

ON NILPOTENT EXTENSIONS OF ALGEBRAIC NUMBER FIELDS I

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Introduction

The lower central series of the absolute Galois group of a field is obtained by iterating the process of forming the maximal central extension of the maximal nilpotent extension of a given class, starting with the maximal abelian extension. The purpose of this paper is to give a cohomological description of this central series in case of an algebraic number field. This description is based on a result of Tate which states that the Schur multiplier of the absolute Galois group of a number field is trivial. We are in a profinite situation throughout which requires some functorial background especially for treating the dual of the Schur multiplier of a profinite group. In a future paper we plan to apply our results to construct a nilpotent reciprocity map.

§ 1. Central extensions and Schur multipliers

Let k be an algebraic number field of finite degree over the rationals \mathbf{Q} , and let k^{ab} (resp. k^{nil}) be the maximal abelian (resp. nilpotent) extension of k in the algebraic closure $\bar{\mathbf{Q}}$ of \mathbf{Q} . For each positive integer c denote by $k^{(c)}/k$ the maximal nilpotent extension of class (at most) c . Hence $k^{(1)} = k^{\text{ab}}$ and $k^{\text{nil}} = \bigcup_{c=1}^{\infty} k^{(c)}$. For convenience we set $k^{(0)} = k$. Put $G^c = \text{Gal}(k^{(c)}/k)$ and $N^c = \text{Gal}(k^{(c)}/k^{(c-1)})$; N^c is a closed normal subgroup of G^c which is contained in the center $Z(G^c)$. Therefore we have a central extension of Galois groups

$$1 \longrightarrow N^{c+1} \longrightarrow G^{c+1} \longrightarrow G^c \longrightarrow 1.$$

We furnish the rational torus group $T = \mathbf{Q}/\mathbf{Z}$ with the discrete topology and consider it as a Galois module with trivial action.

PROPOSITION 1. *For each $c \geq 1$, the compact group N^{c+1} is canonically*

isomorphic to the Pontrjagin dual of the Schur multiplier $H^2(G^c, T)$ of G^c .

Proof. Put $\mathfrak{G} = \text{Gal}(\overline{\mathbf{Q}}/k)$ and $\mathfrak{N}^c = \text{Gal}(\overline{\mathbf{Q}}/k^{(c)})$. Then we have $G^c = \mathfrak{G}/\mathfrak{N}^c$, and the exact Hochschild-Serre sequence

$$\text{Hom}(\mathfrak{G}, T) \xrightarrow{\text{res}} \text{Hom}(\mathfrak{N}^c, T)^{\mathfrak{G}} \xrightarrow{\tau^c} H^2(G^c, T) \longrightarrow H^2(\mathfrak{G}, T),$$

where τ^c is the transgression. The last term $H^2(\mathfrak{G}, T)$ vanishes by a well known result of Tate; see e.g. [Se], § 6. Therefore $H^2(G^c, T)$ is isomorphic to the cokernel of the restriction map which is naturally identified with $\text{Hom}(\mathfrak{N}^c/[\mathfrak{N}^c, \mathfrak{G}], T)$. By definition we have $[\mathfrak{N}^c, \mathfrak{G}] = \mathfrak{N}^{c+1}$ and $\mathfrak{N}^c/\mathfrak{N}^{c+1} = N^{c+1}$. This shows that $H^2(G^c, T)$ is isomorphic to $\text{Hom}(N^{c+1}, T)$. Taking the dual groups we immediately obtain the proposition.

§ 2. The dual of the Schur multiplier

In this paper, $H^2(G, T)$ and its dual for a Galois group G of an infinite algebraic extension plays an important role. When G is finite, $H^2(G, T)$ is a finite abelian group and isomorphic to its dual although not canonically. When G is an infinite profinite group, however, $H^2(G, T)$ is different from its dual. So it seems worthwhile to give a brief survey on the dual of $H^2(G, T)$ for a profinite group G .

Let G be a profinite group and suppose that we have a presentation $G = F/R$ with a free profinite group F and its closed normal subgroup R which is generated by the relations in G . Associated to the exact sequence

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\rho} G \longrightarrow 1,$$

the transgression gives an exact sequence

$$\text{Hom}(F, T) \xrightarrow{\text{res}} \text{Hom}(R, T)^F \xrightarrow{\tau} H^2(G, T).$$

THEOREM 1. *The transgression induces an isomorphism*

$$\tau: \text{Hom}(R \cap [F, F]/[R, F], T) \xrightarrow{\sim} H^2(G, T).$$

Proof. It is easily seen that the cokernel of the restriction map can be identified with

$$\text{Hom}(R \cap [F, F]/[R, F], T).$$

To show that τ is surjective, we can slightly modify the method of

[Ka], pp. 47–48. Let (φ) be an element in $H^2(G, T)$ with a 2-cocycle φ in $Z^2(G, T)$. Since the image of φ is a compact subset of the discrete topological group T , there exists a positive integer m such that the image of φ is contained in the subgroup $C_m = (1/m)\mathbf{Z}/\mathbf{Z}$ of T . Then φ belongs to $Z^2(G, C_m)$ and determines a central extension of profinite groups

$$1 \longrightarrow C_m \longrightarrow H \xrightarrow{\pi} G \longrightarrow 1.$$

