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EXTENSION OF HOLOMORPHIC L^2-FUNCTIONS WITH WEIGHTED GROWTH CONDITIONS

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Introduction

In this article a new contribution to the following question is given: Let $\Omega \subset\subset \mathbf{C}^n$ be a bounded pseudoconvex domain with C^∞-smooth boundary, $q \in \partial\Omega$ a fixed point and H a k-dimensional affine complex plane such that $q \in H$ and H intersects $\partial\Omega$ at q transversally. Let U be a suitably small neighborhood of q, and denote by r a C^∞-defining function of Ω on U. Under which conditions on $\partial\Omega$ near q is it possible to find an exponent $\eta > 0$ such that every holomorphic function f on $\Omega' = H \cap \Omega \cap U$ with

$$\int_{\Omega'} |f|^2 d\lambda' < \infty \tag{0.1}$$

where $d\lambda'$ denotes the Lebesgue-measure on H, can be extended to a holomorphic function $\hat{f}$ on $\Omega \cap U$ such that even

$$\int_{\Omega\cap U} \left| \hat{f} \right|^2 \frac{d\lambda}{|r|^\eta} < \infty. \tag{0.2}$$

More generally, we will also consider certain cases, where $d\lambda'$ and $d\lambda$ are the respective Lebesgue-measures together with a weight factor of the form $e^{-\varphi}$ where φ is allowed to be *not* plurisubharmonic.

One of the main motivations for studying this question in a situation, which is necessarily technically more complicated than in previous work, is the following: in [B-D] (Theorem 3) a $\bar{\partial}$-solving integral operator was constructed on bounded pseudoconvex domains with real-analytic boundary, which is regularizing with respect to the L^1-norm, a result which, so-far, has not been obtained by other methods. In the respective estimation of that kernel (Proof of Theorem 3) a proposition was used which was stated on p. 93 of [B-D] and for the proof of which it was referred to the present article. Theorem 1 of the present article is, in fact, this proposition.

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Similar extension problems as here have been considered in several articles by various authors. In fact, the solution of the Levi problem as given in Hörmander's book [H2] (see Theorem 4.2.9) is already based on a simple extension technique for L^2-holomorphic functions or, more generally, $\bar{\partial}$-closed $(0, q)$-forms. Refined extension results with L^2-control are, for instance, due to T. Yoshioka [Y], T. Ohsawa [O1], S. Nakano [N], T. Takegoshi [O-T], T. Ohsawa [O2] and Diederich-Herbort-Ohsawa [D-H-O].

In [D-H-O] a quantitative version of the following statememt was proved: If Ω is uniformly extendable near q, then there are always holomorphic functions on $\Omega \cap H \cap U$ which are not in L^2 $(\Omega \cap H \cap \mathrm{U})$, but can, nevertheless, be extended to square-integrable holomorphic functions on $\Omega \cap U$. The goal of this article as expressed by the inequalities (0.1) and (0.2) can be understood as in some sense dual to this fact. Namely, here we start with holomorphic L^2-functions f on $\Omega \cap H \cap U$ and extend them to holomorphic functions $\hat{f}$ on $\Omega \cap U$ which are better than just L^2 . In order to deal with this problem a more complicated $\bar{\partial}$-solving machinery has to be applied than in [D-H-O]. We will use as our most essential tool a curvature inequality due to T. Ohsawa and K. Takegoshi [O-T].

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§ 1. Basic notions, notations and results

Let $\Omega \subset\subset \mathbf{C}^n$ be a bounded pseudoconvex domain with C^∞-smooth boundary, $z_0 \in \partial\Omega$ an arbitrary point. By a defining function of Ω near z_0 we mean a C^∞ real-valued function r on a neighborhood U of z_0 such that

$$\Omega \cap U = \{z \in U \mid r(z) < 0\}$$

and $dr(z) \neq 0$ for all $z \in \partial\Omega \cap U$. We talk about a global defining function r of Ω if U is a neighborhood of all of $\partial\Omega$.
In [D-L] the notion of pseudoconvex extendability of finite order was introduced as a summarization of certain properties which in [D-F 2] were already shown to hold for $\partial\Omega$ real-analytic. For the purpose of this paper we need the following modified version of this notion:

Definition. Let Ω be as above, $0 \in \partial\Omega$ and r a defining function of Ω near 0. Furthermore, let H be a k-dimensional complex linear subspace of $\mathbf{C}^n$ which intersects $\partial\Omega$ at 0 transversally and let $N \in \mathbf{N}$. For $\zeta \in \mathbf{C}^n$ we denote by H_ζ the affine subspace of $\mathbf{C}^n$ parallel to H and passing through ζ. Then Ω is said to be uniformly extendable of N^{th} order (in a pseudoconvex way) along the H_ζ near 0 if there exist a radius $R>0$ and a function $\rho(\zeta, z) \in C^\infty(M)$, where $M=(\bar{B}(0\,;R)\cap\bar{\Omega})\times\bar{B}(0\,;2R)$, with the following properties

1) $d_z\,\rho(\zeta, z)\neq 0$ on M

2) There is a $C_1>0$ such that for $\zeta\in B(0;R)\cap\bar{\Omega}$ and $z\in B(0;2R)$ we have

$$C_1\,(-\operatorname{dist}(z, H_\zeta)+r(\zeta)+r(z))\leq\rho(\zeta, z)\leq r(\zeta)+r(z)-\operatorname{dist}^N(z,H_\zeta)$$

3) The sets $\{z\in B(0;\ 2R)\mid\rho(\zeta,\ z)<0\}$ are pseudoconvex for all $\zeta\in B(0;R)\cap\bar{\Omega}$.

In complete analogy to the proof of Theorem 2 in [D-F 2] the following can be shown (we will not give details in this article):

PROPOSITION. *If $\partial\Omega$ is C^ω and of finite type near* 0, *in particular, if $\partial\Omega$ is C^ω everywhere, and if H is as above, then there is an $N\in\mathbf{N}$ such that Ω is uniformly extendable of N^{th} order along the H_ζ near* 0.

Remark. It was shown in [D-F 1] that bounded pseudoconvex domains $\Omega\subset\subset\mathbf{C}^n$ with smooth real-analytic boundaries are of finite type.

Now let $D\subset\Omega$ be a pseudoconvex domain given by

$$D=\{\rho_D:=r+\psi_0(|z|^2)<0\}, \tag{1.1}$$

with a convex increasing smooth function ψ_0 on $\mathbf{R}$, for which, with small $\varepsilon>0$, $\psi_0=0$ on $(-\infty,\varepsilon^2]$. So $\partial D\cap B(0;\ \varepsilon)=\partial\Omega\cap B(O;\ \varepsilon)$. Assume $D\subset\Omega\cap B(0;2\varepsilon)$. We will solve our extension problem on D.

