ON S-CLASS NUMBER RELATIONS OF ALGEBRAIC TORI IN GALOIS EXTENSIONS OF GLOBAL FIELDS

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Introduction

As an interpretation and a generalization of Gauss' genus theory on binary quadratic forms in the language of arithmetic of algebraic tori, Ono [O2] established an equality between a kind of "Euler number $E(K/k)$" for a finite Galois extension K/k of algebraic number fields and other arithmetical invariants associated to K/k. His proof depended on his Tamagawa number formula [O1] and Shyr's formula [Sh] which follows from the analytic class number formula of a torus. Later, two direct proofs were given by Katayama [K] and Sasaki [Sa].

In this paper, we generalize Ono's formula to S-arithmetical one, including the function field case, and give a new direct proof using Nisnevich cohomology. We also give a formula by applying our method to a similar exact sequence of tori associated to two linearly disjoint Galois extensions K_1, K_2/k of global fields. We think that Nisnevich cohomology is a natural and suitable tool to connect class sets of affine group schemes and etale (or flat) cohomology and to study their functorial behavior. ([N1] and [N2]).

The contents of this paper are as follows. In § 1 we introduce our invariants $E_S(K/k)$ and $E_S(K_1, K_2/k)$, and in § 2 we state our main results. In § 3 we prove the surjectivity of the norm map. In § 4 we give a brief review of Nisnevich topology and cohomology for our purpose. In § 5 we prove our theorems, and in § 6 we discuss some examples.

me a problem, a vast generalization of Gauss' genus theory using (non-abelian) algebraic groups. I hope to come back to this beautiful problem some time in the future.

§ 1.   The invariants $E_S(K/k)$ and $E_S(K_1, K_2/k)$

Let k be a global field. For a place v of k, let k_v denote the completion of k at v and $\mathcal{O}_v$ the ring of integers in k_v. Let S be a finite set of places of k which is non-empty and contains all archimedean places, S_∞. Let $\mathcal{O}_S$ be the ring of S-integers in k, $\mathcal{O}_S = \bigcap_{v \notin S} \mathcal{O}_v$, and put $X := \mathrm{Spec}\, \mathcal{O}_S$. For a finite set of places P of k containing S, we put

$$k_A(P) := \prod_{v \in P} k_v \times \prod_{v \notin P} \mathcal{O}_v$$

and

$$k_A := \varinjlim_P k_A(P).$$

Let G be an affine group scheme of finite type over X with the smooth generic fibre. We define the S-class set $C_S(G)$ of G by the set of double cosets:

$$C_S(G) = G(k_A(S))\backslash G(k_A)/G(k)$$

and call its cardinality the S-class number of G over X.

Let K, K_1 and K_2 be finite Galois extensions of k such that $K_1 \cap K_2 = k$, and let S_K and S_i be the set of places of K and K_i $(i = 1, 2)$, respectively, lying above S. Let $\mathcal{O}_{S_K}$ and $\mathcal{O}_{S_i}$ be the normalizations of $\mathcal{O}_S$ in K and K_i, and put $X_K := \mathrm{Spec}\, \mathcal{O}_{S_K}$, $X_i := \mathrm{Spec}\, \mathcal{O}_{S_i}$ $(i = 1, 2)$. Then, let us consider the following two exact sequences of group schemes or etale sheaves on X induced by the norm maps N and N_i associated to the Galois coverings X_K/X and X_i/X $(i = 1, 2)$ respectively:

$$(1.1) \qquad 0 \longrightarrow T \longrightarrow \prod_{X_K/X} \mathbf{G}_{m, X_K} \xrightarrow{\ N\ } \mathbf{G}_{m, X} \longrightarrow 0$$

$$(1.2) \qquad 0 \longrightarrow T' \longrightarrow \prod_{X_1/X} \mathbf{G}_{m, X_1} \times \prod_{X_2/X} \mathbf{G}_{m, X_2} \xrightarrow{\ N_1 \cdot N_2\ } \mathbf{G}_{m, X} \longrightarrow 0$$

where $\mathbf{G}_{m, Y} := \mathrm{Spec}\, \mathbf{Z}[t, t^{-1}] \times_{\mathrm{Spec}\, \mathbf{Z}} Y$ for a scheme Y, $\prod_{Z/Y}$ denotes the Grothendieck functor of scaler restriction ($[D\text{-}G]$), $N_1 \cdot N_2$ means the product of N_1 and N_2, and T, T' are defined as the kernels of N, $N_1 \cdot N_2$ respectively. For the surjectivity of N and $N_1 \cdot N_2$, we will give a proof in § 3.

