ \alpha_1^\vee(x)\right\} \Phi_v(r_v(k_v)).$$

Therefore $I_v$ equals $\mathrm{vol}(\mathcal{O}_v)$ times

$$q_v^{-\frac{1}{2}} \mathcal{A}_v(\sigma, {}^*\chi_v) \Phi_v(k_v) \qquad \text{if } v \notin S_1(\chi)$$

$$\left(\frac{1 - \chi_v \circ \alpha_1^\vee(p_v) q_v^{-\langle \Lambda, \alpha_1^\vee \rangle - 1}}{1 - \chi_v \circ \alpha_1^\vee(p_v) q_v^{-\langle \Lambda, \alpha_1^\vee \rangle}}\right) \mathcal{A}_v(\sigma, \langle \Lambda, \alpha_1^\vee \rangle, \chi_v \circ \alpha_1^\vee(p_v) \Phi_v(k_v)) \qquad \text{if } v \in S_1(\chi)$$

Summing up, we obtain the desired equality for $\Lambda \in (C^+ + \delta) + \sqrt{-1}\mathfrak{a}$. Then

$$\xi(\langle \Lambda, \alpha_1^\vee \rangle + 1, \chi \circ \alpha_1^\vee) \langle \Lambda, \alpha_1^\vee \rangle (\langle \Lambda, \alpha_1^\vee \rangle - 1)$$
$$\times \prod_{v \in S_1(\chi)} (1 - \chi \circ \alpha_1^\vee(p_v) q_v^{-\langle \Lambda, \alpha_1^\vee \rangle - 1}) M(\sigma, \Lambda, \chi) \Phi(k)$$

must be holomorphic on $\mathfrak{a}_{\mathbf{C}}$. Hence, the equality is valid for any $\Lambda \in \mathfrak{a}_{\mathbf{C}}$.

Using the functional equation (1.3), one can describe $M(w, \Lambda, X)\Phi$ for any $w \in W$. If $w_1 \cdots w_m$, $w_j \in \{\sigma, \tau\}$ for $1 \le j \le m$, is a reduced expression of an element of $W$, we define the operator $\mathcal{A}_v(w_1 \ldots w_m, {}^*\chi_v)$ on $I({}^*\chi_v // \mathbf{k}_v)$ by

$$\mathcal{A}_v(w_1, w_2 \ldots w_m\, {}^*\chi_v) \circ \mathcal{A}_v(w_2, w_3 \cdots w_m\, {}^*\chi_v) \circ \cdots \circ \mathcal{A}_v(w_m, {}^*\chi_v).$$

This definition is independent of reduced expressions, actually one sees $\mathcal{A}_v((\sigma\tau)^2, {}^*\chi_v) = \mathcal{A}_v((\tau\sigma)^2, {}^*\chi_v)$ by $(w_\sigma w_\tau)^2 = (w_\tau w_\sigma)^2$. We set

$$\Xi_w(\Lambda, \chi) = \prod_{\substack{1 \le i \le 4 \\ w(\alpha_i) < 0}} \frac{\xi(\langle \Lambda, \alpha_i^\vee \rangle, \chi \circ \alpha_i^\vee)}{\xi(\langle \Lambda, \alpha_i^\vee \rangle + 1, \chi \circ \alpha_i^\vee)}$$

for each $w \in W$ and let $S_u(\chi)$ be the set consisting of $v \in S$ such that $\chi_v$ is unramified. Then, for any $\Lambda \in \mathfrak{a}_{\mathbf{C}}$ one has the following:

$$M(\sigma\tau, \Lambda, \chi)\, \Phi$$
$$= \Xi_{\sigma\tau}(\Lambda, \chi) \left\{\otimes_{v \in S - S_2(\chi) - S_3(\chi)} \mathcal{A}_v(\sigma\tau, {}^*\chi_v)\, \Phi_v\right\}$$

$$\otimes \left\{ \otimes_{v\in S_2(\chi)-S_3(\chi)} \mathscr{A}_v(\sigma,\ \tau^*\chi_v) \circ \mathscr{A}_v(\tau,\ \langle \Lambda,\ \alpha_2^\vee \rangle,\ \chi_v \circ \alpha_2^\vee\ (p_v))\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in (S-S_2(\chi))\cap S_3(\chi)} \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_3^\vee \rangle,\ \chi_v \circ \alpha_3^\vee\ (p_v)) \circ \mathscr{A}_v(\tau,\ {}^*\chi_v)\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_u(\chi)} \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_3^\vee \rangle,\ \chi_v \circ \alpha_3^\vee\ (p_v)) \circ \mathscr{A}_v(\tau,\ \langle \Lambda,\ \alpha_2^\vee \rangle,\right.$$

$$\left. \chi_v \circ \alpha_2^\vee\ (p_v))\ \Phi_v \right\}$$

$$M(\tau\sigma,\ \Lambda,\ \chi)\ \Phi$$

$$= \Xi_{\tau\sigma}(\Lambda,\ \chi) \left\{ \otimes_{v\in S-S_1(\chi)-S_4(\chi)} \mathscr{A}_v(\tau\sigma,\ {}^*\chi_v)\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_1(\chi)-S_4(\chi)} \mathscr{A}_v(\tau,\ \sigma^*\chi_v) \circ \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_1^\vee \rangle,\ \chi_v \circ \alpha_1^\vee\ (p_v))\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in (S-S_1(\chi))\cap S_4(\chi)} \mathscr{A}_v(\tau,\ \langle \Lambda,\ \alpha_4^\vee \rangle,\ \chi_v \circ \alpha_4^\vee\ (p_v)) \circ \mathscr{A}_v(\sigma,\ {}^*\chi_v)\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_u(\chi)} \mathscr{A}_v(\tau,\ \langle \Lambda,\ \alpha_4^\vee \rangle,\ \chi_v \circ \alpha_4^\vee\ (p_v)) \circ \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_1^\vee \rangle,\right.$$

$$\left. \chi_v \circ \alpha_1^\vee\ (p_v))\ \Phi_v \right\}$$

$$M(\sigma\tau\sigma,\ \Lambda,\ \chi)\ \Phi$$

$$= \Xi_{\sigma\tau\sigma}(\Lambda,\ \chi) \left\{ \otimes_{v\in S-S_1(\chi)-S_3(\chi)-S_4(\chi)} \mathscr{A}_v(\sigma\tau\sigma,\ {}^*\chi_v)\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in (S-S_1(\chi)-S_4(\chi))\cap S_3(\chi)} \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_3^\vee \rangle,\ \chi_v \circ \alpha_3^\vee(p_v)) \circ \mathscr{A}_v(\tau\sigma,\ {}^*\chi_v)\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_1(\chi)-S_3(\chi)-S_4(\chi)} \mathscr{A}_v(\sigma\tau,\ \sigma^*\chi_v) \circ \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_1^\vee \rangle,\ \chi_v \circ \alpha_1^\vee(p_v))\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in (S-S_1(x))\cap S_4(\chi)-S_3(\chi)} \mathscr{A}_v(\sigma,\ \tau\sigma^*\chi_v) \right.$$

$$\left. \circ\ \mathscr{A}_v(\tau,\ \langle \Lambda,\ \alpha_4^\vee \rangle,\ \chi_v \circ \alpha_4^\vee\ (p_v)) \circ \mathscr{A}_v(\sigma,\ {}^*\chi_v)\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in (S_1(\chi)-S_4(\chi))\cap S_3(\chi)} \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_3^\vee \rangle,\ \chi_v \circ \alpha_3^\vee(p_v)) \right.$$

$$\left. \circ\ \mathscr{A}_v(\tau,\ \sigma^*\chi_v) \circ \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_1^\vee \rangle,\ \chi_v \circ \alpha_1^\vee(p_v))\ \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_u(\chi)} \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_3^\vee \rangle,\ \chi_v \circ \alpha_3^\vee(p_v)) \right.$$

$$\left. \circ\ \mathscr{A}_v(\tau,\ \langle \Lambda,\ \alpha_4^\vee \rangle,\ \chi_v \circ \alpha_4^\vee(p_v))\ \mathscr{A}_v(\sigma,\ \langle \Lambda,\ \alpha_1^\vee \rangle,\ \chi_v \circ \alpha_1^\vee(p_v))\ \Phi_v \right\}$$

$$M(\tau\sigma\tau,\,\Lambda,\,\chi)\,\Phi$$
$$=\Xi_{\tau\sigma\tau}(\Lambda,\,\chi)\Big\{\otimes_{v\in S-S_2(\chi)-S_3(\chi)-S_4(\chi)}\mathscr{A}_v(\tau\sigma\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in(S-S_2(\chi)-S_3(\chi))\cap S_4(\chi)}\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_4^\vee\rangle,\,\chi_v\circ\alpha_4^\vee(p_v))\circ\mathscr{A}_v(\sigma\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in S_2(x)-S_3(\chi)-S_4(\chi)}\mathscr{A}_v(\tau\sigma,\,\tau^*\chi_v)\circ\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_2^\vee\rangle,\,\chi_v\circ\alpha_2^\vee(p_v))\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in(S-S_2(\chi))\cap S_3(\chi)-S_4(\chi)}\mathscr{A}_v(\tau,\,\sigma\tau^*\chi_v)$$
$$\circ\,\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_3^\vee\rangle,\,\chi_v\circ\alpha_3^\vee(p_v))\circ\mathscr{A}_v(\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in S_u(\chi))}\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_4^\vee\rangle,\,\chi_v\circ\alpha_4^\vee(p_v))$$
$$\circ\,\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_3^\vee\rangle,\,\chi_v\circ\alpha_3^\vee(p_v))\circ\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_2^\vee\rangle,\,\chi_v\circ\alpha_2^\vee(p_v))\,\Phi_v\Big\}$$

$$M((\sigma\tau)^2,\,\Lambda,\,\chi)\,\Phi$$
$$=\Xi_{(\sigma\tau)^2}(\Lambda,\,\chi)\Big\{\otimes_{v\in S-\cup_{1\leq i\leq 4}S_i(\chi)}\mathscr{A}_v((\sigma\tau)^2,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in(S-\cup_{2\leq i\leq 4}S_i(\chi))\cap S_1(x)}\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_1^\vee\rangle,\,\chi_v\circ\alpha_1^\vee(p_v))\circ\mathscr{A}_v(\tau\sigma\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in(S-S_2(\chi)-S_3(\chi))\cap S_4(\chi)-S_1(\chi)}\mathscr{A}_v(\sigma,\,\tau\sigma\tau^*\chi_v)$$
$$\circ\,\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_4^\vee\rangle,\,\chi_v\circ\alpha_4^\vee(p_v))\circ\mathscr{A}_v(\sigma\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in S_2(\chi)-S_1(\chi)-S_3(\chi)-S_4(\chi)}\mathscr{A}_v(\sigma\tau\sigma,\,\tau^*\chi_v)$$
$$\circ\,\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_2^\vee\rangle,\,\chi_v\circ\alpha_2^\vee(p_v))\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in(S-S_2(\chi))\cap S_3(X)-S_1(\chi)-S_4(\chi)}\mathscr{A}_v(\sigma\tau,\,\sigma\tau^*\chi_v)$$
$$\circ\,\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_3^\vee\rangle,\,\chi_v\circ\alpha_3^\vee(p_v))\circ\mathscr{A}_v(\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in((S-S_2(\chi))\cap S_3(\chi)-S_4(\chi))\cap S_1(\chi)}\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_1^\vee\rangle,\,\chi_v\circ\alpha_1^\vee(p_v))$$
$$\circ\,\mathscr{A}_v(\tau,\sigma\tau^*\chi_v)\circ\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_3^\vee\rangle,\,\chi_v\circ\alpha_3^\vee(p_v))\circ\mathscr{A}_v(\tau,\,{}^*\chi_v)\,\Phi_v\Big\}$$
$$\otimes\Big\{\otimes_{v\in S_u(\chi)}\mathscr{A}_v(\sigma,\,\langle\Lambda,\,\alpha_1^\vee\rangle,\,\chi_v\circ\alpha_1^\vee(p_v))\circ\mathscr{A}_v(\tau,\,\langle\Lambda,\,\alpha_4^\vee\rangle,$$

$$\chi_v \circ \alpha_4^\vee (p_v))$$
$$\circ \mathscr{A}_v(\sigma, \langle \Lambda, \alpha_3^\vee \rangle, \chi_v \circ \alpha_3^\vee(p_v)) \circ \mathscr{A}_v(\tau, \langle \Lambda, \alpha_2^\vee \rangle, \chi_v \circ \alpha_2^\vee(p_v))\ \Phi_v \Big\}.$$