Given a holomorphic function f on $D\cap H_\zeta$ as in (0.1) we will construct the holomorphic extension $\hat{f}$ for f, for which (0.2) holds, in the following special form: $\hat{f}=f_1-g$, where f_1 is a smooth extension of f to a "cone" shaped set with support in this set, and g is a smooth function on D which satisfies

$$\bar{\partial}g=\bar{\partial}f_1. \tag{1.2}$$

In order to make this more precise, we introduce, for $\zeta\in B(0;R)$, the orthogonal projection π''_ζ of $\mathbf{C}^n$ onto H and let $\pi'_\zeta=\mathrm{id}-\pi''_\zeta$.
Then, for small enough C_0, $R'>0$, and for all ζ, with $|\zeta|<R'$, the cone

$$K_{C_0}(\zeta):=\{z\in D\mid|\pi'_\zeta(z)|\leq 2c_0|\rho_D(z)|\}$$

is mapped onto $D\cap H_\zeta$ under π''_ζ, and

$$2\,\rho_D(\pi''_\zeta(z))<\rho_D(z)<\frac{1}{2}\rho_D(\pi''_\zeta(z)) \tag{1.3}$$

on $K_{c_0}(\zeta)$.
Let us fix a cut-off function $\chi \in C_0^\infty(\mathbf{R})$ with $0 \leq \chi \leq 1$, $\chi \equiv 1$, on $[-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp}(\chi) \subset [-1, 1]$. For a positive continuous function γ we denote by $L^2(D, \gamma d\lambda^n)$ (resp. $L^2(D \cap H_\zeta, \gamma d\lambda^k)$) the space of measurable functions on D (resp. $D \cap H_\zeta$) which are square-integrable with respect to the measure $\gamma d\lambda^n$ (resp. $\gamma d\lambda^k$). Here, for $1 \leq \nu \leq n$, $d\lambda^\nu$ denotes the Lebesgue measure in complex dimension ν. Our extension theorem is the following (cf. Proposition (p. 93) in [B-D]).

THEOREM 1. *Let $\Omega = \{r < 0\}$ be a bounded pseudoconvex domain in $\mathbf{C}^n$ with C^∞-smooth boundary which contains 0, and let $D \subset \Omega$ be a pseudoconvex domain as in (1.1) with defining function ρ_D. Assume $H \subset H^{k+1}$ are linear subspaces of $\mathbf{C}^n$ of dimensions k and $k+1$, respectively, and H intersects $\partial\Omega$ transversally near 0. Furthermore, suppose Ω is uniformly extendable in a pseudoconvex way of N^{th} order along the affine subspaces H_ζ with an extending function ρ defined on $(\bar{B}(0; R) \cap \bar{\Omega}) \times B(0; 2R)$. Let a, δ be numbers with $0 < a \leq 1$ and $\delta \in (-1 + \frac{2a}{N}, \frac{2a}{N})$.*

Then for small $\varepsilon' > 0$ there exists a family $(E_\zeta)_{\zeta \in B(0;\varepsilon') \cap \Omega}$ of continuous linear extension operators

$$E_\zeta : L^2(D \cap H_{\zeta'} \mid \rho_D |^\delta d\lambda^k) \cap \mathcal{O}(D \cap E_\zeta) \longrightarrow$$
$$L^2(D \cap H_\zeta^{k+1}, (| \rho_D |^{\delta - 2a/N} | \log | \rho_D ||^{-3})(z) | \pi'_\zeta(z)|^{-2(1-a)}$$
$$\times d\lambda^{k+1}(z''')) \cap \mathcal{O}(D \cap H_\zeta^{k+1})$$

of the form

$$E_\zeta(h) = \chi\left(\frac{| \pi'_\zeta(z) |}{c_0 | \rho_D(\pi''_\zeta(z))|}\right) h(\pi''_\zeta(z)) - g_\zeta(z) \tag{1.4}$$

where $g_\zeta \in C^\infty(D \cap H_\zeta^{k+1})$ is a function satisfying

$$\int_{z \in D \cap H_\zeta^{k+1}} | g_\zeta |^2 \left(\frac{| \rho_D |^{-a/N}}{| \pi'_\zeta |^{1-a}}\right)^2 \frac{| \rho_D |^\delta}{| \log | \rho_D ||^3} d\lambda^{k+1} \leq C \| h \|^2_{L^2(D \cap H_\zeta, | \rho_D |^\delta d\lambda^k)} \tag{1.5}$$

with a positive constant C, independent of ζ. The operator norms of the E'_ζ are bounded above by C.

Remark. In case $k = n - 1$, we obtain again Proposition 2 of [D-H-O] up to zero-order terms in $| \rho_D |$ by choosing $a = 1$ and $\delta = \frac{2}{N}$.

By an iteration method on Theorem 1 we can consider the following situation. Suppose that we have an ascending chain of linear subspaces

$$H^k = H \subsetneqq H^{k+1} \subsetneqq \cdots \subsetneqq H^{n-1} \subsetneqq H^n = \mathbf{C}^n$$

such that for each ν the section $\Omega \cap H^{\nu+1}$ is uniformly extendable along H^ν, $k \leq \nu \leq n-1$, of order $N_{\nu+1} \geq 2$ near 0.

Then we have the following results:

THEOREM 2. *Assume Ω and H are as before. Let* $\varepsilon_n := \min\{2\sum_{j=k+1}^{n} \frac{1}{N_j}, 1-\varepsilon''\}$ *with an $\varepsilon'' > 0$ arbitrarily small, and $0 \leq \delta \leq 2/N_{k+1}$. Then there exists a bounded linear extension operator*

$$E : L^2(D \cap H, |\rho_D|^\delta d\lambda^k) \cap \mathcal{O}(D \cap H) \longrightarrow$$
$$L^2(D, |\rho_D|^{\delta-\varepsilon_n} |\log|\rho_D||^{-3(n-k)} d\lambda^n) \cap \mathcal{O}(D),$$

if D is sufficiently small.

THEOREM 3. *Let ε_n be as in Theorem 2, and $\varepsilon'_n = \varepsilon_n/2$. If $\delta > 0$ is small enough, then there exists a bounded linear extension operator*

$$E' : L^2(D \cap H, |\rho_D|^\delta d\lambda^k) \cap \mathcal{O}(D \cap H) \longrightarrow$$
$$L^2(D, |\rho_D|^{\delta-\varepsilon'_n} d^{-1} |\log|\rho_D||^{-3(n-k)}) \cap \mathcal{O}(D).$$

Here d denotes the function $d(z) = \Pi_{\nu=k}^{n-1} \operatorname{dist}(z, H^\nu)$.

§ 2. The apriori estimate for the $\bar\partial$ equation with weights

Let (X, ds^2) be a hermitian manifold of dimension n, and $\omega : X \to \mathbf{R}^+$ be a continuous function. For $q \in \{0,\ldots, n-1\}$ we denote by $L^2_{(n,q)}(X, \omega, ds^2)$ the Hilbert space of all measurable (n, q) forms u for which $|\int_X u \wedge \overline{*u\cdot\omega})|$ is finite. Here, $*$ is the Hodge operator associated to ds^2. If φ is a real-valued continuous function on X, the $\bar\partial$ operator and its formal adjoint have densely defined closures $\bar\partial_\varphi : L^2_{(n,q)}(X, e^{-\varphi}, ds^2) \to L^2_{(n,q+1)}(X, e^{-\varphi}, ds^2)$ and $\bar\partial^*_\varphi : L^2_{(n,q+1)}(X, e^{-\varphi}, ds^2) \to L^2_{(n,q)}(X, e^{-\varphi}, ds^2)$. The domains of $\bar\partial_\varphi$ and $\bar\partial^*_\varphi$ will be denoted $\operatorname{dom}(\bar\partial_\varphi)$ and $\operatorname{dom}(\bar\partial^*_\varphi)$, respectively, and the scalar product and norm on $L^2_{(n,q)}(X, e^{-\varphi}, ds^2)$ by $(\cdot, \cdot)_{ds^2,e^{-\varphi}}$ and by $\| \|_{ds^2,e^{-\varphi}}$.