Since $\rho: F \rightarrow G$ gives a presentation of G with a free profinite group F , there exists a homomorphism η of F to H such that $\pi \circ \eta = \rho$. Then $\eta(R)$ is contained in (the image of) C_m . Take a continuous cross-section $\sigma: G \rightarrow F$ of ρ (cf. [Ko], Satz 1.16, p. 8, or [Sh], Theorem 3, p. 10), and put $\xi = \eta \circ \sigma$. Then we have a 2-cocycle $\psi \in Z^2(G, C_m)$ defined by

$$\psi(x, y) = \xi(xy)^{-1}\xi(x)\xi(y), \quad x, y \in G,$$

because ξ is a cross-section of $\pi: H \rightarrow G$. By the choice of H , ψ is cohomologous to φ . Therefore it belongs to the class (φ) in $H^2(G, T)$. Now let λ be the element of $\text{Hom}(R, T)$ obtained by restricting η to R (combined with the inclusion $C_m \hookrightarrow T$). Then we have

$$\begin{aligned} \psi(x, y) &= \eta(\sigma(xy))^{-1}\eta(\sigma(x))\eta(\sigma(y)) \\ &= \eta(\sigma(xy)^{-1}\sigma(x)\sigma(y)) \\ &= \lambda(\sigma(xy)^{-1}\sigma(x)\sigma(y)) \end{aligned}$$

for $x, y \in G$. Since σ is a cross-section of ρ , this shows that $\tau(\lambda) = (\psi) = (\varphi)$, which proves that τ is surjective.

If we take another presentation $G = F'/R'$ with a free profinite group F' , then we also have an isomorphism

$$\tau': \text{Hom}(R' \cap [F', F']/[R', F'], T) \xrightarrow{\sim} H^2(G, T)$$

by the theorem. It is easy to see, however, that there exists a canonical homomorphism of F to F' which induces a homomorphism

$$\theta: R \cap [F, F]/[R, F] \longrightarrow R' \cap [F', F']/[R', F'].$$

The dual $\hat{\theta}$ satisfies the condition $\tau' = \tau \circ \hat{\theta}$ and is an isomorphism. Therefore θ is an isomorphism of compact abelian groups. This observation allows us to define

$$\mathfrak{M}(G) := R \cap [F, F]/[R, F].$$

Then the statement of the theorem is dualized as follows:

THEOREM 1'. *For a profinite group G , the Schur multiplier $H^2(G, T)$ is canonically dual to $\mathfrak{M}(G)$.*

§ 3. The structure of $\mathfrak{M}(G)$ as a profinite group

Let G be a profinite group and N be its closed normal subgroup. Then a homomorphism

$$\gamma = \gamma_N^G: \mathfrak{M}(G) \longrightarrow \mathfrak{M}(G/N)$$

is canonically determined by the definition. The cokernel of γ is also determined; the following sequence is exact (cf. [B-E], Theorem 1.1, p. 101):

$$(1) \quad \mathfrak{M}(G) \xrightarrow{\gamma} \mathfrak{M}(G/N) \longrightarrow N \cap [G, G]/[N, G] \longrightarrow 1.$$

On the other hand, we have the inflation map

$$\lambda = \lambda_N^G: H^2(G/N, T) \longrightarrow H^2(G, T);$$

its kernel can be determined by the Hochschild-Serre exact sequence

$$(1') \quad 1 \longrightarrow \text{Hom}(N \cap [G, G]/[N, G], T) \longrightarrow H^2(G/N, T) \xrightarrow{\lambda} H^2(G, T).$$

Using Theorem 1' we see

PROPOSITION 2. *The exact sequence (1') is dual to (1).*

Altogether this shows

PROPOSITION 3. *Let the notation and the assumptions be as above. Denote the dual map of γ_N^G by $\hat{\gamma}_N^G$, and let τ_G and $\tau_{G/N}$ be the isomorphisms given in Theorem 1 for G and for G/N , respectively. Then we have a commutative diagram*

$$(2) \quad \begin{array}{ccc} \text{Hom}(\mathfrak{M}(G), T) & \xrightarrow{\tau_G} & H^2(G, T) \\ \hat{\gamma}_N^G \uparrow & & \uparrow \lambda_N^G \\ \text{Hom}(\mathfrak{M}(G/N), T) & \xrightarrow{\tau_{G/N}} & H^2(G/N, T). \end{array}$$

Now let \mathfrak{U} be the family of all open normal subgroups of G . For $U, V \in \mathfrak{U}$, $U \supset V$, we have a homomorphism

$$\gamma_{U,V}: \mathfrak{M}(G/V) \longrightarrow \mathfrak{M}(G/U)$$

together with

$$\gamma_U: \mathfrak{M}(G) \longrightarrow \mathfrak{M}(G/U),$$

and

$$\gamma_V: \mathfrak{M}(G) \longrightarrow \mathfrak{M}(G/V).$$

From the definition we see

$$(3) \quad \gamma_U = \gamma_{U,V} \circ \gamma_V$$

and

$$\gamma_{U,W} = \gamma_{U,V} \circ \gamma_{V,W}$$

for $U, V, W \in \mathfrak{U}$, $U \supset V \supset W$. Therefore we have a projective system of finite abelian groups

$$(4) \quad \{\mathfrak{M}(G/U), \gamma_{U,V} \mid U, V \in \mathfrak{U}, U \supset V\}.$$