In this paper, we are interested in the singular hyperplanes of $M(w, \Lambda, \chi)$ contributing to the spectral decomposition of $L^2(B, K)$. Such a singular hyperplane $S$ has to satisfy the following condition (cf. [14, Theorem 7.1], [17, Theorem 5.12]).

(2.2) $S$ is defined by a linear equation of the form $\langle \Lambda, \alpha_i^\vee \rangle = c$ with positive real constant $c$.

From this and Lemma 1 (1), we obtain:

LEMMA 3. *Let* $S_i = \{\Lambda \in \mathfrak{a}_{\mathbf{C}} \mid \langle \Lambda, \alpha_i^\vee \rangle = 1\}$ *for* $1 \le i \le 4$. *Then* $S_i, 1 \le i \le 4$, *exhaust the singular hyperplanes of* $M(w, \Lambda, \chi)$, $w \in W$, $\chi \in \Omega_S(T)$, *contributing to the spectral decomposition of* $L^2(B, K_S)$.

## 3. Residues of $M(w, \Lambda, \chi)$ (1)

Each $S_i$ is rewritten as $S_i = \mathbf{C}u_i + v_i$, $1 \le i \le 4$, where $u_1 = \beta_2$, $u_2 = \beta_1$, $u_3 = \alpha_1$, $u_4 = \alpha_2$ and $v_i = \alpha_i/2$ for $1 \le i \le 4$. Then we take a coordinate $z_i(\Lambda)$ on $S_i$ as $\Lambda = z_i(\Lambda)u_i + v_i$ for $\Lambda \in S_i$, $1 \le i \le 4$.

Next, for $1 \le i, j \le 4$, let $W_{ij}$ be the set of elements $w \in W$ such that $-\overline{wS_i} = S_j$. Obviously $W_{ij}$ is empty unless $i \equiv j$ modulo 2. For $1 \le i \le 4$, let $W_i$ be the union of $W_{ij}$, $1 \le j \le 4$.

For $\chi \in \Omega_S(T)$, $\Phi \in I(\chi//\mathbf{A})_S$ and $w \in W$, we set

$$M^i(w, \Lambda, \chi) = \frac{\xi(2)}{2\pi c(F)\sqrt{-1}} \int_C M(w, \Lambda + zv_i, \chi)\, dz$$
$$E^i(g, \Phi, \Lambda) = \frac{\xi(2)}{2\pi c(F)\sqrt{-1}} \int_C E(g, \Phi, \Lambda + zv_i)\, dz$$

for $\Lambda \in S_i$, $1 \le i \le 4$, where $C$ is a small contour around the origin in the complex plane. Let $\Lambda_{ij}$ denote the intersection of $S_i$ and $S_j$ for $i \ne j$. One sees that the order of pole of $E^i(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{ij}$ is at most one for any $\Phi \in I(\chi//\mathbf{A})_S$, $\chi \in \Omega_S(T)$, $1 \le j \le 4$, $j \ne i$ (cf. Lemma 11). Then the main theorem of [11] deduces that, for $X \in \Omega_S(T)/W$, the space $L^2_d(B, K_S, X)$ is spanned by

$$\mathrm{Res}_{\Lambda=\Lambda_{ij}} E^i(g, \Phi, \Lambda), \qquad \Phi \in I(\chi//\mathbf{A})_S,\ \chi \in X,\ i = 1,2,\ 1 \le j \le 4,\ j \ne i$$

belonging to $L^2(G(F)\backslash G(\mathbf{A}))$. Here, notice that, since principal singular hyperplanes (in the sense of [17]) are only $S_1$ and $S_2$, it is enough for our purpose to consider only residues of $E^i(g, \Phi, \Lambda)$ for $i = 1,2$ (cf. [17, Chapter 6]). Since the residue $\mathrm{Res}_{\Lambda=\Lambda_{ij}} E^i(g, \Phi, \Lambda)$ is completely determined by its constant term

$$\int_{N(F)\backslash N(\mathbf{A})} \mathrm{Res}_{\Lambda=\Lambda_{ij}} E^i(ng, \Phi, \Lambda)dn$$
$$= \mathrm{Res}_{\Lambda=\Lambda_{ij}} \left\{ \sum_{w\in W_i} e^{\langle w\Lambda+\delta, H(g)\rangle} M^i(w, \Lambda, \chi)\Phi(g) \right\},$$

what we should do is to calculate residues of $M^i(w, \Lambda, \chi)$ at $\Lambda = \Lambda_{ij}$ for $i = 1,2$, $1 \leq j \leq 4$, $j \neq i$.

Let $A_S(F)$ be the set of characters $\mu \in \Omega_S$ such that $\mu^2 = \mu_0$, where $\mu_0$ is the trivial character. For each $\mu \in A_S(F)$, the $W$ orbit of $\chi(\mu, \mu)$ is denoted by $X(\mu)$, that is, $X(\mu) = \{\chi(\mu, \mu)\}$, Then the next proposition follows from direct calculations.

Proposition 1. *Let $X \in \Omega_S(T)/W$ be a $W$ orbit. Assume $X \notin \{X(\mu) \mid \mu \in A_S(F)\}$. Then, for any $\chi \in X$, $w \in W$ and $1 \leq i \neq j \leq 4$,* $\mathrm{Res}_{\Lambda=\Lambda_{ij}} M^i(w, \Lambda, \chi) = 0$

By that mentioned above, we obtain

Corollary. $L^2_d(B, K_S) = \bigoplus_{\mu\in A_S(F)} L^2_d(B, K_S, X(\mu))$.

The major part of remains of this paper will be devoted to a detailed calculation of residues of $M^i(w, \Lambda, \chi)$ for $\chi = \chi(\mu, \mu)$, $\mu \in A_S(F)$. At first, we give an explicit form of $M^i(w, \Lambda, \chi)$. For each $\mu \in A_S(F)$, the subset $S_u(\mu)$ of $S$ is defined to be $S_u(\mu) = \{v \in S \mid \mu_v \text{ is unramified}\}$.

Lemma 4. *Let $\mu \in A_S(F)$ be a nontrivial character, $\chi = \chi(\mu, \mu) \in \Omega_S(T)$ and $\Phi \in I(\chi//\mathbf{A})_S$ an arbitrary element. Then, for any $w \in W_2$, $M^2(w, \Lambda, \chi)\Phi$ is identically zero. Further one has*

$$M^1(\sigma, \Lambda, \chi)\, \Phi = \otimes_{v\in S} \mathscr{A}_v(\sigma,1,1)\, \Phi_v$$

$$M^1(\tau\sigma, \Lambda, \chi)\, \Phi = \frac{\xi(z+\frac{1}{2}, \mu)}{\xi(z+\frac{3}{2}, \mu)} \left\{ \otimes_{v\in S-S_u(\mu)} \mathscr{A}_v(\tau, {}^*\chi_v) \circ \mathscr{A}_v(\sigma, 1, 1)\, \Phi_v \right\}$$
$$\otimes \left\{ \otimes_{v\in S_u(\mu)} \mathscr{A}_v(\tau, z+\frac{1}{2}, \mu_v(p_v)) \circ \mathscr{A}_v(\sigma, 1, 1)\, \Phi_v \right\}$$

$$M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi = \frac{\xi(2z)\xi(z+\frac{1}{2}, \mu)}{\xi(2z+1)\xi(z+\frac{3}{2}, \mu)} \times$$

$$\left\{ \otimes_{v\in S-Su(\mu)} \mathscr{A}_v(\sigma,2z,1)\circ\mathscr{A}_v(\tau,{}^*\chi_v)\circ\mathscr{A}_v(\sigma,1,1)\,\Phi_v\right\}$$

$$\otimes\left\{ \otimes_{v\in Su(\mu)} \mathscr{A}_v(\sigma,2z,1)\circ\mathscr{A}_v(\tau,z+\frac{1}{2},\mu_v(p_v))\circ\mathscr{A}_v(\sigma,1,1)\,\Phi_v\right\}$$

$$M^1((\sigma\tau)^2,\Lambda,\chi)\,\Phi=\frac{\xi(2z)\xi(z-\frac{1}{2},\mu)}{\xi(2z+1)\xi(z+\frac{3}{2},\mu)}\times$$

$$\left\{ \otimes_{v\in S-Su(\mu)} \mathscr{A}_v(\sigma,1,1)\circ\mathscr{A}_v(\tau,{}^*\chi_v)\circ\mathscr{A}_v(\sigma,2z,1)\circ\mathscr{A}_v(\tau,{}^*\chi_v)\,\Phi_v\right\}$$

$$\otimes\left\{ \otimes_{v\in Su(\mu)} \mathscr{A}_v(\sigma,1,1)\circ\mathscr{A}_v(\tau,z+\frac{1}{2},\mu_v(p_v))\right.$$

$$\left.\circ\,\mathscr{A}_v(\sigma,2z,1)\circ\mathscr{A}_v(\tau,z-\frac{1}{2},\mu_v(p_v))\,\Phi_v\right\}$$

*for any* $\Lambda=zu_1+v_1\in S_1$.