The following theorem on the solvability of the $\bar\partial$ equation is well-known ([A-V]):

PROPOSITION 2.1. *Let $v \in L^2_{(n,q+1)}(X, e^{-\varphi}, ds^2)$ be a smooth $\bar\partial$ closed form on X. Suppose there exists a positive continuous function η on X such that, with a positive constant C_v we have the basic estimate*

(BE) $$|(u, v)_{ds^2,e^{-\varphi}}|^2 \leq C_v\, Q_{\varphi,\eta}(u)$$

for all $u \in L^2_{(n,q+1)}(X, e^{-\varphi}, ds^2) \cap \operatorname{dom}(\bar\partial_\varphi) \cap \operatorname{dom}(\bar\partial^*_\varphi)$, *where* $Q_{\varphi,\eta}(u) := \|\sqrt{\eta}\,\bar\partial_\varphi u\|^2_{ds^2,e^{-\varphi}} + \|\sqrt{\eta}\,\bar\partial^*_\varphi u\|^2_{ds^2,e^{-\varphi}}$. *Then there exists a solution* $w \in L^2_{(n,q)}(X, e^{-\varphi}, ds^2)$ *of the equation* $\bar\partial(\sqrt{\eta}\,w) = v$, *satisfying* $\|w\|^2_{ds^2,e^{-\varphi}} \leq C_v$.

If one looks carefully at proof of this theorem, then one observes, that the following holds

PROPOSITION 2.2. *If* Y *is a subspace of* $L^2_{(n,q+1)}(X, e^{-\varphi}, ds^2) \cap$ *Null space of* $\bar\partial_\varphi$ *with* (BE) *holding for each* $v \in Y$, *then there exists a linear operator* $S\colon Y \to L^2_{(n,q)}(X, e^{-\varphi}, ds^2)$ *with* $\bar\partial(\sqrt{\eta}\,S(v)) = v$ *and* $\|S(v)\|^2_{ds^2,e^{-\varphi}} \leq C_v$.

We want to solve (1.2) by using this proposition with suitable φ and η and metric ds^2. Our starting point is a curvature estimate due to Ohsawa-Takegoshi (the formula before Proposition 1 in [O-T], p. 199) which leads to sufficient conditions on the auxiliary functions φ and η for (BE) to hold for a given smooth form $v \in L^2_{(n,1)}(X, e^{-\varphi}, ds^2)$. The lemma which is relevant for our purposes is

PROPOSITION 2.3. *Let* $v \in L^2_{(n,1)}(X, e^{-\varphi}, ds^2)$ *be a smooth form on* X. *Suppose,* ds^2 *is Kähler, and there are smooth functions* φ *and* η *on* X, $\eta > 0$, *such that*

a) $i\,\partial\bar\partial\varphi \geq ds^2$

b) *The length* $|\frac{\bar\partial\eta}{\eta}|_{ds^2}$ *of* $\frac{\bar\partial\eta}{\eta}$ *with respect to* ds^2 *is bounded above by some positive constant* C_1.

c) $-\eta$ *is strictly plurisubharmonic on* X, *and the integral* $J_\varphi(v) := \int_X v \wedge \overline{*}_{-\partial\bar\partial\eta}\, v\, e^{-\varphi}$ *is finite, where* $\overline{*}_{-\partial\bar\partial\eta}$ *is the Hodge operator associated to the Kähler metric with potential* $-\eta$.

Then, for any smooth $(n, 1)$ *form* u *on* X *with compact support, we have*

(BE′) $$|(u, v)_{ds^2,e^{-\varphi}}|^2 \leq 2(1+2C_1^2)\, J_\varphi(v)\, Q_{\varphi,\eta}(u).$$

Proof. Let Λ be the adjoint in $L^2_{(n,1)}(X, e^{-\varphi}, ds^2)$ of the left multiplication by the fundamental form of ds^2. For any $u \in C^{n,1}_0(X) :=$ space of compactly supported smooth $(n, 1)$ forms on X the Ohsawa-Takegoshi curvature formula gives

(2.1) $$Q_{\varphi,\eta}(u) \geq i((\eta\partial\bar\partial\varphi - \partial\bar\partial\eta) \wedge \Lambda u, u)_{ds^2,e^{-\varphi}} + 2\,Re(u, \bar\partial\eta \wedge \bar\partial^*_\varphi u)_{ds^2,e^{-\varphi}}$$

The second member on the right-hand side is in absolute value bounded by

$$\begin{aligned}|(u, \bar\partial\eta \wedge \bar\partial^*_\varphi u)_{ds^2,e^{-\varphi}}| &= |(\sqrt{\eta}\,u, \frac{\bar\partial\eta}{\eta} \wedge \sqrt{\eta}\,\bar\partial^*_\varphi u)_{ds^2,e^{-\varphi}}| \\ &\leq \frac{1}{2}\|\sqrt{\eta}\,u\|^2_{ds^2,e^{-\varphi}} + 2C_1^2\|\sqrt{\eta}\,\bar\partial^*_\varphi u\|^2_{ds^2,e^{-\varphi}}\end{aligned}$$

$$\leq \frac{1}{2} i(\eta\partial\bar{\partial}\varphi \wedge \Lambda u, u)_{ds^2, e^{-\varphi}} + 2C_1^2 Q_{\varphi,\eta}(u)$$

(since, by (a), $\| \sqrt{\eta}\ u \|^2_{ds^2, e^{-\varphi}} \leq i(\eta\partial\bar{\partial}\varphi \wedge \Lambda u, u)_{ds^2, e^{-\varphi}})$. Substituting this into (2.1) we arrive at

(2.2) $$-i(\partial\bar{\partial}\eta \wedge \Lambda u, u)_{ds^2, e^{-\varphi}} \leq (1 + 2C_1^2)\, Q_{\varphi,\eta}(u).$$

Our claim now is

(2.3) $$|(u,v)_{ds^2, e^{-\varphi}}|^2 \leq -2i\, J_\varphi(v)\,(\partial\bar{\partial}\eta \wedge \Lambda u, u)_{ds^2, e^{-\varphi}}.$$

Let for proof of this inequality U be any local coordinate patch and $(\omega_1, \ldots, \omega_n)$ be an orthonormal frame for ds^2 on U; by dV we denote the volume form of ds^2. Let $A = (\eta_{\nu\bar{\mu}})_{\nu,\mu=1}^n$ be the matrix for which

$$-\partial\bar{\partial}\eta = \sum_{\nu,\mu=1}^{n} \eta_{\nu\bar{\eta}}\, \omega_\nu \wedge \bar{\omega}_\mu.$$