We have also an inductive system of Schur multipliers

$$(5) \quad \{H^2(G/U, T), \lambda_{V,U} \mid U, V \in \mathfrak{U}, U \supset V\}$$

where $\lambda_{V,U}$ is the inflation map

$$\lambda_{V,U}: H^2(G/U, T) \longrightarrow H^2(G/V, T),$$

and also a system of homomorphisms

$$\lambda_U: H^2(G/U, T) \longrightarrow H^2(G, T), \quad U \in \mathfrak{U}.$$

Since the action of G on T is trivial, we have

$$(6) \quad H^2(G, T) = \varinjlim_U H^2(G/U, T),$$

(cf. [Sh], Corollary 1, p. 26); and then

$$\text{PROPOSITION 4.} \quad \mathfrak{M}(G) = \varinjlim_{U \in \mathfrak{U}} \mathfrak{M}(G/U).$$

Proof. Put $H = \varinjlim \mathfrak{M}(G/U)$. Then by (3) and the universal property of H , we have a continuous homomorphism $\varphi: \mathfrak{M}(G) \rightarrow H$. Because of (2), the two systems (4) and (5) are dual to each other. By (6), therefore, $H^2(G, T)$ is the dual group of H . Then by Theorem 1' and (2), we conclude that φ is an isomorphism.

§ 4. The structure of $\text{Gal}(k^{(c+1)}/k^{(c)})$

Let us go back to Galois groups of nilpotent extensions of an algebraic number field k . An open normal subgroup U of G^c corresponds to a

finite normal subextension K/k of $k^{(c)}/k$ in such a way that $U = \text{Gal}(k^{(c)}/K)$. Therefore we obtain from Theorem 1' and Proposition 4

THEOREM 2. (i) *For each $c \geq 1$ there is a canonical isomorphism*

$$\iota^c: \mathfrak{M}(G^c) \xrightarrow{\sim} N^{c+1} = \text{Gal}(k^{(c+1)}/k^{(c)}).$$

(ii) $\mathfrak{M}(G^c)$ is determined by finite normal subextensions K/k of $k^{(c)}/k$ as

$$\mathfrak{M}(G^c) = \varinjlim_K \mathfrak{M}(\text{Gal}(K/k)).$$

For a finite normal subextension K/k of $k^{(c)}/k$ with $U = \text{Gal}(k^{(c)}/K)$ denote by

$$\gamma_K = \gamma_U: \mathfrak{M}(G^c) \longrightarrow \mathfrak{M}(\text{Gal}(K/k))$$

the natural homomorphism determined by (ii) of the theorem. We denote the maximal central extension of K/k in $\bar{\mathbf{Q}}$ by $MC(K/k)$. This is a subfield of $k^{(c+1)}$ because K/k is a subextension of $k^{(c)}/k$.

In Section 6 we shall prove

THEOREM 3. *For each finite normal subextension K/k of $k^{(c)}/k$, there exists a canonical isomorphism*

$$\iota_K: \text{Im}(\gamma_K) \xrightarrow{\sim} \text{Gal}(MC(K/k) \cdot k^{(c)}/k^{(c)})$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{M}(G^c) & \xrightarrow{\iota^c} & N^{c+1} = \text{Gal}(k^{(c+1)}/k^{(c)}) \\ \gamma_K \downarrow & & \downarrow \text{res} \\ \text{Im}(\gamma_K) & \xrightarrow{\iota_K} & \text{Gal}(MC(K/k) \cdot k^{(c)}/k^{(c)}). \end{array}$$

§ 5. Base-change for abundant central extensions

In this section let K/k be a Galois extension of algebraic number fields of finite degree. Put $\mathfrak{g} = \text{Gal}(K/k)$. The maximal central extension $MC(K/k)$ of K/k contains $K \cdot k^{\text{ab}}$. There exists a canonical isomorphism

$$\iota_{K/k}: \mathfrak{M}(\mathfrak{g}) \xrightarrow{\sim} \text{Gal}(MC(K/k)/K \cdot k^{\text{ab}})$$

(see e.g. [Mi]). Since $MC(K/k)$ is a finite extension of $K \cdot k^{\text{ab}}$, there is a finite central extension L of K/k such that $MC(K/k)$ is equal to the composite field $L \cdot k^{\text{ab}}$. Such an L is called an abundant central extension

of K/k . Put $L^* = L \cap K \cdot k^{\text{ab}}$. Then the isomorphism $\iota_{K/k}$ induces an isomorphism of $\mathfrak{M}(\mathfrak{g})$ onto $\text{Gal}(L/L^*)$ if L is abundant. Suppose that another finite Galois extension K_1/k , $K_1 \supset K$, is given, and put $G = \text{Gal}(K_1/k)$ and $N = \text{Gal}(K_1/K)$. Then $\mathfrak{g} = G/N$. The homomorphism γ of (1) in Section 3 gives a basic relation between $\mathfrak{M}(G)$ and $\mathfrak{M}(\mathfrak{g})$. Let \hat{K}_1 be the maximal central extension of K/k in K_1 , i.e. $\hat{K}_1 = K_1 \cap MC(K/k)$, and let K_1^* be the genus field, i.e. $K_1^* = K_1 \cap K \cdot k^{\text{ab}}$. Then by the definition we see that $\text{Gal}(\hat{K}_1/K_1^*)$ is isomorphic to $N \cap [G, G]/[N, G]$, the third term of the exact sequence (1). Now let us denote the composite field $L \cdot K_1$ by L_1 , and put $L_1^{**} = L_1 \cap K_1 \cdot k^{\text{ab}}$. Since L_1 is a central extension of K_1/k , $\text{Gal}(L_1/L_1^{**})$ is a homomorphic image of $\mathfrak{M}(G)$ under the map induced by the canonical homomorphism $\iota_{K_1/k}$ for the Galois extension K_1/k .