LEMMA 5. *Let* $\chi=\chi(\mu_0,\mu_0)\in\Omega_S(T)$ *be the trivial character and* $\Phi\in I(\chi//\mathbf{A})_S$ *an arbitrary element. Then one has the following.*

*For* $\Lambda=zu_1+v_1\in S_1$,

$$M^1(\sigma,\Lambda,\chi)\,\Phi=\otimes_{v\in S}\mathscr{A}_v(\sigma,1,1)\,\Phi_v$$

$$M^1(\tau\sigma,\Lambda,\chi)\,\Phi=\frac{\xi(z+\frac{1}{2})}{\xi(z+\frac{3}{2})}\left\{\otimes_{v\in S}\mathscr{A}_v(\tau,z+\frac{1}{2},1)\circ\mathscr{A}_v(\sigma,1,1)\,\Phi_v\right\}$$

$$M^1(\sigma\tau\sigma,\Lambda,\chi)\,\Phi=\frac{\xi(2z)\xi(z+\frac{1}{2})}{\xi(2z+1)\xi(z+\frac{3}{2})}\times$$

$$\left\{\otimes_{v\in S}\mathscr{A}_v(\sigma,2z,1)\circ\mathscr{A}_v(\tau,z+\frac{1}{2},1)\circ\mathscr{A}_v(\sigma,1,1)\,\Phi_v\right\}$$

$$M^1((\sigma\tau)^2,\Lambda,\chi)\,\Phi=\frac{\xi(2z)\xi(z-\frac{1}{2})}{\xi(2z+1)\xi(z+\frac{3}{2})}\left\{\otimes_{v\in S}\mathscr{A}_v(\sigma,1,1)\right.$$

$$\left.\circ\,\mathscr{A}_v(\tau,z+\frac{1}{2},1)\circ\mathscr{A}_v(\sigma,2z,1)\circ\mathscr{A}_v(\tau,z-\frac{1}{2},1)\,\Phi_v\right\}$$

*For* $\Lambda=zu_2+v_2\in S_2$,

$$M^2(\tau,\Lambda,\chi)\,\Phi=\otimes_{v\in S}\mathscr{A}_v(\tau,1,1)\,\Phi_v$$

$$M^2(\sigma\tau, \Lambda, \chi)\, \Phi = \frac{\xi(z+1)}{\xi(z+2)} \left\{ \otimes_{v \in S} \mathscr{A}_v(\sigma, z+1, 1) \circ \mathscr{A}_v(\tau, 1, 1)\, \Phi_v \right\}$$

$$M^2(\tau\sigma\tau, \Lambda, \chi)\, \Phi = \frac{\xi(z)}{\xi(z+2)} \left\{ \otimes_{v \in S} \mathscr{A}_v(\tau, z, 1) \circ \mathscr{A}_v(\sigma, z+1, 1) \circ \mathscr{A}_v(\tau, 1, 1)\, \Phi_v \right\}$$

$$M^2((\sigma\tau)^2, \Lambda, \chi)\, \Phi = \frac{\xi(z-1)}{\xi(z+2)} \left\{ \otimes_{v \in S} \mathscr{A}_v(\sigma, z-1, 1) \circ \mathscr{A}_v(\tau, z, 1) \circ \mathscr{A}_v(\sigma, z+1, 1) \circ \mathscr{A}_v(\tau, 1, 1)\, \Phi_v \right\}$$

These lemmas are easily proved. Consequently, we obtain the following:

PROPOSITION 2. *Let $\mu \in A_S(F)$ be a nontrivial character. Then $L^2_d(B, K_S, X(\mu))$ is spanned by* $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$, $\Phi \in I(\chi(\mu, \mu)//\mathbf{A})_S$, *where $\beta_1 = \Lambda_{13} = \Lambda_{14}$ is a fundamental weight of $G$.*

*Proof.* From Lemma 4, it follows

$$\mathrm{Res}_{\Lambda=\Lambda_{12}} E^1(g, \Phi, \Lambda) = 0$$

$$\mathrm{Res}_{\Lambda=\Lambda_{2j}} E^2(g, \Phi, \Lambda) = 0, \quad 1 \le j \le 4, j \ne 2$$

for all $\Phi \in I(\chi(\mu, \mu)//\mathbf{A})_S$. On the other hand, by Lemma 4 again, it is possible that $E^1(g, \Phi, \Lambda)$ has a simple pole at $\Lambda = \beta_1$. If so, the constant term of $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$ equals

$$e^{\langle -\beta_1+\delta, H(g)\rangle} \{\mathrm{Res}_{\Lambda=\beta_1} M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi + \mathrm{Res}_{\Lambda=\beta_1} M^1((\sigma\tau)^2, \Lambda, \chi)\, \Phi\}.$$

Thus, Langlands' $L^2$-ness criterion deduces $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$ is square integrable for any $\Phi \in I(\chi(\mu,\mu)//\mathbf{A})_S$. This implies the assertion.

Residues of $M^1(\sigma\tau\sigma, \Lambda, \chi(\mu, \mu))$ and $M^1((\sigma\tau)^2, \Lambda, \chi(\mu, \mu))$ at $\Lambda = \beta_1$ for nontrivial $\mu \in A_S(F)$ are given as follows.

$$\mathrm{Res}_{\Lambda=\beta_1} M^1(\sigma\tau\sigma, \Lambda, \chi(\mu, \mu))\, \Phi$$
$$= \frac{c_F(\mu)}{2} \left\{ \otimes_{v \in S - S_u(\mu)} \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, {}^*\chi_v) \circ \mathscr{A}_v(\sigma, 1, 1)\, \Phi_v \right\}$$
$$\otimes \left\{ \otimes_{v \in S_u(\mu)} \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 1, \mu_v(p_v)) \circ \mathscr{A}_v(\sigma, 1, 1)\, \Phi_v \right\}$$

$$\mathrm{Res}_{\Lambda=\beta_1} M^1((\sigma\tau)^2, \Lambda, \chi(\mu, \mu))\, \Phi$$

$$= \frac{\varepsilon(\mu)c_F(\mu)}{2}\left\{\otimes_{v\in S-S_{u(\mu)}}\mathscr{A}_v(\sigma, 1, 1)\circ\mathscr{A}_v(\tau, {}^*\chi_v)\circ\mathscr{A}_v(\sigma, 1, 1)\circ\mathscr{A}_v(\tau, {}^*\chi_v)\,\Phi_v\right\}$$

$$\otimes\left\{\otimes_{v\in S_{u(\mu)}}\mathscr{A}_v(\sigma, 1, 1)\circ\mathscr{A}_v(\tau, 1, \mu_v(p_v))\circ\mathscr{A}_v(\sigma, 1, 1)\circ\mathscr{A}_v(\tau, 0, \mu_v(p_v))\,\Phi_v\right\},$$

where $c_F(\mu) = c(F)\xi(1, \mu)\xi(2)^{-1}\xi(2, \mu)^{-1}$. Hence, in order to describe $L^2_d(B, K_S, X(\mu))$ as a representation space of $K/K_S$ in more detail, we have to investigate the intertwining operators $\mathscr{A}_v(\sigma, 1, 1)$, $\mathscr{A}_v(\tau, {}^*\chi_v), \ldots$. To do so, we use a result of Gurtis and Fossum. Thus, in next section, we recall the representation theory of $G(\mathbf{k}_v)$.

## 4. Principal series of $G(\mathbf{F}_q)$

Throughout this section, $\mathbf{k}$ denotes the finite field of cardinality $q$. For a character ${}^*\chi$ of $T(\mathbf{k})$, let $I({}^*\chi/\!/\mathbf{k})$ be the representation of $G(\mathbf{k})$ induced by the trivial extension to $B(\mathbf{k})$ of ${}^*\chi$ and $H(G,B;{}^*\chi)$ denote the centralizer ring of $I({}^*\chi/\!/\mathbf{k})$. Dropping the lower index $v$, operators $\mathscr{A}(w, {}^*\chi)$ and $\mathscr{A}(w, z, r)$ for $w \in \{\sigma,\tau\}$ on $I({}^*\chi/\!/\mathbf{k})$ are similarly defined as in Section 2. In this section, we explain an irreducible decomposition of $I({}^*\chi/\!/\mathbf{k})$ for some particular ${}^*\chi$ and then represent the operators $\mathscr{A}(w, {}^*\chi)$ and $\mathscr{A}(w, z, r)$ by linear combinations of those projections to irreducible subspaces of $I({}^*\chi/\!/\mathbf{k})$ which are constructed by using a theorem of Curtis and Fossum.

For a character $\mu$ of $\mathbf{k}^*$, the character ${}^*\chi(\mu)$ of $T(\mathbf{k})$ is defined to be ${}^*\chi(\mu)(t(a, b)) = \mu(ab)$ for $t(a, b) \in T(\mathbf{k})$. Then we consider the following three cases.

(#-1) $\mu$ is the trivial character.

(#-2) $q \equiv 1 \bmod 4$ and $\mu$ is the quadratic character.

(#-3) $q \equiv 3 \bmod 4$ and $\mu$ is the quadratic character.

Before going to case by case consideration, we state a result deduced from general theory (cf. [5]).

LEMMA 6. *Let ${}^*\chi$ be an arbitrary character of $T(\mathbf{k})$. Then there is a bijection $\eta \mapsto \theta(\eta)$ from the set of equivalence classes of irreducible representations of $H(G, B;{}^*\chi)$ to the set of equivalence classes of irreducible constituents of $I({}^*\chi/\!/\mathbf{k})$ such that the character of $\eta$ is equal to the restriction to $H(G, B;{}^*\chi)$ of the character* *of $\theta(\eta)$. Here, notice that $H(G, B;{}^*\chi)$ is considered as a subalgebra of the group ring of $G(\mathbf{k})$. Furthermore, the multiplicity of $\theta(\eta)$ in $I({}^*\chi//\mathbf{k})$ is equal to the degree of $\eta$.*

Now we start with the case (♯-1).

(♯-1) ${}^*\chi = {}^*\chi(\mu)$, $\mu$ is the trivial character.

The self-intertwining operators $\alpha_\sigma$ and $\alpha_\tau$ of $I({}^*\chi//\mathbf{k})$ are defined to be

$$(4.1)\qquad \alpha_\sigma(\Phi)(g) = \sum_{x\in\mathbf{k}} \Phi(w_\sigma^{-1}n_\sigma(x)g), \quad \alpha_\tau(\Phi)(g) = \sum_{x\in\mathbf{k}} \Phi(w_\tau^{-1}n_\tau(x)g)$$

for $\Phi \in I({}^*\chi//\mathbf{k})$. Further, if $w = w_1 \cdots w_m$, $w_j \in \{\sigma, \tau\}$, $1 \le j \le m$ is a reduced expression, we define $\alpha_w$ by $\alpha_{w_1} \cdots \alpha_{w_m}$, which does not depend on reduced expressions. Then $H(G, B;{}^*\chi)$ is generated by $\alpha_\sigma$ and $\alpha_\tau$ together with the relations

$$\alpha_\sigma^2 = q\alpha_e + (q-1)\alpha_\sigma, \quad \alpha_\tau^2 = q\alpha_e + (q-1)\alpha_\tau, \quad (\alpha_\sigma\alpha_\tau)^2 = (\alpha_\tau\alpha_\sigma)^2,$$

where $\alpha_e$ is the identity map of $I({}^*\chi//\mathbf{k})$. Irreducible representations of $H(G, B;{}^*\chi)$ are exhausted by the following ones up to equivalence.

$$\eta_1 : \begin{cases} \alpha_\sigma \mapsto q \\ \alpha_\tau \mapsto q \end{cases}, \quad \eta_2 : \begin{cases} \alpha_\sigma \mapsto -1 \\ \alpha_\tau \mapsto q \end{cases}, \quad \eta_3 : \begin{cases} \alpha_\sigma \mapsto q \\ \alpha_\tau \mapsto -1 \end{cases}, \quad \eta_4 : \begin{cases} \alpha_\sigma \mapsto -1 \\ \alpha_\tau \mapsto -1 \end{cases}$$

$$\eta_5 : \alpha_\sigma \mapsto \frac{1}{q+1}\begin{pmatrix} q-1 & 1 \\ 2q(q^2+1) & q(q-1) \end{pmatrix}, \quad \alpha_\tau \mapsto \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}$$