For any form $w \in C_o^{(n,1)}(X)$ we write on U

$$w = \sum_{\nu=1}^{n} w_\nu\, \omega_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_\nu,$$

and denote by $\hat{w}$ the column vector entries $w_1, \ldots, w_n$ and $w^t\hat{w}$ its transpose. Then we have on U:

(α) $$u \wedge \bar{*} v e^{-\varphi} = {}^t\hat{u}\bar{\hat{v}}\ e^{-\varphi}\, dV$$

(β) $$-i\partial\bar{\partial}\eta \wedge \Lambda u \wedge \bar{*}\ u\, e^{-\varphi} = \frac{1}{2} {}^t\hat{u} A \bar{\hat{u}}\ e^{-\varphi}\, dV$$

(γ) $$v \wedge \bar{*}_{-\partial\bar{\partial}\eta}\ v\, e^{-\varphi} = {}^t\hat{v}\, A^{-1}\bar{\hat{v}}\ e^{-\varphi} dV$$

Now by the Cauchy-Schwarz inequality we can estimate

$$|{}^t\hat{u}\ \bar{\hat{v}}\,|\, e^{-\varphi} \leq ({}^t\hat{v}\, A^{-1}\bar{\hat{v}}\ e^{-\varphi})^{\frac{1}{2}} ({}^t\hat{v}\, A\ \bar{\hat{v}}\ e^{-\varphi})^{\frac{1}{2}}.$$

By means of a standard partition of unity argument we obtain (2.3) from this. Obviously (BE′) is implied by (2.2) and (2.3) □

§3. Proof of Theorem 1

We begin by normalizing the holomorphic coordinates in such a way that, if we write $z = (z'', z')$, $z'' = (z_1, \ldots, z_k)$, $z' = (z_{k+1}, \ldots, z_n)$, $z''' = (z'', z_{k+1})$, $z^* = (z_{k+2}, \ldots, z_n)$, then $H = \{z \in \mathbf{C}^n \mid z' = 0\}$, $H^{k+1} = \{z \in \mathbf{C}^n \mid z^* = 0\}$, and hence $H_\zeta = \{z' = \zeta'\}$, $H_\zeta^{k+1} = \{z^* = \zeta^*\}$. The projections π''_ζ and π'_ζ now have the form $\pi''_\zeta(z) = (z'', \zeta')$ and $\pi'_\zeta(z) = (0'', z' - \zeta')$. Furthermore, we assume that the Re z_1-axis points in the direction of the outer normal to $\partial\Omega$ at 0. Notice that, because of the transversality of H and $\partial\Omega$, for any $\tilde{\zeta} \in \bar{B}(0; \varepsilon') \cap \bar{\Omega}$ there is always a $\zeta \in B(0; \varepsilon) \cap \partial\Omega$ such that $H_{\tilde{\zeta}} = H_\zeta$. We fix such a ζ. For each $f \in L^2(D \cap H_\zeta, |\rho_D|^\delta d\lambda^k) \cap \mathcal{O}(D \cap H_\zeta)$ we introduce a smooth $\bar{\partial}$-closed $(n, 1)$-form on $X := D \cap H_\zeta^{k+1} \setminus H_\zeta$, by

$$v_f := \bar{\partial}\{\chi\Big(\frac{|z_{k+1} - \zeta_{k+1}|}{c_0|\rho_D(z'', \zeta')|}\Big) f(z'', \zeta') dz_1 \wedge \cdots \wedge dz_{k+1}\}. \tag{3.1}$$

For small enough c_0 we have $\operatorname{supp}(v_f) \subset K_{c_0}(\zeta)$. In order to be able to apply Proposition 2.3 we first provide X with a complete Kähler metric and choose a smooth function φ on X satisfying $i\partial\bar{\partial}\varphi \geq ds^2$ (which is hypothesis (a) in Proposition 2.3). For $0 < \delta' << 1 - \frac{2a}{N} + \delta$ we let

$$\varphi_1 = -\delta' \log(-\rho_D(z''', \zeta^*)) + |z'''|^2 + V_{H_\zeta}(z'''), \tag{3.2}$$

where $V_{H_\zeta}(z''') = -\log\log\frac{1}{|z_{k+1} - \zeta_{k+1}|}$.

Then φ_1 is the potential of a complete Kähler metric ds^2 on X. With a smooth plurisubharmonic function Ψ which will be chosen later, we put

$$\varphi := \varphi_1 + \Psi. \tag{3.3}$$

For a small number $\beta > 0$ we define

$$\eta := -(-\rho_D)^{\frac{2a}{N} + \delta' - \delta}(1 - \beta\log(-\rho_D(z''', \zeta^*)))^3 V_{H_\zeta} \tag{3.4}$$

and will prove later that, if we replace ρ_D by $\rho_D e^{-L|z|^2}$ with a large positive number L, then η will, (after shrinking D, resp. ε) satisfy the conditions (b) and (c) of Proposition 2.3 uniformly with respect to ζ with an explicit estimate $J_\varphi(v_f)$ in terms of the norm $\|f\|^2_{L^2(D \cap H_\zeta, |\rho_D|^\delta d\lambda^k)}$. Our key lemma now is:

LEMMA 3.1. *Let $0 < p < 1$ and $m \in \mathbf{N}_0$. Then the positive numbers β, ε, and $\varepsilon' < \varepsilon$ and the defining function ρ_D for D can be chosen such that for any $\zeta \in \bar{B}(0; \varepsilon') \cap \partial\Omega$ the function*

$$\tilde{\eta} := -(-\rho_D)^P(1 - \beta\log(-\rho_D(z''', \zeta^*)))^{3m} V_{H_\zeta} \tag{3.5}$$

is strictly plurisubharmonic on X and satisfies

(i) $$\Big|\frac{\bar{\partial}\tilde{\eta}}{\tilde{\eta}}\Big| \leq C_1$$

(ii) $$-i\frac{(\partial\bar{\partial})'''\tilde{\eta}}{\tilde{\eta}} \geq$$

$$iC_2\Big(\partial\bar{\partial}|z'''|^2 + \frac{\partial'''\rho_D \wedge \bar{\partial}'''\rho_D}{\rho_D^2}(z'', \zeta^*) + \frac{1}{-V_{H_\zeta}}\partial'''V_{H_\zeta} \wedge \bar{\partial}'''V_{H_\zeta}\Big)$$

where the positive constants C_1, C_2 depend on p, m and ε, but not on ζ, and $\bar{\partial}'''$ is the

$\bar\partial$ operator with respect to z'''.