THEOREM 4. *Let the notation and the assumptions be as above. Then L_1 is a central extension of K_1/k with the following properties:*

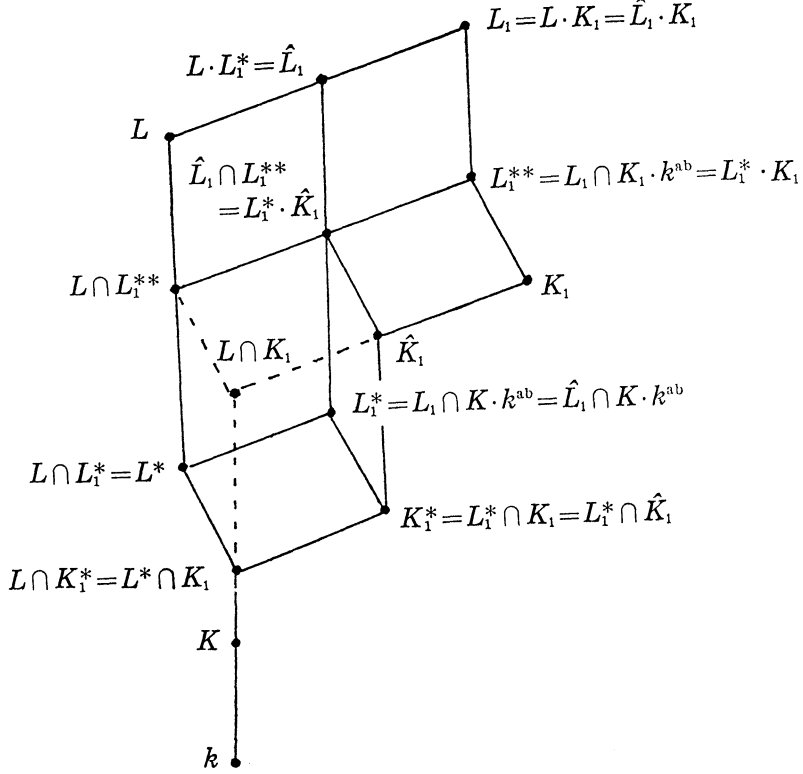
- (i) $\text{Gal}(L_1/L_1^{**})$ is canonically isomorphic to $\text{Gal}(L/L \cap L_1^{**})$;
- (ii) $\text{Gal}(L \cap L_1^{**}/L^*)$ is canonically isomorphic to $\text{Gal}(\hat{K}_1/K_1^*)$;
- (iii) We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathfrak{M}(G) & \longrightarrow & \mathfrak{M}(\mathfrak{g}) & \longrightarrow & N \cap [G, G]/[N, G] & \longrightarrow & 1 \\ \downarrow & & \downarrow \wr & & \downarrow \wr & & \\ \text{Gal}(L_1/L_1^{**}) & \longrightarrow & \text{Gal}(L/L^*) & \longrightarrow & \text{Gal}(L \cap L_1^{**}/L^*) & \longrightarrow & 1. \end{array}$$

Proof. Put $\mathfrak{G} = \text{Gal}(L_1/k)$ and $\mathfrak{N} = \text{Gal}(L_1/K)$; \mathfrak{N} is a normal subgroup with quotient $\mathfrak{g} = \mathfrak{G}/\mathfrak{N}$. Put $\mathfrak{A} = \text{Gal}(L_1/K_1)$ and $\mathfrak{B} = \text{Gal}(L_1/L)$; then $\text{Gal}(L_1/L \cap K_1)$ is a direct product $\mathfrak{B} \times \mathfrak{A}$ because $L_1 = L \cdot K_1$. Let \hat{L}_1 be the subfield of L_1 determined by the condition $\text{Gal}(L_1/\hat{L}_1) = [\mathfrak{N}, \mathfrak{G}]$; this is the maximal central extension of K/k in L_1 , i.e. $\hat{L}_1 = L_1 \cap MC(K/k)$; it contains L . Therefore we have $[\mathfrak{N}, \mathfrak{G}] \subset \mathfrak{B}$. Since \mathfrak{A} is a normal subgroup of \mathfrak{G} contained in \mathfrak{N} , the commutator $[\mathfrak{A}, \mathfrak{G}]$ is contained in $\mathfrak{A} \cap [\mathfrak{N}, \mathfrak{G}]$; hence we have $[\mathfrak{A}, \mathfrak{G}] = 1$ because $\mathfrak{A} \cap \mathfrak{B} = 1$; this shows that \mathfrak{A} lies in the center of \mathfrak{G} , which means that L_1 is a central extension of K_1/k .