Let us denote by $h_{k,S}$, h_{K,S_K}, h_{K_i,S_i} $h_{T,S}$ and $h_{T',S}$ the S-class numbers of $\mathbf{G}_{m,X}$, $\prod_{X_K/X} \mathbf{G}_{m,X_K}$, $\prod_{X_i/X} \mathbf{G}_{m,X_i}$, T and T' respectively. Here, $h_{k,S}$, $h_{K,S}$ and h_{K_i,S_i} are nothing but the S-class numbers of k, K and K_i. Now we define our invariants $E_S(K/k)$ and $E_S(K_1, K_2/k)$ by the alternative products of the S-class numbers taken along the sequences (1.1) and (1.2) respectively, namely,

$$(1.3) \qquad E_S(K/k) := \frac{h_{K,S}}{h_{k,S} \cdot h_{T,S}}$$

$$(1.4) \qquad E_S(K_1, K_2/k) := \frac{h_{K_1,S_1} \cdot h_{K_2,S_2}}{h_{k,S} \cdot h_{T',S}}.$$

We remark that $E_{S_\infty}(K/k)$ coincides with Ono's $E(K/k)$ for the number field case ([O2]).

§2. Statement of results

To state our results, let us prepare more notations. For a place v of k, we choose a place w of K and a place z of $K_1 K_2$ lying above v, and denote by w_1 and w_2 the restrictions of z to K_1 and K_2 respectively, and put $M_v := K_{1w_1} \cap K_{2w_2}$. Let $\mathcal{O}_w$ and $\mathcal{O}_{M_v}$ be the ring of integers in K_w and M_v respectively, and let S'_r be the set of finite places of k which ramify in K_1/k or K_2/k. For a Galois extension E/F of fields, let E'/F denote the maximal abelian subextension of E/F. For norm maps, we put $N := N_{K/k}$, $N_i := N_{K_i/k}$ $(i = 1, 2)$, $N_v := N_{K_w/k_v}$, and $N_{M_v} := N_{M_v/k_v}$ for simplicity. $R^\times$ denotes the group of invertible elements in a ring R. Finally, $[*]$ denotes the cardinality of a set $*$.

THEOREM. *Notations being as above, we have*

$$(2.1) \qquad E_S(K/k) = \frac{[\text{Ш}_k(T)] \prod_{v \in S}[K'_w : k_v] \prod_{v \notin S} e(K'_w/k_v)}{[K' : k](\mathcal{O}_S^\times : N\mathcal{O}_{S_K}^\times)}$$

$$(2.2) \qquad E_S(K_1, K_2/k) = \frac{[\text{Ш}_k(T')] \prod_{v \in S}[M'_v : k_v] \prod_{v \notin S} e(M'_v/k_v)}{(\mathcal{O}_S^\times : N_1\mathcal{O}_{S_1}^\times N_2\mathcal{O}_{S_2}^\times)(J : J \cap \varDelta' \prod_{S \cup S'_r} N_{M_v} M_v^\times)}$$

where $J := \prod_{v \in S} k_v^\times \times \prod_{S'_r \backslash S} \mathcal{O}_v^\times$, $\varDelta' := \{x \in k^\times \mid x \in N_{M_v} M_v^\times, v \notin S \cup S'_r\}$, $e(K'_w/k_v)$ and $e(K'_S/k_v)$ denote the ramification indices of K'_w/k_v and M'_v/k_v respectively, and the Tate-Shafarevich groups over k, $\text{Ш}_k(T)$ and $\text{Ш}_k(T')$, are given by the kernels of the natural maps:

$$\text{Ш}_k(T) = \text{Ker}(k^\times/NK^\times \longrightarrow \prod_v k_v^\times/N_v K_w^\times),$$

$$\text{Ш}_k(T') = \text{Ker}(k^{\times}/N_1 K_1^{\times} N_2 K_2^{\times} \longrightarrow \prod_v k_v^{\times}/N_{M_v} M_v^{\times}).$$

In the following, we put $I := J/J \cap \varDelta' \prod_{S \cup S_i} N_{M_v} M_v^{\times}$ for simplicity.

COROLLARY 1. *Assume k is a number field and $S = S_{\infty}$. Then we have*

(2.3) $$E_{S_{\infty}}(K/k) = \frac{[\text{Ш}_k(T)] \prod_v e(K_w'/k_v)}{[K' : k](\mathcal{O}_k^{\times} : N\mathcal{O}_K^{\times})}$$

(2.4) $$E_{S_{\infty}}(K_1, K_2/k) = \frac{[\text{Ш}_k(T')] \prod_v e(M_v'/k_v)}{(\mathcal{O}_k^{\times} : N_1 \mathcal{O}_{K_1}^{\times} N_2 \mathcal{O}_{K_2}^{\times})[I]}$$

where we put $\mathcal{O}_v = k_v$ for $v \in S_{\infty}$, and $\mathcal{O}_k$, $\mathcal{O}_K$ and $\mathcal{O}_{K_i}$ are the rings of integers in k, K and K_i $(i = 1, 2)$ respectively.