Proof. Since for all small enough δ' (independently of ζ) one has

$$i\,\partial\bar\partial(-\delta'\log(-\rho_D(z''',\zeta^*))+|z'''|^2)\geq i\frac{\delta'}{2}\frac{\partial'''\rho_D\wedge\bar\partial'''\rho_D}{p_D^2}(z''',\zeta^*)$$

it follows that

$$ds^2\geq i\Big(\frac{\delta'}{2}\frac{\partial'''\rho_D\wedge\bar\partial'''\rho_D}{\rho_D^2}(z''',\zeta^*)+\partial'''V_{H_\zeta}\wedge\bar\partial'''V_{H_\zeta}\Big).$$

We can now check (i). A computation gives

$$\frac{\bar\partial'''\tilde\eta}{\tilde\eta}=\Big(p-\frac{3\beta m}{1-\beta\log(-\rho_D)}\Big)\frac{\bar\partial'''\rho_D}{\rho_D}(z''',\zeta^*)+\frac{\bar\partial'''V_{H_\zeta}}{-V_{H_\zeta}}.$$

For sufficiently small $\beta>0$ and $\varepsilon'<\varepsilon'<\varepsilon<\frac{1}{3}e^{-e}$ we have

$$0<3\beta\mathrm{m}/1-\beta\log(-\rho_D)<p/2 \textit{ on } D, \textit{ and } -V_{H_\zeta}\geq 1, \textit{ when } |\zeta|<\varepsilon';$$

hence

$$\Big|\frac{\bar\partial'''\tilde\eta}{\tilde\eta}\Big|^2_{ds^2}\leq 2p^2\Big|\frac{\bar\partial'''\rho_D}{\rho_D}(z''',\zeta^*)\Big|^2_{ds^2}+2\Big|\bar\partial'''V_{H_\zeta}\Big|^2_{ds^2}.$$
$$\leq\frac{4}{\delta'}p^2+2.$$

This proves (i). To obtain (ii) we need to choose the defining function for D suitably. By the arguments of [D-F 3] we can find a constant $L\gg 1$ such that, for $\varepsilon\ll 1$ the function $\sigma=-(-p_D)^{1-(1-p)^2}$ is strictly plurisubharmonic on D and $i\,\partial\bar\partial\sigma\geq ic_3|\sigma|\,\partial\bar\partial|z|^2$. The numbers L and $c_3>0$ do not depend on ζ. If we use the notation $U_\beta=1-\beta\log(-\rho_D)$ and $\psi=U_\beta^{3m}\cdot(-V_{H_\zeta})$ we have

$$\tilde\eta=(-\sigma)^{1-\mu}\psi(z''',\zeta^*)$$

where $\mu=\dfrac{1-p}{2-p}$ lies in (0,1). Explicit computation and evaluation at (z''',ζ^*) now gives the formula

$$-i\frac{(\partial\bar\partial)'''\tilde\eta}{\tilde\eta}=i(1-\mu)\Big((1-\frac{3m\beta}{pU_\beta})\frac{(\partial\bar\partial)'''\sigma}{-\sigma}+ \tag{3.6}$$

$$\Big[\mu+\frac{3\mathrm{m}\beta}{pU_\beta}(1-2\mu-\frac{(3\mathrm{m}-1)(1-\mu)\beta}{pU_\beta})\Big]\frac{\partial'''\sigma\wedge\bar\partial'''\sigma}{\sigma^2}$$

$$-(1-\frac{3m\beta}{pU_\beta}\Big(\frac{\partial'''V_{H_\zeta}}{V_{H_\zeta}}\wedge\frac{\bar\partial'''\sigma}{\sigma}+\frac{\partial'''\sigma}{\sigma}\wedge\frac{\bar\partial'''V_{H_\zeta}}{V_{H_\zeta}}\Big)$$

$$+\frac{1}{1-\mu}\frac{1}{-V_{H_\zeta}}(\partial'''V_{H_\zeta}\wedge\bar\partial'''V_{H_\zeta})\Big)$$

on X. If ε is small enough, then $U_\beta \geq 1$ on $D \cap H_\zeta^{k+1}$ for any choice of $\beta > 0$; then we choose $\beta < p/6m$ so small that

$$\frac{3m\beta}{p}\left(1 - 2\mu - \frac{(3m-1)(1-\mu)\beta}{p}\right) > -\frac{\mu}{2}.$$

Now

$$i\left(\frac{\partial''' V_{H_\zeta}}{V_{H_\zeta}} \wedge \frac{\bar\partial''' \sigma}{\sigma} + \frac{\partial''' \sigma}{\sigma} \wedge \frac{\partial''' V_{H_\zeta}}{V_{H_\zeta}}\right)$$

$$\leq \frac{\mu}{4} i \frac{\partial''' \sigma \wedge \bar\partial''' \sigma}{2} + \frac{4}{\mu} i \frac{\partial''' V_{H_\zeta} \wedge \bar\partial''' V_{H_\zeta}}{V_{H_\zeta}^2}$$

at $(z''', \zeta^*) \in X$. This will imply (because of (3.5) and $i\,\partial\bar\partial\sigma \geq -c_3\sigma\,\partial\bar\partial\,|z|^2$):

$$(3.7) \qquad -i\frac{(\partial\bar\partial)'''\tilde\eta}{\tilde\eta} \geq i(1-\mu)\Big(\frac{1}{2}c_3(\partial\bar\partial)'''\,|z'''|^2 + \frac{\mu}{4}\frac{\partial'''\sigma \wedge \bar\partial'''\sigma}{\sigma_2}$$

$$+ \frac{1}{1-\mu}\frac{1}{-V_{H_\zeta}}\left(1 - \frac{4}{\mu}\frac{1-\mu}{-V_{H_\zeta}}\right)\partial''' V_{H_\zeta} \wedge \bar\partial''' V_{H_\zeta}\Big)$$

on X, where we also have $-V_{H_\zeta} \geq \log\log\frac{1}{3_\varepsilon}$, if $|\zeta| < \varepsilon$.

Hence, for $\varepsilon < \frac{1}{3}\exp\left(-\exp\left(8(1-\mu)/\mu\right)\right)$ we can estimate on X

$$-i(\partial\bar\partial)'''\tilde\eta \geqq i\frac{(1-\mu)\mu}{4}\tilde\eta\Big(c_3(\partial\bar\partial)'''|z'''|^2 + \frac{\partial'''\sigma \wedge \bar\partial'''\sigma}{\sigma^2}$$

$$+ \frac{1}{-V_{H_\zeta}}\partial''' V_{H_\zeta} \wedge \bar\partial''' V_{H_\zeta}\Big).$$

Since $\partial'''\sigma/\sigma = \frac{1+\mu}{p}\partial'''\rho_D/\sigma_D$, inequality (ii) now follows a constant $C_2 > 0$ independent of ζ. □

The key lemma applies to the function η defined by (3.4) . (It has the form $\tilde\eta$ with $m = 1$, and $p = \frac{2a}{N} + \delta' - \delta$. The assumptions on δ and N, as well as the choice of δ' make sure that $0 < p < 1$). By virtue of Proposition 2.2 we have for any form $u \in C_0^{(n,1)}(X)$

$$(\mathrm{BE}') \qquad |(u, v_f)_{ds^2, e^{-\varphi}}|^2 \leq 2(1 + 2C_1^2)\, J_\varphi(v_f)\, Q_{\varphi,\eta}(u).$$