Now let us see (i). We have $\text{Gal}(L_1/L_1^{**}) = \mathfrak{A} \cap [\mathfrak{G}, \mathfrak{G}]$ because $L_1^{**} = L_1 \cap K_1 \cdot k^{\text{ab}} = K_1 \cdot (L_1 \cap k^{\text{ab}})$ and $\text{Gal}(L_1/L_1 \cap k^{\text{ab}}) = [\mathfrak{G}, \mathfrak{G}]$. Therefore $\text{Gal}(L_1/L \cap L_1^{**}) = \mathfrak{B} \times \text{Gal}(L_1/L_1^{**})$ because $\text{Gal}(L_1/L \cap K_1) = \mathfrak{B} \times \mathfrak{A}$. Hence it is obvious that the projection of $\text{Gal}(L_1/L \cap L_1^{**})$ onto $\text{Gal}(L/L \cap L_1^{**})$ maps $\text{Gal}(L_1/L_1^{**})$ isomorphically onto $\text{Gal}(L/L \cap L_1^{**})$. This proves (i). Note that $\text{Gal}(L_1/L_1^{**})$ is a homomorphic image of $\mathfrak{M}(G)$ because L_1^{**} is the genus field of the central extension L_1 of K_1/k . Put $L_1^* = L_1 \cap K \cdot k^{\text{ab}}$; this

Diagram



is the genus field of the central extension $\hat{L}_1 = L_1 \cap MC(K/k)$ of K/k . Since $L_1^* = K \cdot (L_1 \cap k^{ab})$, we have $\text{Gal}(L_1/L_1^*) = \mathfrak{R} \cap [\mathfrak{S}, \mathfrak{S}]$. By the assumption, L is an abundant central extension of K/k . Therefore \hat{L}_1 is contained in $L \cdot k^{ab}$ and hence equal to $L \cdot L_1^*$. Then $\text{Gal}(\hat{L}_1/L_1^*)$ is naturally isomorphic to $\text{Gal}(L/L^*)$ because $L \cap L_1^* = L \cap K \cdot k^{ab} = L^*$. Under this isomorphism the intermediate field $L \cap L_1^{**}$ of L/L^* corresponds to $(L \cap L_1^{**}) \cdot L_1^*$. Furthermore, we also have natural isomorphisms

$$\text{Gal}(\hat{L}_1/L) \cong \text{Gal}((L \cap L_1^{**}) \cdot L_1^*/L \cap L_1^{**}) \cong \text{Gal}(L_1^*/L^*).$$

Since there is a natural isomorphism of $\text{Gal}(L_1/L)$ onto $\text{Gal}(L_1^{**}/L \cap L_1^{**})$, we conclude that the intermediate field $(L \cap L_1^{**}) \cdot L_1^*$ coincides with $\hat{L}_1 \cap L_1^{**}$. Therefore $\text{Gal}(L \cap L_1^{**}/L^*)$ is naturally isomorphic to $\text{Gal}(\hat{L}_1 \cap L_1^{**}/L_1^*)$. Next let us look at the extension L_1^{**}/K_1^* . By definition we easily see $L_1^* \cap K_1 = K_1^*$. Hence $\text{Gal}(L_1/K_1^*) = (\mathfrak{R} \cap [\mathfrak{S}, \mathfrak{S}]) \cdot \mathfrak{A}$ because $\text{Gal}(L_1/L_1^*) = \mathfrak{R} \cap [\mathfrak{S}, \mathfrak{S}]$ and $\text{Gal}(L_1/K_1) = \mathfrak{A}$. Therefore we obtain

$$\begin{aligned}
\text{Gal}(L_1^*/K_1^*) &= (\mathfrak{N} \cap [\mathfrak{G}, \mathfrak{G}]) \cdot \mathfrak{A} / (\mathfrak{N} \cap [\mathfrak{G}, \mathfrak{G}]) \\
&\cong \mathfrak{A} / \mathfrak{A} \cap (\mathfrak{N} \cap [\mathfrak{G}, \mathfrak{G}]) \\
&= \mathfrak{A} / \mathfrak{A} \cap [\mathfrak{G}, \mathfrak{G}] \\
&= \text{Gal}(L_1^{**}/K_1).
\end{aligned}$$

In particular we have $[L_1^{**}: K_1] = [L_1^*: K_1^*]$. It is clear that L_1^{**} contains $L_1^* \cdot K_1$. Since $[L_1^* \cdot K_1: K_1] = [L_1^*: L_1^* \cap K_1] = [L_1^*: K_1^*] = [L_1^{**}: K_1]$, we conclude $L_1^{**} = L_1^* \cdot K_1$. This implies that $\text{Gal}(\hat{L}_1 \cap L_1^{**}/L_1^*)$ is naturally isomorphic to $\text{Gal}(\hat{K}_1/K_1^*)$ because $\hat{K}_1 = (\hat{L}_1 \cap L_1^{**}) \cap K_1$. Combining this with the result obtained above, we have shown that $\text{Gal}(L \cap L_1^{**}/L^*)$ is naturally isomorphic to $\text{Gal}(\hat{K}_1/K_1^*)$ as is claimed in (ii).