Let $\theta(\eta_i)$ be the irreducible representation of $G(\mathbf{k})$ corresponding to $\eta_i$ by Lemma 6 and $V_i$ the $\theta(\eta_i)$-isotypic subspace of $I({}^*\chi//\mathbf{k})$ for $1 \le i \le 5$. Then one has

$$I({}^*\chi//\mathbf{k}) = \bigoplus_{i=1}^{5} V_i, \quad \begin{cases} V_i \cong \theta(\eta_i) & \text{for } 1 \le i \le 4 \\ V_5 \cong \theta(\eta_5) \oplus \theta(\eta_5). \end{cases}$$

If $P_i$ denotes the projection from $I({}^*\chi//\mathbf{k})$ onto $V_i$ for $1 \le i \le 5$, then [5, Theorem (2.4)] allows us to represent $P_i$ by linear combinations of $\alpha_w$, $w \in W$. Actually one has

$$P_1 = \frac{1}{(q+1)^2(q^2+1)} \sum_{w\in W} \alpha_w$$

$$P_2 = \frac{1}{2(q+1)^2}\{q\alpha_e - \alpha_\sigma + q\alpha_\tau - \alpha_{\sigma\tau} - \alpha_{\tau\sigma} + q^{-1}\alpha_{\sigma\tau\sigma} - \alpha_{\tau\sigma\tau} + q^{-1}\alpha_{(\sigma\tau)^2}\}$$

$$P_3 = \frac{1}{2(q+1)^2}\{q\alpha_e + q\alpha_\sigma - \alpha_\tau - \alpha_{\sigma\tau} - \alpha_{\tau\sigma} - \alpha_{\sigma\tau\sigma} + q^{-1}\alpha_{\tau\sigma\tau} + q^{-1}\alpha_{(\sigma\tau)^2}\}$$

$$P_4 = \frac{1}{(q+1)^2(q^2+1)}\{q^4\alpha_e - q^3\alpha_\sigma - q^3\alpha_\tau + q^2\alpha_{\sigma\tau}$$

$$+ q^2\alpha_{\tau\sigma} - q\alpha_{\sigma\tau\sigma} - q\alpha_{\tau\sigma\tau} + \alpha_{(\sigma\tau)^2}\}$$

$$P_5 = \frac{1}{2(q^2+1)}\{2q\alpha_e + (q-1)\alpha_\sigma + (q-1)\alpha_\tau$$

$$+ q^{-1}(q-1)\alpha_{\sigma\tau\sigma} + q^{-1}(q-1)\alpha_{\tau\sigma\tau} - 2q^{-1}\alpha_{(\sigma\tau)^2}\}.$$

We express $\mathscr{A}(\sigma, 1, 1)$ and $\mathscr{A}(\tau, 1, 1)$ by linear combinations of projections to irreducible subspaces. By definition,

$$\mathscr{A}(\sigma, 1, 1) = \frac{1}{q+1}(\alpha_e + \alpha_\sigma), \qquad \mathscr{A}(\tau, 1, 1) = \frac{1}{q+1}(\alpha_e + \alpha_\tau).$$

Thus, it follows from easy calculation

$$\mathscr{A}(\sigma, 1, 1) = P_1 + P_3 + Q_\sigma, \quad Q_\sigma = P_5\mathscr{A}(\sigma, 1, 1)$$

$$\mathscr{A}(\tau, 1, 1) = P_1 + P_3 + Q_\tau, \quad Q_\tau = P_5\mathscr{A}(\tau, 1, 1).$$

Then $Q_\sigma$ and $Q_\tau$ are considered as elements in $\mathrm{End}_{G(\mathbf{k})}(V_5)$ and satisfy the following:

$$Q_\sigma Q_\tau Q_\sigma = \frac{2q}{(q+1)^2} Q_\sigma, \qquad Q_\tau Q_\sigma Q_\tau = \frac{2q}{(q+1)^2} Q_\tau$$

We define four elements of $\mathrm{End}_{G(\mathbf{k})}(V_5)$ by

$$P_\sigma = \frac{1}{(q^2+1)}(-2qQ_\sigma Q_\tau + (q+1)^2 Q_\sigma),$$

$$P_\tau = \frac{1}{(q^2+1)}(-2qQ_\tau Q_\sigma + (q+1)^2 Q_\tau)$$

$$R_+ = \frac{\sqrt{2q}(q+1)}{q^2+1}(Q_\tau Q_\sigma - Q_\tau), \quad R_- = \frac{\sqrt{2q}(q+1)}{q^2+1}(Q_\sigma Q_\tau - Q_\sigma),$$

where $\sqrt{2q}$ is the positive square root of $2q$. Then these elements satisfy

$$P_\sigma^2 = R_- R_+ = P_\sigma, \quad P_\tau^2 = R_+ R_- = P_\tau,$$

$$P_\sigma P_\tau = P_\tau P_\sigma = (R_+)^2 = (R_-)^2 = 0.$$

Therefore $P_\sigma, P_\tau, R_+, R_-$ becomes a base of the four dimensional space $\mathrm{End}_{G(\mathbf{k})}(V_5)$ and both $P_\sigma$ and $P_\tau$ are projections to irreducible subspaces in $V_5$. Representing $Q_\sigma$ and $Q_\tau$ by these we obtain

$$\mathscr{A}(\sigma, 1, 1) = P_1 + P_3 + P_\sigma + \frac{\sqrt{2q}}{q+1} R_- \tag{4.2}$$

$$\mathscr{A}(\tau, 1, 1) = P_1 + P_2 + P_\tau + \frac{\sqrt{2q}}{q+1} R_+.$$

We set particularly

$$P_5^+ = Q_\sigma = P_\sigma + \frac{\sqrt{2q}}{q+1} R_-.$$

Then $P_5^+ I(^*\chi//\mathbf{k}) \cong \theta(\eta_5)$ and one has

$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, 1, 1) \circ \mathscr{A}(\sigma, 1, 1) = P_1 + \frac{2q}{(q+1)^2} P_5^+ \tag{4.3}$$

Since $\mathscr{A}(w, z, r)$, $w \in \{\sigma, \tau\}$, equals

$$\frac{1}{1 - rq^{-z-1}} \{(1 - rq^{-z})(1 + q^{-1}) \mathscr{A}(w, 1, 1) + q^{-1}(rq^{-z+1} - 1)\alpha_e\},$$

one has

$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, 1, -1) \circ \mathscr{A}(\sigma, 1, 1) = P_1 - \frac{2q}{q^2+1} P_3 \tag{4.4}$$

$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, 1, -1) \circ \mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, 0, -1) = P_1 + \frac{2q}{q^2+1} P_3.$$

(#-2) $q \equiv 1 \bmod 4$, $^*\chi = {}^*\chi(\mu)$, $\mu$ is the quadratic character.

We define the self-intertwining operators $\alpha_w$, $w \in W$ of $I(^*\chi//\mathbf{k})$ as in (4.1). Then the centralizer ring $H(G, B; ^*\chi)$ is generated by $\alpha_\sigma$ and $\alpha_\tau$ together with the relations

$$\alpha_\sigma^2 = q\alpha_e + (q-1)\alpha_\sigma, \quad \alpha_\tau^2 = q\alpha_e, \quad (\alpha_\sigma\alpha_\tau)^2 = (\alpha_\tau\alpha_\sigma)^2$$

Irreducible representations of $H(G, B; ^*\chi)$ are exhausted by the following ones up to equivalence.

$$\eta_1' : \begin{cases} \alpha_\sigma \mapsto q, \\ \alpha_\tau \mapsto q^{\frac{1}{2}}, \end{cases} \quad \eta_2' : \begin{cases} \alpha_\sigma \mapsto -1 \\ \alpha_\tau \mapsto q^{\frac{1}{2}} \end{cases}, \quad \eta_3' : \begin{cases} \alpha_\sigma \mapsto q \\ \alpha_\tau \mapsto -q^{\frac{1}{2}}, \end{cases} \quad \eta_4' : \begin{cases} \alpha_\sigma \mapsto -1 \\ \alpha_\tau \mapsto -q^{\frac{1}{2}} \end{cases}$$

$$\eta_5' : \alpha_\sigma \mapsto \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_\tau \mapsto \begin{pmatrix} 0 & q^{\frac{1}{2}} \\ q^{\frac{1}{2}} & 0 \end{pmatrix},$$

where $q^{\frac{1}{2}}$ is the positive square root of $q$. Let $\theta(\eta_i')$ be the irreducible representation of $G(\mathbf{k})$ corresponding to $\eta_i'$ and $V_i'$ the $\theta(\eta_i')$-isotypic subspace of $I(^*\chi//\mathbf{k})$ for $1 \leq i \leq 5$. Then it follows from Lemma 6

$$I(^*\chi//\mathbf{k}) = \bigoplus_{i=1}^{5} V_i', \quad \begin{cases} V_i' \cong \theta(\eta_i') & \text{for } 1 \leq i \leq 4 \\ V_5' \cong \theta(\eta_5') \oplus \theta(\eta_5'). \end{cases}$$

If $P_i'$ denote projections from $I(^*\chi//\mathbf{k})$ to $V_i'$ for $1 \leq i \leq 5$, then by [5], one has

$$P_1' = \frac{1}{2q(q+1)^2} \{q\alpha_e + q\alpha_\sigma + q^{\frac{1}{2}}\alpha_\tau + q^{\frac{1}{2}}\alpha_{\sigma\tau} + q^{\frac{1}{2}}\alpha_{\tau\sigma} + q^{\frac{1}{2}}\alpha_{\sigma\tau\sigma} + \alpha_{\tau\sigma\tau} + \alpha_{(\sigma\tau)^2}\}$$

$$P_2' = \frac{1}{2(q+1)^2}\{q^2\alpha_e - q\alpha_\sigma + q^{\frac{3}{2}}\alpha_\tau - q^{\frac{1}{2}}\alpha_{\sigma\tau} - q^{\frac{1}{2}}\alpha_{\tau\sigma} + q^{-\frac{1}{2}}\alpha_{\sigma\tau\sigma}$$
$$- \alpha_{\tau\sigma\tau} + q^{-1}\alpha_{(\sigma\tau)^2}\}$$

$$P_3' = \frac{1}{2q(q+1)^2}\{q\alpha_e + q\alpha_\sigma - q^{\frac{1}{2}}\alpha_\tau - q^{\frac{1}{2}}\alpha_{\sigma\tau} - q^{\frac{1}{2}}\alpha_{\tau\sigma} - q^{\frac{1}{2}}\alpha_{\sigma\tau\sigma}$$
$$+ \alpha_{\tau\sigma\tau} + \alpha_{(\sigma\tau)^2}\}$$

$$P_4' = \frac{1}{2(q+1)^2}\{q^2\alpha_e - q\alpha_\sigma - q^{\frac{3}{2}}\alpha_\tau + q^{\frac{1}{2}}\alpha_{\sigma\tau} + q^{\frac{1}{2}}\alpha_{\tau\sigma} - q^{-\frac{1}{2}}\alpha_{\sigma\tau\sigma}$$
$$- \alpha_{\tau\sigma\tau} + q^{-1}\alpha_{(\sigma\tau)^2}\}$$

$$P_5' = \frac{1}{(q+1)^2}\{2q\alpha_e + (q-1)\alpha_\sigma + q^{-1}(q-1)\alpha_{\tau\sigma\tau} - 2q^{-1}\alpha_{(\sigma\tau)^2}\}.$$