Estimation of $J_\varphi(v_f)$. Let us now estimate the integral

$$J_\varphi(v_f) = \int_X v_f \wedge \bar{*}_{-(\partial\bar\partial)'''\eta}\, v_f\, e^{-\varphi}$$

$$= \int_X |v_f|^2_{-(\partial\bar\partial)'''\eta} e^{-\varphi} d\lambda^{k+1}$$

in terms of $\|f\|^2_{L^2(D\cap H_\zeta, |\rho_D|^\delta d\lambda^k)}$. Here $|\cdot|_{-(\partial\bar\partial)'''_\eta}$ denotes the length of a form with respect to the Kähler metric with potential $-\eta$. By computation we obtain

$$v_f = \pm \frac{1}{c_0} \chi_1 f(z'', \zeta') \frac{|z_{k+1}-\zeta_{k+1}|}{|\rho_D(z'', \zeta')|} \times \tag{3.8}$$

$$\times \left[(\log \frac{1}{|z_{k+1}-\zeta_{k+1}|}) \bar{\partial}''' V_{H_\zeta} + \frac{\bar{\partial}''\rho_D}{\rho_D}(z'', \zeta') \right] \wedge \omega_{k+1}$$

where $\chi_1 = \chi'(|z'-\zeta'|/c_0|\rho_D(z'', \zeta')|)$, $\bar{\partial}'' = \bar{\partial}_{z''}$, and $\omega_{k+1} = dz_1 \wedge \cdots \wedge dz_{k+1}$. Therefore:

$$|v_f|^2_{-(\partial\bar{\partial})'''\eta} \leq 2\, \chi_1^2 |f(z'', \zeta')|^2 \times \tag{3.9}$$

$$\times \left[(\log \frac{1}{|z_{k+1}-\zeta_{k+1}|})^2 \left| \bar{\partial}''' V_{H_\zeta} \right|^2_{-(\partial\bar{\partial})'''\eta} + \left| \frac{\bar{\partial}''\rho_D}{\rho_D}(z''.\zeta') \right|^2_{-(\partial\bar{\partial})'''\eta} \right].$$

By (ii) in Lemma 3.1 we have $\left| \bar{\partial}''' V_{H_\zeta} \right|^2_{-(\partial\bar{\partial})'''\eta} \leq -V_H / C_2\eta$.

In order to estimate the second term in the brackets on the right side of (3.8) we write

$$\bar{\partial}''\rho_D(z'', \zeta') = \bar{\partial}''' \rho_D(z''', \zeta^*) - \frac{\partial\rho_D}{\partial\bar{z}_{k+1}}(z''', \zeta^*)\, d\bar{z}_{k+1}$$
$$+ \{\bar{\partial}\rho_D(z'', \zeta') - \bar{\partial}''\rho_D(z'', \zeta^*)\}.$$

The form within { } has coefficients which are bounded on $D \cap H_\zeta^{k+1}$ by c_4 $|z_{k+1}-\zeta_{k+1}|$ with some positive constant c_4 independent of ζ. Thus, again by (ii) of Lemma 3.1

$$|\bar{\partial}''\rho_D(z'', \zeta') - \bar{\partial}''\rho_D(z'', \zeta^*)|^2_{-\partial\bar{\partial})'''\eta} \leq \frac{c_4^2}{c_2} \frac{|z_{k+1}-\zeta_{k+1}|^2}{\eta}$$

and, on $\mathrm{supp}(v_f)$, $\subset K_{c_0}(\zeta)$ because of (1.3):

$$\frac{|\bar{\partial}''\rho_D(z'', \zeta')|^2_{-\partial\bar{\partial})'''\eta}}{\rho_D(z'', \zeta')^2} \leq 8 \frac{|\bar{\partial}'''\rho_D|^2_{-\partial\bar{\partial})'''\eta}}{\rho_D{}^2}(z''', \zeta^*) \tag{3.10}$$

$$+ 8 \left| \frac{\partial\rho_D}{\partial\bar{z}_{k+1}}(z''', \zeta^*) \right|^2 \frac{|d\bar{z}_{k+1}|^2_{-(\partial\bar{\partial})'''\eta}}{\rho_D(z''', \zeta^*)^2} + \frac{8c_0^2c_4^2}{C_2\eta}.$$

Since

$$i\,\partial''' V_{H_\zeta} \wedge \bar{\partial}''' V_{H_\zeta} = \frac{i\, dz_{k+1} \wedge d\bar{z}_{k+1}}{4\,|z_{k+1}-\zeta_{k+1}|^2 \log^2 \frac{1}{|z_{k+1}-\zeta_{k+1}|}}$$

we obtain from (3.10) and (ii) of Lemma 3.1 at once

$$\left| \frac{\bar{\partial}'' \rho_D(z'', \zeta')}{\rho_D(z'', \zeta^*)} \right|^2_{-(\partial\bar{\partial})'''\eta} \leq c_5 \frac{-V_{H_\zeta}}{\eta} \log^2 \frac{1}{|z_{k+1} - \zeta_{k+1}|}$$

on $\mathrm{supp}(v_f)$, with a universal positive constant c_5. Finally (3.9) and (3.10) imply

$$|v_f|^2_{-(\partial\bar{\partial})'''\eta} e^{-\varphi} \leq c_6 |f(z'', \zeta')|^2 |\rho_D(z''', \zeta^*)|^\delta \frac{e^{-\Psi}}{|\rho_D(z''', \zeta^*)|^{2a/N}}. \tag{3.11}$$

We shall now choose the plurisubharmonic weight function Ψ in a suitable way, using the uniform extendability of Ω along H_ζ. The goal is to cancel the denominator in (3.11). For this we need

PROPOSITION 3.2. *Let ζ be as before. Then there exists a smooth function $\tilde{\sigma}(\zeta; \cdot)$ on $B(0; 3\varepsilon)$ with the following properties:* (a) *The surface $\{\tilde{\sigma}(\zeta; \cdot) = 0\}$ is smooth and pseudoconvex from the side $\{\tilde{\sigma}(\zeta; \cdot) = 0\}$,* (b) *With a positive constant C_1 (independent of ζ) the estimate*

$$C_1(-|z' - \zeta'| + \rho_D(z)) \leq \tilde{\sigma}(\zeta; z) \leq -|z' - \zeta'|^N + \rho_D(z)$$

is satisfied for any $z \in B(0; 2\varepsilon)$.

Proof. The construction of $\tilde{\sigma}$ from the given extending function ρ follow from a simple patching argument. One only has to use the fact that $\partial D \backslash \partial\Omega$ is everywhere strictly pseudoconvex and therefore even extendable of order two. We leave the details to the reader. □

We now can construct Ψ in the following way:

LEMMA 3.3. *There exists a smooth function σ in an open neighborhood of $\bar{D}$ which is negative on D, such that the function*

$$\Psi(z''') := \frac{2}{N}(-a \log(-\sigma(z''', \zeta^*)) + N \log |z_{k+1} - \zeta_{k+1}|)$$

is plurisubharmonic on $D \cap H_\zeta^{k+1}$, for any $\zeta \in \partial\Omega \cap B(0; \varepsilon')$ and satisfies

$$e^{-\Psi} \leq C_1' \frac{|\rho_D(z''', \zeta^*)|^{2a/N}}{|z_{k+1} - \zeta_{k+1}|^2} \tag{3.12}$$

on $\mathrm{supp}(v_f)$, where C_1' is a positive constant independent of ζ, and furthermore,

$$e^{-\Psi} \geq |z_{k+1} - \zeta_{k+1}|^{2(1-a)} \tag{3.13}$$

on $D \cap H_\zeta^{k+1}$.