The last Galois group is isomorphic to

$$\text{Gal}(K_1/K_1^*)/\text{Gal}(K_1/\hat{K}_1) = N \cap [G, G]/[N, G].$$

Since all the isomorphisms of Galois groups discussed above are natural and group-theoretic, (iii) is also clear.

§ 6. The proof of Theorem 3

We use the same notation as in Section 4. Let K/k be a finite normal subextension of $k^{(c)}/k$, $c \geq 1$; its Galois group is denoted by $\mathfrak{g} = \text{Gal}(K/k)$; $U = \text{Gal}(k^{(c)}/K)$ is an open normal subgroup of G^c . We fix an abundant central extension L of K/k ; $MC(K/k) \cdot k^{(c)}$ is equal to $L \cdot k^{(c)}$. Put $K_0 = L \cap k^{(c)}$; then $\text{Gal}(L/K_0)$ is canonically isomorphic to $\text{Gal}(MC(K/k) \cdot k^{(c)}/k^{(c)})$. Put $\tilde{U} = \text{Gal}(k^{(c)}/K_0)$, $G = \text{Gal}(K_0/k)$ and $N = \text{Gal}(K_0/K)$. We use Theorem 4 for $L/K/k$ and $K_1 = K_0$; in this case we have $L \supset K_1 \supset L^*$; N lies in the center of G , and $[N, G] = 1$; therefore $K_1 = \hat{K}_1$, $K_1^* = L^*$; moreover we have $L_1 = L$, $L_1^* = L^*$ and $L_1^{**} = K_1 = L \cap L_1^{**}$; hence the image of the homomorphism

$$\gamma_N^G: \mathfrak{M}(G) \longrightarrow \mathfrak{M}(\mathfrak{g})$$

is mapped isomorphically onto $\text{Gal}(L/K_0)$ by the isomorphism

$$\text{res} \circ \iota_{K/k}: \mathfrak{M}(\mathfrak{g}) \xrightarrow{\sim} \text{Gal}(MC(K/k)/K \cdot k^{\text{ab}}) \xrightarrow{\sim} \text{Gal}(L/L^*).$$

Let us express this using U and \tilde{U} ; we have $\mathfrak{M}(G) = \mathfrak{M}(G^c/\tilde{U})$, $\mathfrak{M}(\mathfrak{g}) = \mathfrak{M}(G^c/U)$ and $\gamma_N^G = \gamma_{U, \tilde{U}}$ in the notation of Section 3; furthermore, the image of $\gamma_{U, \tilde{U}}$, $\text{Im}(\gamma_{U, \tilde{U}})$, is mapped isomorphically onto the subgroup of $\text{Gal}(MC(K/k)/K \cdot k^{\text{ab}})$ by $\iota_{K/k}$ which is canonically isomorphic to $\text{Gal}(L/K_0)$ and hence also to $\text{Gal}(MC(K/k) \cdot k^{(c)}/k^{(c)})$. Next let V be an open normal

subgroup of G^c contained in U ; put $\tilde{V} = V \cap \tilde{U}$; then we have

$$\gamma_{U, \tilde{V}} = \gamma_{U, V} \circ \gamma_{V, \tilde{V}} = \gamma_{U, \tilde{V}} \circ \gamma_{\tilde{V}, \tilde{V}};$$

hence, in order to show $\text{Im } \gamma_U = \text{Im}(\gamma_{U, \tilde{V}})$ for $\gamma_U (= \gamma_K) = \lim_{V \subset U} \gamma_{U, V}$, it is sufficient to prove that $\text{Im}(\gamma_{U, \tilde{V}})$ in $\mathfrak{M}(\mathfrak{g})$ is mapped isomorphically onto $\text{Gal}(L/K_0)$ by $\text{res} \circ \iota_{K/k}$. This time we use Theorem 4 for $L/K/k$ and the extension K_1/k determined by the condition $\text{Gal}(K_1/k) = G^c/\tilde{V}$. We have $K_1 \supset K_0$ by the choice of V ; since K_1 is a subfield of $k^{(c)}$, we also have $L \cap K_1 = K_0$ and hence $L^* \subset L \cap K_1$. For $c \geq 1$, $L_1^{**} = L_1^* \cdot K_1$ is contained in $k^{(c)}$ because $k^{(c)} \supset K \cdot k^{\text{ab}}$ and $L_1^* = L_1 \cap K \cdot k^{\text{ab}}$; therefore $L \cap L_1^{**} \subset L \cap k^{(c)} = K_0$; conversely, it is clear that $L \cap L_1^{**} \supset K_0 = L \cap K_1$; thus we have $L \cap L_1^{**} = K_0$. It now follows from Theorem 4 that the image of the homomorphism

$$\gamma_{U, \tilde{V}}: \mathfrak{M}(G^c/\tilde{V}) \longrightarrow \mathfrak{M}(G^c/U)$$

is isomorphically mapped onto $\text{Gal}(L/K_0)$ by $\text{res} \circ \iota_{K/k}$. This proves that $\text{Im } \gamma_U, \gamma_U = \gamma_K$, coincides with $\text{Im}(\gamma_{U, \tilde{V}})$ and also that there exists a canonical isomorphism

$$\iota_K: \text{Im } \gamma_K \longrightarrow \text{Gal}(MC(K/k) \cdot k^{(c)}/k^{(c)}).$$

The rest of Theorem 3 will easily be seen in a straightforward way by dualizing the diagram of Theorem 4.