We set

$$Q_+ = \frac{1}{2}(\alpha_e + \mathscr{A}(\tau, {}^*\chi)) \circ P_5', \quad Q_- = \frac{1}{2}(\alpha_e - \mathscr{A}(\tau, {}^*\chi)) \circ P_5',$$
$$R_+ = 2\mathscr{A}(\sigma, 1, 1)Q_+ - Q_+, \quad R_- = 2\mathscr{A}(\sigma, 1, 1)Q_- - Q_-.$$

Since $\mathscr{A}(\sigma, 1, 1) = \frac{1}{q+1}(\alpha_e + \alpha_\sigma)$, $\mathscr{A}(\tau, {}^*\chi) = q^{-\frac{1}{2}}\alpha_\tau$ and $\mathscr{A}(\tau, {}^*\chi)^2 = \alpha_e$ one has

$$Q_+^2 = R_-R_+ = Q_+, \qquad Q_-^2 = R_+R_- = Q_-$$
$$Q_+Q_- = Q_-Q_+ = R_+^2 = R_-^2 = 0.$$

Therefore $\{Q_+, Q_-, R_+, R_-\}$ gives a basis of $\mathrm{End}(V_5')$. Then

$$\mathscr{A}(\sigma, 1, 1) = P_1' + P_3' + \frac{1}{2}(Q_+ + Q_- + R_+ + R_-)$$
$$\mathscr{A}(\tau, {}^*\chi) = P_1' + P_2' - P_3' - P_4' + Q_+ - Q_-.$$

Hence we obtain

$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, {}^*\chi) \circ \mathscr{A}(\sigma, 1, 1) = P_1' - P_3' \tag{4.5}$$
$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, {}^*\chi) \circ \mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, {}^*\chi) = P_1' + P_3'.$$

(#-3) $q \equiv 3 \bmod 4$, ${}^*\chi = {}^*\chi(\mu)$, $\mu$ is the quadratic character.

This case is similar to the case (#-2), hence we will omit the details. Define $\alpha_w$, $w \in W$, as in (4.1). Then the centralizer ring $H(G,B;{}^*\chi)$ is generated by $\alpha_\sigma$ and $\alpha_\tau$ together with the relations

$$\alpha_\sigma^2 = q\alpha_e + (q-1)\alpha_\sigma, \quad \alpha_\tau^2 = -q\alpha_e, \quad (\alpha_\sigma\alpha_\tau)^2 = (\alpha_\tau\alpha_\sigma)^2.$$

Irreducible representations of $H(G,B;\, {}^*\chi)$ are exhausted by the following ones up to equivalence.

$$\eta_1'' : \begin{cases} \alpha_\sigma \mapsto q \\ \alpha_\tau \mapsto -(-q)^{\frac{1}{2}}, \end{cases} \quad \eta_2'' : \begin{cases} \alpha_\sigma \mapsto -1 \\ \alpha_\tau \mapsto -(-q)^{\frac{1}{2}}, \end{cases} \quad \eta_3'' : \begin{cases} \alpha_\sigma \mapsto q \\ \alpha_\tau \mapsto (-q)^{\frac{1}{2}}, \end{cases}$$

$$\eta_4'' : \begin{cases} \alpha_\sigma \mapsto -1 \\ \alpha_\tau \mapsto (-q)^{\frac{1}{2}}, \end{cases} \quad \eta_5'' : \alpha_\sigma \mapsto \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_\tau \mapsto \begin{pmatrix} 0 & -(-q)^{\frac{1}{2}} \\ -(-q)^{\frac{1}{2}} & 0 \end{pmatrix},$$

where $(-q)^{\frac{1}{2}}$ lies in the upper half plane of the complex plane. Let $\theta(\eta_i'')$ be the irreducible representation of $G(\mathbf{k})$ corresponding to $\eta_i''$ and $V_i''$ the $\theta(\eta_i'')$-isotypic subspace of $I({}^*\chi//\mathbf{k})$ for $1 \leq i \leq 5$. Then it follows from Lemma 6

$$I({}^*\chi//\mathbf{k}) = \bigoplus_{i=1}^{5} V_i'', \quad \begin{cases} V_i'' \cong \theta(\eta_i'') & \text{for } 1 \leq i \leq 4 \\ V_5'' \cong \theta(\eta_5'') \oplus \theta(\eta_5''). \end{cases}$$

Let $P_i''$ be the projection from $I({}^*\chi//\mathbf{k})$ to $V_i''$ for each $1 \leq i \leq 5$. Then, from the argument similar to the case (#-2), it follows

$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, {}^*\chi) \circ \mathscr{A}(\sigma, 1, 1) = -\sqrt{-1}(P_1'' - P_3'') \tag{4.6}$$

$$\mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, {}^*\chi) \circ \mathscr{A}(\sigma, 1, 1) \circ \mathscr{A}(\tau, {}^*\chi) = -(P_1'' + P_3'').$$

## 5. Residues of $M(w, \Lambda, \chi)$ (2)

We return to calculations of residues of $M(w, \Lambda, \chi)$. In this section, we fix a nontrivial character $\mu \in A_S(F)$ and put $\chi = \chi(\mu, \mu)$.

Define four subsets of $S$ as follows:

$$S_r^+(\mu) = \{v \in S \mid {}^*\mu_v \text{ is the quadratic character and } {}^*\mu_v(-1) = 1\}$$

$$S_r^-(\mu) = \{v \in S \mid {}^*\mu_v \text{ is the quadratic character and } {}^*\mu_v(-1) = -1\}$$

$$S_u^+(\mu) = \{v \in S \mid \mu_v \text{ is trivial}\}$$

$$S_u^-(\mu) = \{v \in S \mid \mu_v \text{ is unramified and } \mu_v(p_v) = -1\}$$

Then $S$ is the disjoint union of these subsets. Notice that if $v \in S$ lies above 2 then $v$ is contained in $S_u(\mu) = S_u^+(\mu) \cup S_u^-(\mu)$. We apply results in Section 4 to $I({}^*\chi_v//\mathbf{k}_v)$ for each $v \in S$. $I({}^*\chi_v//\mathbf{k}_v)$ takes the case (#-1), (#-2) or (#-3) according as $v \in S_u(\mu)$, $S_r^+(\mu)$ or $S_r^-(\mu)$. Then, using the notations of Section 4 with respect to $\mathbf{k} = \mathbf{k}_v$ and ${}^*\chi = {}^*\chi_v$, we define irreducible subspaces $Y_1^\mu(v)$ and $Y_0^\mu(v)$ of $I({}^*\chi_v//\mathbf{k}_v)$ as follows:

$$Y_1^{\mu}(v)=\begin{cases} V_1 & \text{if } v\in S_u(\mu),\\ V_3' & \text{if } v\in S_r^+(\mu),\\ V_1'' & \text{if } v\in S_r^-(\mu),\end{cases}\qquad Y_0^{\mu}(v)=\begin{cases} P_5^+V_5 & \text{if } v\in S_u^+(\mu)\\ V_3 & \text{if } v\in S_u^-(\mu)\\ V_1' & \text{if } v\in S_r^+(\mu)\\ V_3'' & \text{if } v\in S_r^-(\mu)\end{cases}$$

Let $R_i^{\mu}(v)$ be the projection of $I(^*\chi_v//\mathbf{k}_v)$ to $Y_i^{\mu}(v)$ for $i=0,1$.

LEMMA 7. *Using the labelling of Srinivasan* ([20]) *for $v\nmid 2$ and Enomoto* ([7]) *for $v\mid 2$,*

$$Y_1^{\mu}(v)\cong\begin{cases} \textit{the trivial representation} & \textit{if } v\in S_u(\mu)\\ \theta_3 & \textit{if } v\in S_r(\mu)\end{cases}$$

$$Y_0^{\mu}(v)\cong\begin{cases} \theta_9 & \textit{if } v\in S_u^+(\mu) \textit{ and } v\nmid 2\\ \theta_1 & \textit{if } v\in S_u^+(\mu) \textit{ and } v\mid 2\\ \theta_{11} & \textit{if } v\in S_u^-(\mu) \textit{ and } v\nmid 2,\\ \theta_2 & \textit{if } v\in S_u^-(\mu) \textit{ and } v\mid 2\\ \theta_4 & \textit{if } v\in S_r(\mu)\end{cases}$$

*where $S_r(\mu)$ is the union of $S_r^+(\mu)$ and $S_r^-(\mu)$.*

We remark that Enomoto's character table contains some misprints. The degrees of $\theta_1$ and $\theta_2$ are correctly $\frac{1}{2}q_v(q_v+1)^2$ and $\frac{1}{2}q_v(q_v^2+1)$, respectively. This is easily checked from the defining equations of $\theta_1$ and $\theta_3$ in [7, p. 83].

*Proof.* First we assume $v\in S_u(\mu)$, hence one has the case (#-1). When $q=q_v$ is odd, the correspondence $\eta_i\mapsto\theta(\eta_i)$ is well known (cf. [23]). When $q=q_v$ is even, one can compute explicitly $\theta_3(Bw_\sigma B)$ and $\theta_3(Bw_\tau B)$ by using the tables of conjugacy classes and characters in [7]. Then one has $\theta_3(Bw_\sigma B)=\eta_2(\alpha_\sigma)$ and $\theta_3(Bw_\tau B)=\eta_2(\alpha_\tau)$, hence $V_2\cong\theta(\eta_2)=\theta_3$. Further, from the formula in [5], it follows $\dim\theta(\eta_2)=\dim\theta(\eta_3)=\frac{1}{2}q(q^2+1)$ and $\dim\theta(\eta_5)=\frac{1}{2}q(q+1)^2$. Thus the character table concludes $V_3\cong\theta(\eta_3)=\theta_2$ and $P_5^+V_5\cong\theta(\eta_5)=\theta_1$.

Next, assume $v\in S_r^+(\mu)$, hence one has the case (#-2). The formula in [5] deduces that $\dim\theta(\eta_1')=\dim\theta(\eta_3')=\frac{1}{2}(q_v^2+1)$, hence $\{\theta(\eta_1'),\theta(\eta_3')\}=\{\theta_3,\theta_4\}$. Furthermore, Littlewood's formula (cf. [12]) deduces $\theta(\eta_1')(g)\geq\theta(\eta_3')(g)$ for any $g\in G(\mathbf{k}_v)$. Then, by the character table in [20], it is known that $V_1'\cong\theta(\eta_1')=\theta_4$ and $V_3'\cong\theta(\eta_3')=\theta_3$.