Proof. For large enough $A > 0$ the function

$$\sigma(z) := e^{A(4\varepsilon^2 - |z|^2)} \tilde{\sigma}(\zeta; z)$$

will work (cf. [D-H-O], Lemma 2, part b)). We have on $D \cap H_\zeta^{k+1}$

$$e^{-\Psi} = \frac{(-\sigma)^{2a/N}}{|z_{k+1} - \zeta_{k+1}|^2} \tag{3.14}$$

Thus (3.12), (3.13) follow from part (b) of Proposition 3.2 with z replaced by (z''', ζ^*).

The estimation of $J_\varphi(v_f)$ can now be finished as follows: We substitute (3.12) into (3.11) and replace $|\rho_D(z''', \zeta^*)|^\delta$ by $2^{|\delta|}|\rho_D(z'', \zeta')|^\delta$ (possible because of (1.3)). Integration over $D \cap H_\zeta^{k+1}$ by means of Fubini's theorem will give us the desired estimate

$$J_\varphi(v_f) \leq \|f\|^2_{L^2(D \cap H_{\zeta'}, |\rho_D|^\delta d\lambda^k)} \tag{3.15}$$

where c_7 is a positive universal constant, independent of ζ.

The extension operator. Since the metric ds^2 is complete Kähler, (BE) is satisfied for all $u \in L^2_{(n,1)}(X, e^{-\varphi}, ds^2) \cap \mathrm{dom}\,(\bar\partial_\varphi) \cap \mathrm{dom}\,(\bar\partial^*_\varphi)$. This follows from Proposition 5 in [A-V]. We apply our Proposition 2.2 to the space

$$Y = \{v_f \mid f \in L^2(D \cap H_{\zeta'} \mid \rho_D|^\delta d\lambda^k) \cap \mathcal{O}(D \cap H_\zeta)\}$$

and represent the solution operator S (*with* $q = 0$) as

$$S(v_f) = S'(f)\, dz_1 \wedge \ldots \wedge dz_{k+1}.$$

Our claim is that

$$E_\zeta(f) := \chi\left(\frac{|z_{k+1} - \zeta_{k+1}|}{c_0|\rho_D(z'', \zeta')|}\right) f(z'', \zeta') - \sqrt{\eta}\, S'(f)$$

is the desired extension operator. Clearly $E_\zeta(f)$ is holomorphic on $D \cap H_\zeta^{k+1} \setminus H_\zeta$ $(= X)$. From the definition of φ and Ψ we get

$$\frac{\eta|\rho_D|^\delta|\log|\rho_D||^{-3}}{|z_{k+1} - \zeta_{k+1}|^2} \leq e^{4\varepsilon^2}\left|\frac{\rho_D}{\sigma}\right|^{\frac{2a}{N}} e^{-\varphi}. \tag{3.16}$$

Furthermore $|\sigma| \geq |\rho_D|$ (because of Proposition 3.2b) . Thus

$$\int_{z''' \in X} \frac{|\rho_D|^\delta|\log|\rho_D||^{-3}}{|z_{k+1} - \zeta_{k+1}|^2}\, \eta\, |S'(f)|^2 d\lambda^{k+1} \leq$$
$$e^{4\varepsilon^2}\int_X |S'(f)|^2 e^{-\varphi} d\lambda^{k+1} < \infty.$$

This implies $\sqrt{\eta}\, S'(f)(z''', \zeta^*) \longrightarrow 0$, *as* $z_{k+1} \to \zeta_{k+1}$, and so $E_\zeta(f)$ is a holomorphic extension for f to $D \cap H_\zeta^{k+1}$.

Finally, we check the weighted L^2 estimte for $E_\zeta(f)$, (see the formula before

(3.16). Namely

$$\int_{D\cap H_\zeta^{k+1}} \chi\Big(\frac{|z_{k+1}-\zeta_{k+1}|}{c_0|\rho_D(z'',\zeta')|}\Big)^2 |f(z'',\zeta')|^2 \frac{|\rho_D(z''',\zeta^*)|^{\delta-2a/N}\,d\lambda^{k+1}}{|z_{k+1}-\zeta_{k+1}|^{2(1-a)}|\log|\rho_D(z''',\zeta^*)||^3}$$

$$\le 2^{|\delta|+2a/N}\int_{\{z'':(z'',\zeta')\in D\}} |f(z'',\zeta')|^2|\rho_D(z'',\zeta')|^{\delta-\frac{2a}{N}}\int_{z_{k+1}\in A(z'')}\frac{d\lambda^1}{|z_{k+1}-\zeta_{k+1}|^{2(1-a)}}\,d\lambda^k$$

$$\le c_8\|f\|^2_{L^2(D\cap H_{\zeta'}|\rho_D|^\delta d\lambda^k)}\ (\textit{with } A(z'')=\{|z_{k+1}-\zeta_{k+1}|<c_0|\rho_D(z'',\zeta')|\}),$$

by Fubini's theorem, with a universal positive constant c_8. Also by (3.2), (3.3), (3.4), and (3.13) :

$$\int_{D\cap H_\zeta^{k+1}}\Big(\frac{\eta|S'(f)|^2}{|z_{k+1}-\zeta_{k+1}|^{2(1-a)}|\log|\rho_D||^3}|\rho_D|^{\delta-2a/N}(z''',\zeta^*)\,d\lambda^{k+1}$$

$$\le e^{4\varepsilon^2}\int_{D\cap H_\zeta^{k+1}}|S'(f)|^2e^{-\varphi}d\lambda^{k+1}\le c_9\,J_\varphi(v_f)$$

$$\le c_{10}\|f\|^2_{L^2(D\cap H_{\zeta'}|\rho_D|^\delta d\lambda^k)}.$$

This finishes the proof of Theorem 1. □

Remark. We can state our Thcorem 1 in a slightly more general way, namely:

THEOREM 1′. *Let the hypotheses concerning Ω, H, H^{k+1}, D, a, δ, ε, ε', and N be as in Theorem 1. Furthermore fix a number $m\in\mathbf{N}_0$ and suppose V is plurisubharmonic on Ω and satisfies $V\circ\pi'_\zeta\le V$ on $D\cap H_\zeta^{k+1}\cap\pi''^{-1}_\zeta(D\cap H_\zeta)$, $|\zeta|<\varepsilon'$. Then, after shrinking ε' if necessary, there exists a family $(E_\zeta)_{\zeta\in\Omega\cap B(0;\varepsilon')}$ of bounded linear extension operators*

$$E_\zeta:=L^2(D\cap H_\zeta,|\rho_D|^\delta|\log|\rho_D||^{-3m}e^{-V}d\lambda^k)\cap\mathcal{O}(D\cap H_\zeta)$$

$$\longrightarrow L^2(D\cap H_\zeta^{k+1},|\rho_D|^{\delta-\frac{2a}{N}}|\pi'_\zeta|^{-2(1-a)}|\log|\rho_D||^{-3m}e^{-V}d\lambda^{k+1})\cap\mathcal{O}(D\cap H_\zeta^{k+1}).$$

the operator norms of which are bounded uniformly in ζ.