§ 7. The canonical 2-cohomology classes

We fix an algebraic number field k of finite degree. Let K be a finite Galois extension of k , $K_{\mathbb{A}}^{\times}$ be the idele group of K , $K_{\infty+}^{\times}$ be the connected component of the identity element of the Archimedean part of $K_{\mathbb{A}}^{\times}$ and $K^{\#}$ be the closure of $K^{\times} \cdot K_{\infty+}^{\times}$ in $K_{\mathbb{A}}^{\times}$. We have the Artin map of K ,

$$\alpha_K: K_{\mathbb{A}}^{\times}/K^{\#} \longrightarrow \text{Gal}(K^{\text{ab}}/K),$$

which is a topological isomorphism, and the natural exact sequence

$$E(K/k): 1 \longrightarrow K_{\mathbb{A}}^{\times}/K^{\#} \longrightarrow \text{Gal}(K^{\text{ab}}/k) \longrightarrow \text{Gal}(K/k) \longrightarrow 1.$$

The structure of $\text{Gal}(K^{\text{ab}}/k)$ is then determined by the canonical 2-cohomology class $\bar{\xi}_{K/k}$ of $\text{Gal}(K/k)$ with values in $K_{\mathbb{A}}^{\times}/K^{\#}$. More specifically, the cohomology group $H^2(\text{Gal}(K/k), K_{\mathbb{A}}^{\times}/K^{\#})$ is a cyclic group generated by $\bar{\xi}_{K/k}$; its order is either $\frac{1}{2} \cdot [K:k]$ if there exists a ramified real Archimedean prime in K/k or $[K:k]$ otherwise (cf. Katayama [Kt] and also

Iyanaga [Iy]). In this sense $\text{Gal}(K^{\text{ab}}/k)$ is determined by K/k .

Let F/k , $F \supset K$, be another finite Galois extension, $F_{\mathbb{A}}^{\times}/F^{\#}$ be as above for F and

$$N_{F/K}: F_{\mathbb{A}}^{\times}/F^{\#} \longrightarrow K_{\mathbb{A}}^{\times}/K^{\#}$$

be the norm map of F/K . Then we have a commutative diagram

$$\begin{array}{ccc} F_{\mathbb{A}}^{\times}/F^{\#} & \xrightarrow{\alpha_F} & \text{Gal}(F^{\text{ab}}/F) \\ N_{F/K} \downarrow & & \downarrow \text{restriction} \\ K_{\mathbb{A}}^{\times}/K^{\#} & \xrightarrow{\alpha_K} & \text{Gal}(K^{\text{ab}}/K). \end{array}$$

Therefore $N_{F/K}$ and the homomorphisms defined by restricting automorphisms of F^{ab} or of F to K^{ab} or to K , respectively, give a homomorphism of the exact sequence $E(F/k)$ to $E(K/k)$.

Now suppose that an infinite Galois extension \tilde{k}/k is given. Put $G = \text{Gal}(\tilde{k}/k)$. If we make K/k run over all finite Galois subextensions of \tilde{k}/k , we have projective systems $\{K_{\mathbb{A}}^{\times}/K^{\#}, N_{F/K}\}$, $\{\text{Gal}(K^{\text{ab}}/K)\}$, $\{\text{Gal}(K^{\text{ab}}/k)\}$, $\{\text{Gal}(K/k)\}$ and $\{E(K/k)\}$, and also

$$\begin{aligned} \text{Gal}(\tilde{k}^{\text{ab}}/k) &= \varprojlim_K \text{Gal}(K^{\text{ab}}/k), \\ \text{Gal}(\tilde{k}^{\text{ab}}/\tilde{k}) &= \varprojlim_K \text{Gal}(K^{\text{ab}}/K), \\ G &= \text{Gal}(\tilde{k}/k) = \varprojlim_K \text{Gal}(K/k). \end{aligned}$$

We put

$$\mathfrak{A}(\tilde{k}) = \varprojlim_K K_{\mathbb{A}}^{\times}/K^{\#}.$$

It is clear that $\mathfrak{A}(\tilde{k})$ depends only on \tilde{k} . Each $K_{\mathbb{A}}^{\times}/K^{\#}$ is naturally considered as a G -module. Therefore $\mathfrak{A}(\tilde{k})$ has a G -module structure. Through inner automorphisms of $\text{Gal}(\tilde{k}^{\text{ab}}/k)$, $\text{Gal}(\tilde{k}^{\text{ab}}/\tilde{k})$ becomes a G -module.