Finally, assume $v\in S_r^-(\mu)$. From the similar arguments to the second case, it follows $\{\theta(\eta_1''),\theta(\eta_3'')\}=\{\theta_3,\theta_4\}$ and Imaginary$(\theta(\eta_1'')(g))\geq$ Imaginary$(\theta(\eta_3'')$

$(g))$ for any $g \in G(\mathbf{k}_v)$. The character table in [20] allows us to conclude $V_1'' \cong \theta(\eta_1'') = \theta_3$, $V_3'' \cong \theta(\eta_3'') = \theta_4$.

The following lemma is an immediate consequence of (4.3), (4.4), (4.5) and (4.6). Here, note that the cardinality of $S_r^-(\mu)$ is even since $\mu$ is an even character.

LEMMA 8. *Let $\mu \in A_S(F)$ be a nontrivial character, $\chi = \chi(\mu, \mu)$ and $\Phi \in I(\chi//\mathbf{A})_S$ an arbitrary element. Then*

$$\mathrm{Res}_{\Lambda=\beta_1} M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi$$

$$= \frac{c_F(\mu)(-1)^{|S_r^+(\mu)|+|S_r^-(\mu)|/2}}{2} \left\{ \otimes_{v\in S_u^+(\mu)} \left(R_1^\mu(v) + \frac{2q_v}{(q_v+1)^2} R_0^\mu(v)\right) \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_u^-(\mu)} \left(R_1^\mu(v) - \frac{2q_v}{q_v^2+1} R_0^\mu(v)\right) \Phi_v \right\} \otimes \left\{ \otimes_{v\in S_r(\mu)} (R_1^\mu(v) - R_0^\mu(v))\, \Phi_v \right\}$$

$$\mathrm{Res}_{\Lambda=\beta_1} M^1((\sigma\tau)^2, \Lambda, \chi)\, \Phi$$

$$= \frac{c_F(\mu)\varepsilon(\mu)}{2} \left\{ \otimes_{v\in S_u^+(\mu)} \left(R_1^\mu(v) + \frac{2q_v}{(q_v+1)^2} R_0^\mu(v)\right) \Phi_v \right\}$$

$$\otimes \left\{ \otimes_{v\in S_u^-(\mu)} \left(R_1^\mu(v) + \frac{2q_v}{q_v^2+1} R_0^\mu(v)\right) \Phi_v \right\} \otimes \left\{ \otimes_{v\in S_r(\mu)} (R_1^\mu(v) + R_0^\mu(v))\, \Phi_v \right\}.$$

## 6. Decomposition of $L_d^2(B, K_S, X(\mu))$ for nontrivial $\mu$

We take a nontrivial $\mu \in A_S(F)$ and put $\chi = \chi(\mu, \mu)$. Let $\Gamma(S)$ be the set of all maps from $S$ to $\{0,1\}$. For each $\lambda \in \Gamma(S)$, $\lambda_S(\mu) = \otimes_{v\in S} Y_{\lambda(v)}^\mu(v)$ is an irreducible subspace in $\otimes_{v\in S} I(^*\chi_v//\mathbf{k}_v)$ and $R_\lambda^\mu = \otimes_{v\in S} R_{\lambda(v)}^\mu(v)$ the projection of $\otimes_{v\in S} I(^*\chi_v//\mathbf{k}_v)$ to $\lambda_S(\mu)$. By the isomorphism (1.1), $\lambda_S(\mu)$ is identified with a subspace in $I(\chi//\mathbf{A})_S$. The subset $\Gamma(S, \mu)$ of $\Gamma(S)$ is defined to be

$$\Gamma(S, \mu) = \{\lambda \in \Gamma(S) \mid \varepsilon(\mu) = (-1)^{|S_r(\mu)|+\frac{1}{2}|S_r^-(\mu)|+|\lambda^{-1}(0)\cap S_r(\mu)|+|\lambda^{-1}(0)\cap S_u^-(\mu)|}\},$$

where $\lambda^{-1}(0)$ is the inverse image of 0 by $\lambda$. By Lemma 8, we obtain

PROPOSITION 3. *Let $\mu \in A_S(F)$ be a nontrivial character, $\lambda \in \Gamma(S)$ and $\Phi \in \lambda_S(\mu)$ an arbitrary element. Then the constant term of $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$ is equal to the following:*

$$\begin{cases} \varepsilon(\mu)c_F(\mu)q_\mu(\lambda)e^{\langle -\beta_1+\delta, H(g)\rangle}\Phi(g) & \text{if } \lambda \in \Gamma(S, \mu) \\ 0 & \text{if } \lambda \notin \Gamma(S, \mu) \end{cases},$$

*whene*

$$c_F(\mu) = \frac{c(F)\xi(1, \mu)}{\xi(2)\xi(2, \mu)}, \quad q_\mu(\lambda) = \prod_{v \in \lambda^{-1}(0) \cap S_u^+(\mu)} \frac{2q_v}{(q_v + 1)^2} \times \prod_{v \in \lambda^{-1}(0) \cap S_u^-(\mu)} \frac{2q_v}{q_v^2 + 1}$$

COROLLARY. *For any $\Phi \in I(\chi(\mu, \mu)//\mathbf{A})_S$, one has*

$$\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda) = \sum_{\lambda \in \Gamma(S,\mu)} \mathrm{Res}_{\Lambda=\beta_1} E^1(g, R_\lambda^\mu \Phi, \Lambda).$$

*Proof.* The constant term of the left hand side is equal to that of the right hand side. Hence, Langlands' lemma implies the assertion.

Let $L_d^2(B, K_S, X(\mu))_\lambda$ be the space spanned by $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$, $\Phi \in \lambda_S(\mu)$ for each $\lambda \in \Gamma(S, \mu)$. Combining with Proposition 2, one has

THEOREM 1. *Let $\mu \in A_S(F)$ be a nontrivial character. Then one has a $K/K_S$-irreducible decomposition*

$$L_d^2(B, K_S, X(\mu)) = \bigoplus_{\lambda \in \Gamma(S,\mu)} L_d^2(B, K_S, X(\mu))_\lambda.$$

*For each $\lambda \in \Gamma(S, \mu)$, the constant term map gives rise to a $K/K_S$-isomorphism from $L_d^2(B, K_S, X(\mu))$ onto $\lambda_S(\mu)$.*

## 7. Decomposition of $L_d^2(B, K_S, X(\mu))$ for trivial $\mu$

Throughout this section, $\mu_0$ and $\chi = \chi(\mu_0, \mu_0)$ denote the trivial characters. We prove the following:

THEOREM 2. *$L_d^2(B, K_S, X(\mu_0))$ consists of constant functions.*

We must calculate residues of $E^1(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{12}, \Lambda_{13}$ and of $E^2(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{12}, \Lambda_{23}, \Lambda_{24}$. In what follows, $E_0^i(g, \Phi, \Lambda)$ denote the constant terms of $E^i(g, \Phi, \Lambda)$ for $i = 1,2$.

LEMMA 9. *$E^2(g, \Phi, \Lambda)$ is holomorphic at $\Lambda = \Lambda_{23} = \alpha_2/2$ and $\Lambda = \Lambda_{24} = \beta_2$ for any $\Phi \in I(\chi//\mathbf{A})_S$.*

*Proof.* It is sufficient to show that $E^2{}_0(g, \Phi, \Lambda)$ is holomorphic at $\Lambda = \alpha_2/2$ and $\beta_2$. By Lemma 5, one has

$$\mathrm{Res}_{\Lambda=\alpha_2/2}\, E_0^2(bk, \Phi, \Lambda)$$

$$= \frac{c(F)}{\xi(2)} e^{\langle -\beta_1+\delta, H(b)\rangle} \{ \otimes_{v\in S} \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 1, 1)\, \Phi_v(k_v)$$
$$- \otimes_{v\in S} \mathscr{A}_v(\tau, 0, 1) \circ \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 1,1)\, \Phi_v(k_v)\}$$

$$\mathrm{Res}_{\Lambda=\beta_2}\, E_0^2(bk, \Phi, \Lambda)$$

$$= \frac{c(F)}{\xi(2)} e^{\langle -\beta_2+\delta, H(b)\rangle} \{ \otimes_{v\in S} \mathscr{A}_v(\tau, 1, 1) \circ \mathscr{A}_v(\sigma, 2, 1) \circ \mathscr{A}_v(\tau, 1, 1) \Phi_v(k_v)$$
$$- \otimes_{v\in S} \mathscr{A}_v(\sigma, 0, 1) \circ \mathscr{A}_v(\tau, 1, 1) \circ \mathscr{A}_v(\sigma, 2, 1) \circ \mathscr{A}_v(\tau, 1, 1)\, \Phi_v(k_v)\}$$

for any $b \in B(\mathbf{A})$ and $k \in K$. Since both $\mathscr{A}_v(\tau, 0, 1)$ and $\mathscr{A}_v(\sigma, 0, 1)$ are the identity map of $I({}^*\chi_v//\mathbf{k}_v)$ for any $v \in S$, the residues are identically zero.

LEMMA 10. *The residues of $E^i(g, \Phi, \Lambda)$, $i = 1,2$ at $\Lambda = \Lambda_{12} = \delta$ are constant functions for any $\Phi \in I(\chi//\mathbf{k})_S$.*

*Proof.* $\mathrm{Res}_{\Lambda=\delta}\, E_0^i(bk, \Phi, \Lambda)$ equals

$$\frac{c(F)}{\xi(2)} \{ \otimes_{v\in S} \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 2, 1) \circ \mathscr{A}_v(\sigma, 3, 1) \circ \mathscr{A}_v(\tau, 1, 1))\, \Phi_v(k_v)\}$$

for any $b \in B(\mathbf{A})$, $k \in K$ and $i = 1, 2$. Since

$$\mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 2, 1) \circ \mathscr{A}_v(\sigma, 3, 1) \circ \mathscr{A}_v(\tau, 1, 1)$$

is the projection to the space of constant functions for any $v \in S$, we obtain the assertion.

LEMMA 11. *The order of pole of $E^1(g, \Phi, \Lambda)$ at $\Lambda = \Lambda_{13} = \beta_1$ is at most one for any $\Phi \in I(\chi//\mathbf{A})_S$.*

*Proof.* By Lemma 5, one has

$$\lim_{z_1(\Lambda)\to\frac{1}{2}} (z_1(\Lambda) - \frac{1}{2})^2 E_0^1(k, \Phi, \Lambda)$$
$$= \frac{c(F)^2}{2\xi(2)^2} \{ \otimes_{v\in S} \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 1, 1) \circ \mathscr{A}_v(\sigma, 1, 1)\, \Phi_v(k_v)$$
$$- \otimes_{v\in S} \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 1, 1) \circ \mathscr{A}_v(\sigma, 1, 1) \circ \mathscr{A}_v(\tau, 0, 1)\, \Phi_v(k_v)\}.$$

This vanishes since $\mathscr{A}_v(\tau, 0,1)$ is the identity map for any $v \in S$.

By Lemmas 9, 10, 11 and the main theorem of [14], it is known that the space $L^2_d(B, K_S, X(\mu_0))$ is spanned by the constant functions and those residues of $E^1(g, \Phi, \Lambda)$, $\Phi \in I(\chi//\mathbf{A})_S$ at $\Lambda = \beta_1$ which are square integrable. Hence, in order to finish the proof of Theorem 2, we must show the following.