The proof of this theorem is almost the same as for Theorem 1. Just replace the weight function φ of (3.3) by

$$\varphi=\varphi_1+\Psi+V$$

and in (3.4) let

$$\eta=-(-\rho_D)^{\frac{2a}{N}+\delta'-\delta}(1-\log(-\rho_D))^{3m+3}V_{H_\zeta}.$$

Then all the arguments will go through as before. Any difficulties which come from lack of smoothness of V can be overcome by a standard smoothing argument similar to that of [O-T].

§ 4. Proofs of Theorems 2 and 3

Proof of Theorem 2. For $k \leq \nu \leq n$ we let $\varepsilon_\nu = \min\{2\sum_{j=k+1}^{\nu}\frac{1}{N_j},\ 1-\varepsilon''\}$, and $\varepsilon_k = 0$. Obviously Theorem 2 will be implied by the following statement

$E(\nu)$: *There exists a bounded linear extension operator*

$$E_\nu : L^2(D\cap H, |\rho_D|^\delta d\lambda^k) \cap \mathcal{O}(D\cap H) \longrightarrow$$
$$L^2(D\cap H^\nu, |\rho_D|^{\delta-\varepsilon_\nu}|\log|\rho_D||^{-3(\nu-k)}d\lambda^\nu) \cap \mathcal{O}(D\cap H^\nu).$$

We proceed by induction (on ν). $E(k)$ is trivial. Let us assume $E(\nu)$ is true and $\nu < n$. We need to construct a bounded linear extension operator

$$E_{\nu,\nu+1} : L^2(D\cap H^\nu, |\rho_D|^{\delta-\varepsilon_\nu}|\log|\rho_D||^{-3(\nu-k)}d\lambda^\nu) \cap \mathcal{O}(D\cap H^\nu) \longrightarrow$$
$$L^2(D\cap H^{\nu+1}, |\rho_D|^{\delta-\varepsilon_{\nu+1}}|\log|\rho_D||^{-3(\nu+1-k)}d\lambda^{\nu+1}) \cap \mathcal{O}(D\cap H^{\nu+1}).$$

Note that the gain in the L^2 estimate of the extension is now $\varepsilon_{\nu+1}-\varepsilon_\nu$ which is in general less than $2/N_{\nu+1}$. (Indeed, if $\varepsilon_{\nu+1}=\varepsilon_\nu = 1-\varepsilon''$, then we cannot expect any gain at all). The operator $E_{\nu,\nu+1}$ can now be constructed by pursuing the estimates in the proof of Theorem 1 step by step, setting $a=1$, $\zeta=0$, $m=\nu-k$, replacing H by H^ν, H^{k+1} *by* $H^{\nu+1}$, δ *by* δ_ν, and using the weight functions

$$\varphi_1 = -\delta'\log(-\rho_D|H^{\nu+1}) + |\pi_{H^{\nu+1}}(\cdot)|^2 + V_{H^\nu} \tag{4.1}$$

where $\delta' \in (0, \varepsilon''')$, $\pi_{H^{\nu+1}}$ is the orthogonal projection onto $H^{\nu+1}$,

$$V_{H^\nu} = -\log\log 1/(\mathrm{dist}(\cdot, H^\nu)|H^{\nu+1}),$$
$$\Psi = -(\varepsilon_{\nu+1}-\varepsilon_\nu)\log(-\sigma|H^{\nu+1}) + 2\log(\mathrm{dist}(\cdot, H^\nu)|H^{\nu+1}),$$

σ being the function from Lemma 3.3, and

$$\eta = -(-\rho_D|H^{\nu+1})^{\delta'+\varepsilon_{\nu+1}-\delta}(1-\beta\log(-\rho_D|H^{\nu+1}))^{3(\nu+1-k)}V_{H^\nu}.$$

(Note that for $0<\delta\leq 2/N_{k+1}$, $0<\delta'<\varepsilon''$, Lemma 3.1 applies to this η *!*). The induction step is now complete. Just choose $E_{\nu+1} = E_{\nu,\nu+1}\circ E_\nu$.

Proof of Theorem 3. The argument is similar to the one above. For $\nu = k,\ldots, n$ we let $\varepsilon'_\nu = \varepsilon_\nu/2$, ε_ν being as in the proof of Theorem 2, and $d_\nu = \Pi_{j=k}^{\nu-1}\mathrm{dist}(\cdot, H^j)$, $d_k = 1$. Inductively (on ν) we show the statement

$E'(\nu)$: *There exists a bounded linear extension operator*

$$E'_\nu : L^2(D \cap H, |\rho_D|^\delta d\lambda^k) \cap \mathcal{O}(D \cap H) \longrightarrow$$
$$L^2(D \cap H^\nu, |\rho_D|^{\delta-\varepsilon_\nu} |\log|\rho_D||^{-3(\nu-k)} d_\nu^{-1} d\lambda^\nu) \cap \mathcal{O}(D \cap H^\nu).$$

Again $E'(k)$ is trivial. Suppose $E'(\nu)$ holds, and $\nu < n$. If we repeat the proof of Theorem 1 with $a = 1/2$, $\zeta = 0$, $m = \nu - k$, replacing δ by $\delta'_\nu := \delta - \varepsilon'_\nu$, H by H^ν, H^{k+1} by $H^{\nu+1}$ and work with the weight functions

$$\varphi'_1 = \varphi_1,$$

φ_1 being as in (4.1),

$$\Psi' = -(\varepsilon'_{\nu+1} - \varepsilon'_\nu) \log(-\sigma | H^{\nu+1}) + 2\log(\mathrm{dist}(\cdot, H^\nu) | H^{\nu+1})$$

where σ is as in Lemma 3.3,

$$\varphi' = \varphi'_1 + \log d_\nu + \Psi'$$

and

$$\eta' = -(-\rho_D | H^{\nu+1})^{\varepsilon'_{\nu+1}+\delta'-\delta}(1 - \beta \log(-\rho_D | H^{\nu+1}))^{3(\nu+1-k)} V_{H^\nu},$$

we obtain a bounded linear extension operator

$$E'_{\nu,\nu+1} : L^2(D \cap H^\nu, |\rho_D|^{\delta-\varepsilon'_\nu} |\log \rho_D||^{-3(\nu-k)} d_\nu^{-1} d\lambda^\nu) \cap \mathcal{O}(D \cap H^\nu) \longrightarrow$$
$$L^2(D \cap H^{\nu+1}, |\rho_D|^{\delta-\varepsilon'_{\nu+1}} |\log|\rho_D||^{-3(\nu+1-k)} d_{\nu+1}^{-1} d\lambda^{\nu+1}) \cap \mathcal{O}(D \cap H^{\nu+1}).$$

As before, the induction step follows with $E'_{\nu+1} = E'_{\nu,\nu+1} \circ E'_\nu$.

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