PROPOSITION 5. *Let the notation and the assumptions be as above. The Artin maps α_K for finite Galois subextensions K/k of \tilde{k}/k give a G -isomorphism*

$$\alpha_{\tilde{k}} = \lim_K \alpha_K: \mathfrak{A}(\tilde{k}) \longrightarrow \text{Gal}(\tilde{k}^{\text{ab}}/\tilde{k}).$$

The exact sequence

$$E(\tilde{k}/k): 1 \longrightarrow \mathfrak{A}(\tilde{k}) \longrightarrow \text{Gal}(\tilde{k}^{\text{ab}}/k) \longrightarrow \text{Gal}(\tilde{k}/k) \longrightarrow 1$$

determined naturally by $\alpha_{\bar{k}}$ is the projective limit of $\{E(K/k)\}$. Therefore the canonical classes $\bar{\xi}_{K/k}$ determine the canonical 2-cohomology class $\bar{\xi}_{\bar{k}/k}$ in $H^2(G, \mathfrak{A}(\bar{k}))$ which gives the extension $\text{Gal}(\bar{k}^{\text{ab}}/k)$ of $\text{Gal}(\bar{k}/k)$ by $\mathfrak{A}(\bar{k})$.

The proof is almost obvious because \varprojlim is an exact functor in the category of compact groups (e.g. [E-S], Theorem 5.6, p. 226, or [Ko], Satz 1.9, p. 6).

COROLLARY. *Let $MC(\bar{k}/k)$ be the maximal central extension of \bar{k}/k . Then we have*

$$\text{Gal}(MC(\bar{k}/k)/\bar{k}) = \mathfrak{A}(\bar{k})/\mathfrak{A}(\bar{k})^{dG}$$

where

$$\mathfrak{A}(\bar{k})^{dG} = \langle x^{1-\sigma} \mid x \in \mathfrak{A}(\bar{k}), \sigma \in G \rangle$$

(the right-hand side means the topologically generated closed subgroup).

Let us apply these results to the case where \bar{k}/k is the nilpotent extension $k^{(c)}/k$, $c \geq 1$, and $G = G^c$. Then since $k^{(c+1)} = MC(k^{(c)}/k)$ we obtain from Theorem 2 the following result:

THEOREM 5. (i) *For each $c \geq 1$, there exists a surjective homomorphism $\alpha^c: \mathfrak{A}(k^{(c)}) \rightarrow \mathfrak{M}(G^c)$ with $\text{Ker } \alpha^c = \mathfrak{A}(k^{(c)})^{dG^c}$ such that the homomorphism*

$$\iota^c \circ \alpha^c: \mathfrak{A}(k^{(c)}) \longrightarrow N^{c+1} = \text{Gal}(k^{(c+1)}/k^{(c)})$$

coincides with the homomorphism induced naturally from the Artin map

$$\alpha_{k^{(c)}}: \mathfrak{A}(k^{(c)}) \longrightarrow \text{Gal}(k^{(c), \text{ab}}/k^{(c)}).$$

(ii) *The group extension*

$$1 \longrightarrow \mathfrak{M}(G^c) \xrightarrow{\iota^c} G^{c+1} \longrightarrow G^c \longrightarrow 1$$

determined by $\iota^c: \mathfrak{M}(G^c) \rightarrow \text{Gal}(k^{(c+1)}/k^{(c)})$ corresponds to the image of the canonical class $\bar{\xi}_{k^{(c)}/k}$ under the induced homomorphism

$$(\alpha^c)^*: H^2(G^c, \mathfrak{A}(k^{(c)})) \longrightarrow H^2(G^c, \mathfrak{M}(G^c)).$$

Remark. For g and $h \in G^c$, take \tilde{g} and $\tilde{h} \in G^{c+1}$ over g and h , respectively. Then the commutator $[\tilde{g}, \tilde{h}]$ depends only on g and h because the group extension of Theorem 5, (ii), is central. Hence from the definition we obtain an epimorphism

$$\varphi^c: N^c \otimes G^c \longrightarrow \mathfrak{M}(G^c)$$

such that

$$\iota^c(\varphi(n, g)) = [\tilde{n}, \tilde{g}]$$

for $n \in N^c$ and $g \in G^c$. Let f be a 2-cocycle which belongs to the cohomology class $(\alpha^c)^*(\bar{\xi}_{k^c/k}) \in H^2(G^c, \mathfrak{M}(G^c))$. Then it is easily seen that for $n \in N^c$ and $g \in G^c$ we have

$$\varphi(n, g) = f(n, g) \cdot f(g, n)^{-1}$$

because N^c lies in the center of G^c . For a finite abelian extension K/k with $G = N = \text{Gal}(K/k)$, a detailed analysis of the analogous map is given by Furuta [Fu].

Remark. The main body of our present results relies on two facts; one of them is that Schur multipliers of profinite groups and their duals have good functorial properties; the other is a result of Tate, $H^2(\mathfrak{G}, \mathbf{Q}/\mathbf{Z}) = 0$ for $\mathfrak{G} = \text{Gal}(\bar{\mathbf{Q}}/k)$, from which we not only deduce Proposition 1 but also the existence of abundant central extensions of a finite Galois sub-extension of $\bar{\mathbf{Q}}/k$. It is, therefore, easy to see that parallel results also hold for a local number field k_v and its algebraic closure \bar{k}_v , and for an algebraic number field k and its maximal p -ramified p -extension $\tilde{k}^{(p)}$ when the Leopoldt conjecture holds for k and p , because we have $H^2(\mathfrak{G}, \mathbf{Q}/\mathbf{Z}) = 0$ for $\mathfrak{G} = \text{Gal}(\bar{k}_v/k_v)$ and for $\mathfrak{G} = \text{Gal}(\tilde{k}^{(p)}/k)$, e.g. [He] or [Ng] for the latter case.

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