PROPOSITION 4. *Let $\Phi \in I(\chi//\mathbf{A})_S$. If $\mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$ is square integrable on $G(F)\backslash G(\mathbf{A})$, then it is identically zero.*

*Proof.* Let $z = z_1(\Lambda)$ be the coordinate of $\Lambda$ on $S_1$. We note that $M^1(\sigma\tau\sigma, \Lambda, \chi)$ and $M^1((\sigma\tau)^2, \Lambda, \chi)$ may have double pole at $\Lambda = \beta_1$. By Lemma 5, the residue of $E^1_0(bk, \Phi, \Lambda)$ at $\Lambda = \beta_1$ equals

$$\begin{aligned}
&\sum_{w\in W_1} \mathrm{Res}_{\Lambda=\beta_1}\left\{e^{\langle w\Lambda+\delta,H(b)\rangle} M^1(w, \Lambda, \chi)\, \Phi(k)\right\}\\
&\qquad = e^{\langle -\alpha_2/2+\delta,H(b)\rangle}\, \mathrm{Res}_{\Lambda=\beta_1} M^1(\tau\sigma, \Lambda, \chi)\, \Phi(k)\\
&+ \lim_{z\to\frac{1}{2}} \frac{d}{dz}\,(z-\frac{1}{2})^2 e^{\langle\sigma\tau\sigma\Lambda+\delta,H(b)\rangle} M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi(k)\\
&+ \lim_{z\to\frac{1}{2}} \frac{d}{dz}\,(z-\frac{1}{2})^2 e^{\langle(\sigma\tau)^2\Lambda+\delta,H(b)\rangle} M^1((\sigma\tau)^2, \Lambda, \chi)\, \Phi(k)\\
&\qquad = e^{\langle -\alpha_2/2+\delta,H(b)\rangle}\, \mathrm{Res}_{\Lambda=\beta_1} M^1(\tau\sigma, \Lambda, \chi)\, \Phi(k)\\
&+ \left\{\lim_{z\to\frac{1}{2}} (z-\frac{1}{2})^2 M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi(k)\right\} \frac{d}{dz} e^{\langle\sigma\tau\sigma\Lambda+\delta,H(b)\rangle}\Big|_{\Lambda=\beta^1}\\
&\qquad + e^{\langle -\beta_1+\delta,H(b)\rangle}\, \mathrm{Res}_{\Lambda=\beta_1} M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi(k)\\
&+ \left\{\lim_{z\to\frac{1}{2}} (z-\frac{1}{2})^2 M^1((\sigma\tau)^2, \Lambda, \chi)\, \Phi(k)\right\} \frac{d}{dz} e^{\langle(\sigma\tau)^2\Lambda+\delta,H(b)\rangle}\Big|_{\Lambda=\beta^1}\\
&\qquad + e^{\langle -\beta_1+\delta,H(b)\rangle}\, \mathrm{Res}_{\Lambda=\beta_1} M^1((\sigma\tau)^2, \Lambda, \chi)\, \Phi(k).
\end{aligned}$$

Since the second and fourth terms are cancelled out each other, one has

$$\begin{aligned}
&\mathrm{Res}_{\Lambda=\beta_1} E^1_0(bk, \Phi, \Lambda)\\
&= e^{\langle -\alpha_2/2+\delta,H(b)\rangle}\, \mathrm{Res}_{\Lambda=\beta_1} M^1(\tau\sigma, \Lambda, \chi)\, \Phi(k)\\
&\qquad + e^{\langle -\beta_1+\delta,H(b)\rangle}\, \mathrm{Res}_{\Lambda=\beta_1} \{M^1(\sigma\tau\sigma, \Lambda, \chi)\, \Phi(k) + M^1((\sigma\tau)^2, \Lambda, \chi)\, \Phi(k)\}
\end{aligned}$$

for $b \in B(\mathbf{A})$, $k \in K$. Then it follows from Langlands $L^2$-ness criterion that the residue of $E^1(g, \Phi, \Lambda)$ at $\Lambda = \beta_1$ is square integrable on $G(F)\backslash G(\mathbf{A})$ if and only if the first term of the right hand side vanishes. Therefore, the next lemma completes the proof of Proposition 4.

LEMMA 12. *Let* $\Phi \in I(\chi//\mathbf{A})_S$. *If* $\mathrm{Res}_{\Lambda=\beta_1} M^1(\tau\sigma, \Lambda, \chi)\,\Phi$ *is identically zero, then so is* $\mathrm{Res}_{\Lambda=\beta_1}\{M^1(\sigma\tau\sigma, \Lambda, \chi)\,\Phi + M^1((\sigma\tau)^2, \Lambda, \chi)\,\Phi\}$.

*Proof.* We assume $\mathrm{Res}_{\Lambda=\beta_1} M^1(\tau\sigma, \Lambda, \chi)\,\Phi$ is identically zero. Then, by Lemma 5, one has $\otimes_{v\in S}\mathscr{A}_v(\tau, 1, 1)\circ\mathscr{A}_v(\sigma, 1, 1)\,\Phi_v = 0$. Hence, there exists at least one place $u \in S$ such that $\mathscr{A}_u(\tau, 1, 1)\circ\mathscr{A}_u(\tau, 1, 1)\,\Phi_u = 0$. We fix such a place $u$. For $\mathbf{k} = \mathbf{k}_u$ and $^*\chi = {}^*\chi_u$, we use the same notations as in Section 4 case (#-1). Then, by (4.2),

$$\mathscr{A}_v(\tau, 1, 1)\circ\mathscr{A}_v(\sigma, 1, 1) = P_1 + \left(P_\tau + \frac{\sqrt{2q_u}}{q_u+1}R_+\right)P_5^+.$$

Therefore, $\Phi_u$ must belong to the space $(P_2 + P_3 + P_4 + P_5^-)I(^*\chi_u//\mathbf{k}_u)$, where

$$P_5^- = P_\tau - \frac{\sqrt{2q_u}}{q_u+1}R_-$$

is a projection satisfying $P_5^+P_5^- = P_5^-P_5^+ = 0$.

Assume $\Phi_u \in (P_2 + P_4)I(^*\chi//\mathbf{k}_u)$. Then one has

$$\mathscr{A}_u(\sigma, 2z, 1)\circ\mathscr{A}_u(\tau, z+\frac{1}{2}, 1)\circ\mathscr{A}_u(\sigma, 1, 1)\,\Phi_u = 0$$

$$\mathscr{A}_u(\sigma, 1, 1)\circ\mathscr{A}_u(\tau, z+\frac{1}{2}, 1)\circ\mathscr{A}_u(\sigma, 2z, 1)\circ\mathscr{A}_u(\tau, z-\frac{1}{2}, 1)\,\Phi_u = 0,$$

and hence $M^1(\sigma\tau\sigma, \Lambda, \chi)\,\Phi = M^1((\sigma\tau)^2, \Lambda, \chi)\,\Phi = 0$. This implies the assertion.

Assume $\Phi_u \in P_3I(^*\chi_u//\mathbf{k}_u)$. Then one has

$$\begin{aligned}
&\mathscr{A}_u(\sigma, 2z, 1)\circ\mathscr{A}_u(\tau, z+\frac{1}{2}, 1)\circ\mathscr{A}_u(\sigma, 1, 1)\,\Phi_u \\
&\quad = -\frac{(q_u^{z-\frac{1}{2}}-1)q_u}{q_u^{z+\frac{3}{2}}-1}\,\Phi_u \\
&\mathscr{A}_u(\sigma, 1, 1)\circ\mathscr{A}_u(\tau, z+\frac{1}{2}, 1)\circ\mathscr{A}_u(\sigma, 2z, 1)\circ\mathscr{A}_u(\tau, z-\frac{1}{2}, 1)\,\Phi_u \\
&\quad = \frac{(q_u^{z-\frac{3}{2}}-1)(q_u^{z-\frac{1}{2}}-1)q_u^2}{(q_u^{z+\frac{1}{2}}-1)(q_u^{z+\frac{3}{2}}-1)}\,\Phi_u,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Res}_{\Lambda=\beta_1} M^1(\sigma\tau\sigma, \Lambda, \chi)\,\Phi \\
&\quad = \mathrm{Res}_{z=\frac{1}{2}}\frac{\xi(2z)\xi(z+\frac{1}{2})}{\xi(2z+1)\xi(z+\frac{3}{2})}\left\{-\frac{(q_u^{z-\frac{1}{2}}-1)q_u}{q_u^{z+\frac{3}{2}}-1}\,\Phi_u\right\}\otimes
\end{aligned}$$

$$\{\otimes_{v\in S-\{u\}} \mathscr{A}_v(\sigma, 2z, 1)\circ \mathscr{A}_v(\tau, z+\frac{1}{2}, 1)\circ \mathscr{A}_v(\sigma, 1,1)\,\Phi_v\}$$

$$= -\frac{c(F)^2 q_u \log q_u}{2\xi(2)^2(q_u^2-1)}\,\Phi_u \otimes$$

$$\{\otimes_{v\in S-\{u\}} \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, 1, 1)\circ \mathscr{A}_v(\sigma, 1, 1)\,\Phi_v\}$$

$$\mathrm{Res}_{\Lambda=\beta_1} M^1((\sigma\tau)^2, \Lambda, \chi)\,\Phi$$

$$= \mathrm{Res}_{z=\frac{1}{2}} \frac{\xi(2z)\xi(z-\frac{1}{2})}{\xi(2z+1)\xi(z+\frac{3}{2})}\left\{\frac{(q_u^{z-\frac{3}{2}}-1)(q_u^{z-\frac{1}{2}}-1)q_u^2}{(q_u^{z+\frac{1}{2}}-1)(q_u^{z+\frac{3}{2}}-1)}\,\Phi_u\right\}\otimes$$

$$\{\otimes_{v\in S-\{u\}} \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, z+\frac{1}{2}, 1)\circ \mathscr{A}_v(\sigma, 2z, 1)\circ \mathscr{A}_v(\tau, z-\frac{1}{2}, 1)\,\Phi_v\}$$

$$= \frac{c(F)^2 q_u \log q_u}{2\xi(2)^2(q_u^2-1)}\,\Phi_u \otimes$$

$$\{\otimes_{v\in S-\{u\}} \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, 1, 1)\circ \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, 0, 1)\,\Phi_v\}.$$

Since $\mathscr{A}_v(\tau, 0,1)$ is the identity map for any $v \in S-\{u\}$, $\mathrm{Res}_{\Lambda=\beta_1}\{M^1(\sigma\tau\sigma, \Lambda, \chi)\,\Phi + M^1((\sigma\tau)^2, \Lambda, \chi)\Phi\}$ vanishes.

Assume $\Phi_u \in P_5^- I(^*\chi_u//\mathbf{k}_u)$. From $P_5^+\Phi_u = 0$, it follows $M^1(\sigma\tau\sigma, \Lambda, \chi)\,\Phi = 0$. On the other hand, one has

$$\mathrm{Res}_{\Lambda=\beta_1} M^1((\sigma\tau)^2, \Lambda, \chi)\,\Phi$$

$$= \mathrm{Res}_{z=\frac{1}{2}} \frac{\xi(2z)\xi(z-\frac{1}{2})}{\xi(2z+1)\xi(z+\frac{3}{2})}\left\{\frac{(q_u^{z-\frac{3}{2}}-1)(q_u^{2z-1}-1)q_u^2(q_u+1)}{(q_u^{2z+1}-1)(q_u^{z+\frac{3}{2}}-1)}+\right.$$

$$\left.\frac{(q_u^{2z-1}-1)(q_u^{z-\frac{1}{2}}-1)q_u(q_u^2-1)}{(q_u^{2z+1}-1)(q_u^{z+\frac{3}{2}}-1)}-\frac{(q_u^{z-\frac{1}{2}}-1)^2 q_u(q_u+1)}{(q_u^{z+\frac{3}{2}}-1)(q_u^{z+\frac{1}{2}}-1)}\right\}$$

$$\times \frac{(q_u^2+1)\sqrt{2q_u}}{(q_u+1)^3} R_-\Phi_u$$

$$\otimes\{\otimes_{v\in S-\{u\}} \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, z+\frac{1}{2}, 1)\circ \mathscr{A}_v(\sigma, 2z, 1)\circ \mathscr{A}_v(\tau, z-\frac{1}{2}, 1)\,\Phi_v\}$$

$$= -\frac{c(F)^2}{2\xi(2)^2}\left\{-\frac{2q_u\log q_u}{q_u^2-1}+\frac{2q_u\log q_u}{q_u^2-1}\right\}\frac{(q_u^2+1)\sqrt{2q_u}}{(q_u+1)^3} R_-\Phi_u$$

$$\otimes\{\otimes_{v\in S-\{u\}} \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, 1, 1)\circ \mathscr{A}_v(\sigma, 1, 1)\circ \mathscr{A}_v(\tau, 0, 1)\,\Phi_v\}$$

$$= 0.$$

This completes the proof of Lemma and hence Theorem 2.

## 8. Residual automorphic representations

In this section, we give another representation theoretic interpretation of Theorem 1. For this, it is convenient to consider various $S$ simultaneously. Hence, we set $A_\infty(F) = \cup_S A_S(F)$, where $S$ runs over finite subsets of $V_f$. For each $\mu \in A_\infty(F)$, $S_r(\mu)$ also denotes the set of $v \in V_f$ such that $\mu_v$ is ramified.

We fix a nontrivial $\mu \in A_\infty(F)$. Let

$$I(\mu, \beta_1) = \mathrm{Ind}(B(\mathbf{A}) \uparrow G(\mathbf{A}) : e^{\langle \beta_1, H(\cdot)\rangle}\chi(\mu, \mu))$$

be the normalized induced representation of $G(\mathbf{A})$. This $I(\mu, \beta_1)$ has a restricted tensor product decomposition: $I(\mu, \beta_1) = \otimes_v I_v(\mu, \beta_1)$ (cf. [15]). For any finite set $S \subset V_f$ containing $S_r(\mu)$, we denote by $I(\mu, \beta_1)_S$ the subspace spanned by right $K_S$-invariant elements of $I(\mu, \beta_1)$. Further, $I^1(\mu, \beta_1)$ denotes the $G(\mathbf{A})$-submodule of $I(\mu, \beta_1)$ generated by $I(\mu, \beta_1)_S$ for all finite sets $S \subset V_f$ containing $S_r(\mu)$. Then the residue map

$$I(\chi(\mu, \mu)//\mathbf{A})_S \to L_d^2(B, K_S) : \Phi \mapsto \mathrm{Res}_{\Lambda=\beta_1} E^1(g, \Phi, \Lambda)$$

induces an intertwining operator from $I^1(\mu, \beta_1)$ into the space associated with residual spectrums $L_d^2(B)$. We write $\pi(\mu)$ for the image of this intertwining operator. Namely, $\pi(\mu)$ is an automorphic representation generated by $L_d^2(B, K_S, X(\mu))$ for all finite sets $S \subset V_f$ containing $S_r(\mu)$.

Proposition 5. *Let $\mu \in A_\infty(F)$ be a nontrivial character and $\otimes_v \pi_v(\mu)$ a restricted tensor product decomposition of $\pi(\mu)$. If $u \notin S_r(\mu)$, then $\pi_u(\mu)$ is a spherical irreducible representation of $G(F_u)$.*

*Proof.* From the above construction, $\pi_u(\mu)$ is clearly spherical. We show the irreduciblity of it. In the following, we denote by $\pi_u(\mu)_L$ for an open subgroup $L \subset K_u$ the subspace consisting of $L$-invariant elements of $\pi_u(\mu)$.

First, assume $u$ is a finite place. Then it follows from [3. Corollary 3.3.7] that the $u$-component of a restricted tensor product of $I^1(\mu, \beta_1)$ coincides with $I_u(\mu, \beta_1)$, so that $\pi_u(\mu)$ is isomorphic to a quotient representation of $I_u(\mu, \beta_1)$. We take a finite set $S \subset V_f$ such that $S_r(\mu) \cup \{u\} \subset S$ and $|S_u^-(\mu)| \geq 2$. Then, by Theorem 1,

$$\pi(\mu)_{K_S} = L_d^2(B, K_S, X(\mu)) \cong \bigoplus_{\lambda \in \Gamma(S,\mu)} \lambda_S(\mu)$$

and

$$\pi_u(\mu)_{\mathrm{Ker}(r_u)} = Y_0^\mu(u) \oplus Y_1^\mu(u).$$

Let $L_u = r_u^{-1}(B(\mathbf{k}_u))$ be an Iwahori subgroup of $K_u$. By the Frobenius reciprocity law and Lemma 7, it is seen

$$\dim \pi_u(\mu)_{L_u} = \begin{cases} 3 & \text{if } u \in S_u^+(\mu) \\ 2 & \text{if } u \in S_u^-(\mu) \end{cases}.$$

On the other hand, from [2. Lemma 4.7] and [18. Chapters 3 and 6], it follows that $I_u(\mu, \beta_1)$ has a composition series of the form

$$\{0\} = J_4 \subset J_3 \subset J_2 \subset J_1 \subset J_0 = I_u(\mu, \beta_1)$$

$$J_0/J_1 \text{ is spherical}$$

and

$$\dim(J_0/J_1)_{L_u} = \begin{cases} 3 & \text{if } u \in S_u^+(\mu) \\ 2 & \text{if } u \in S_u^-(\mu) \end{cases}.$$

This implies $\pi_u(\mu) \cong J_0/J_1$.

Next let $\pi_\infty(\mu)$ (resp. $\pi_f(\mu)$) be the infinite part (resp. finite part) of $\pi(\mu)$. We show $\pi_\infty(\mu)$ is irreducible. Since $L_d^2(B)$ decomposes to the sum of irreducible subspaces, so is $\pi(\mu)$. Hence, if $\pi_\infty(\mu)$ is reducible, it decomposes to the sum of proper subspaces $\pi_\infty^1(\mu)$ and $\pi_\infty^2(\mu)$. Then, by Theorem 1, one has

$$(\pi_\infty^1(\mu)_{K_\infty} \oplus \pi_\infty^2(\mu)_{K_\infty}) \otimes \pi_f(\mu)_{K_{S,f}} \cong \bigoplus_{\lambda \in \Gamma(S,\mu)} \lambda_S(\mu)$$

for any finite set $S \subset V_f$ containing $S_r(\mu)$, where $K_{S,f}$ is the finite part of $K_S$. Since $\pi_f(\mu)_{K_{S,f}} \cong \oplus_{\lambda \in \Gamma(S,\mu)} \lambda_S(\mu)$ and the multiplicity of $\lambda_S(\mu)$ in $\pi(\mu)_{K_S}$ is one for any $\lambda \in \Gamma(S, \mu)$, either $\pi_\infty^1(\mu)_{K_\infty}$ or $\pi_\infty^2(\mu)_{K_\infty}$ must be trivial. We assume $\pi_\infty^2(\mu)_{K_\infty}$ is trivial. Then $\pi(\mu)_{K_S} = \pi_\infty^1(\mu)_{K_\infty} \otimes \pi_f(\mu)_{K_{S,f}}$. Since $\pi(\mu)$ is generated by $\pi(\mu)_{K_S}$ for finite sets $S \subset V_f$ one has $\pi(\mu) = \pi_\infty^1(\mu) \otimes \pi_f(\mu)$, and hence $\pi_\infty(\mu) = \pi_\infty^1(\mu)$.

It seems that $\pi(\mu)$ is irreducible. However, at present, we know only the upper bounds of the number of irreducible components of $\pi(\mu)$.

THEOREM 3. *Let $\mu \in A_\infty(F)$ be a nontrivial element. The number of irreducible components of $\pi(\mu)$ is less than or equal to $2^{|S_r(\mu)|}$.*

*Proof.* By Proposition 5, it is sufficient to show that the number of irreducible components of $\pi_v(\mu)$ is at most 2 for any $v \in S_r(\mu)$. Thus we fix a $v \in S_r(\mu)$, Since the $v$-component of a restricted tensor product of $I^1(\mu, \beta_1)$ equals $I_v(\mu, \beta_1)$ by [3, Corollary 3.3.7], $\pi_v(\mu)$ is a quotient representation of $I_v(\mu, \beta_1)$. Further, by [3, Theorem 7.2.4], each constituent $J$ of $\pi_v(\mu)$ has a non-zero vector fixed by $\mathrm{Ker}(r_v)$. In other words, the subspace of $\mathrm{Ker}(r_v)$-invariant elements of $J$ contains necessarily at least one nontrivial representation of $G(\mathbf{k}_v)$. On the other hand, if we take a finite set $S \subset V_f$ such that $S_r(\mu) \subset S$ and $|S_u^-(\mu)| \geq 2$, then one has $\pi(\mu)_{K_S} \cong \bigoplus_{\lambda \in \Gamma(S,\mu)} \lambda_S(\mu)$ and hence $\pi_v(\mu)_{\mathrm{Ker}(r_v)} \cong Y_0^\mu(v) \oplus Y_1^\mu(v)$ by Theorem 1. This implies that the number of irreducble constituents of $\pi_v(\mu)$ is at most 2.

THEOREM 4. *Let $\mu \in A_\infty(F)$ be a nontrivial character. Then each irreducible constituent of $\pi(\mu)$ is of multiplicity one in $L_d^2(B)$.*

*Proof.* By Proposition 5 and the proof of Theorem 3, it is known that each irreducible constituent of $\pi(\mu)$ has a non-zero vector fixed under $K_S$ for sufficiently large $S$. Then the assertion follows from the fact that $\lambda_S(\mu)$ is of multiplicity one in $L_d^2(B, K_S)$ for any $\lambda \in \Gamma(S, \mu)$.

Finally, we remark about an $L$-function of $\pi(\mu)$. Proposition 5 implies that the set of irreducible constituents of $\pi(\mu)$ becomes an $L$-packet of automorphic representations. We write $L(s, \pi(\mu))$ for the standard (degree 5) $L$-function attached to this $L$-packet. If we define the factor $L_v(s, \pi(\mu))$ attached to a ramified place $v \in S_r(\mu)$ by $(1 - q_v^{-s})^{-1}$, then the simple calculation gives

$$L(s, \pi(\mu)) = \zeta_F(s)L(s, \mu)^2 L(s-1, \mu)\, L(s+1, \mu),$$

where $\zeta_F(s)$ is the Dedekind zeta function of $F$.

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