Remark. (2.3) is nothing but Ono's Theorem [O2], § 2.

COROLLARY 2. *Assume that S is enough large such that $h_{k,S} = h_{K,S_K} = h_{K_i,S_i} = h_{T,S} = h_{T',S} = 1$ $(i = 1, 2)$ and S contains all ramified places of k. Then we have the following formulas for the Tate-Shafarevich groups.*

(2.5) $$[\text{Ш}_k(T)] = \frac{[K' : k](\mathcal{O}_S^{\times} : N\mathcal{O}_{S_K}^{\times})}{\prod_{v \in S} [K_w' : k_v]}$$

(2.6) $$[\text{Ш}_k(T')] = \frac{(\mathcal{O}_S^{\times} : N_1 \mathcal{O}_{S_1}^{\times} N_2 \mathcal{O}_{S_2}^{\times})[I]}{\prod_{v \in S} [M_v' : k_v]}$$

where $I = J/J \cap \varDelta' \prod_{v \in S} N_{M_v} M_v^{\times}$.

COROLLARY 3. *If K/k is cyclic, and if K_1/k or K_2/k is cyclic, then we have*

(2.7) $$E_S(K/k) = \frac{\prod_{v \in S} [K_w : k_v] \prod_{v \notin S} e(K_w/k_v)}{[K : k](\mathcal{O}_S^{\times} : N\mathcal{O}_{S_K}^{\times})}$$

(2.8) $$E_S(K_1, K_2/k) = \frac{\prod_{v \in S} [M_v : k_v] \prod_{v \notin S} e(M_v/k_v)}{(\mathcal{O}_S^{\times} : N_1 \mathcal{O}_{S_1}^{\times} N_2 \mathcal{O}_{S_2}^{\times})[I]}$$

where $e(K_w/k_v)$ and $e(M_v/k_v)$ denote the ramification indices of K_w/k_v and M_v/k_v respectively.

§ 3. The surjectivity of the norm map

Let A be a Dedekind domain with the field of fractions F. Let E be a finite Galois extension of F and let B be the integral closure of A in E. Put $X := \text{Spec } A$ and $Y := \text{Spec } B$.

The purpose of this section is to show the surjectivity of the norm map:

$$N: \prod_{Y/X} \mathbf{G}_{m,Y} \longrightarrow \mathbf{G}_{m,X}$$

defined in the following way. For each U in X_{et}, define

$$N(U): \mathrm{Hom}_X(U, \prod_{Y/X} \mathbf{G}_{m,Y}) = (B \otimes_A \Gamma(U, \mathcal{O}_U))^\times \longrightarrow \Gamma(U, \mathcal{O}_U))^\times$$
$$= \mathrm{Hom}_X(U, \mathbf{G}_{m,X})$$

by $N(U)(b \otimes \gamma) := N_{B/A}(b)\gamma^n$, where $N_{B/A}: B \to A$ is the norm map associated to the Galois extension E/F and $n = [E:F]$.

Then, we can easily see that these maps $\{N(U)\}$ commute with restriction maps and so N defines a morphism of sheaves on X_{et}.

To show the surjectivity of N, let us see the map $N_{\bar{x}}$ induced on the stalks at each geometric point $\bar{x}$ associated to $x \in X$:

$$N_{\bar{x}}: (\prod_{Y/X} \mathbf{G}_{m,Y})_{\bar{x}} = (B \otimes_A A_x^{\mathrm{sh}})^\times \longrightarrow (A_x^{\mathrm{sh}})^\times = (\mathbf{G}_{m,X})_{\bar{x}}$$

where A_x^{sh} denotes the strict henselization of A at x.

Let y_i $(1 \le i \le g)$ be the points in Y lying above x with $f = [\kappa(y_i) : \kappa(x)]$ = the degree of the extension of the residue fields, and let B_x^{sh} be the strict henselization of B at y_i. Finally let F_x^{sh} and $E_{y_i}^{\mathrm{sh}}$ denote the fields of fractions of A_x^{sh} and $B_{y_i}^{\mathrm{sh}}$ respectively.

Then, via a canonical isomorphism

$$B \otimes_A A_x^{\mathrm{sh}} \overset{\sim}{\longrightarrow} \prod_{i=1}^{g} \prod_{j=1}^{f} B_{y_i,j}^{\mathrm{sh}} \quad B_{y_i,j}^{\mathrm{sh}} = B_{y_i}^{\mathrm{sh}}$$

we can see

$$N_{\bar{x}} = \prod_{i=1}^{g} \prod_{j=1}^{f} N_{i,j}$$

where $N_{i,j}: B_{y_i,j}^{\mathrm{sh}} = B_{y_i}^{\mathrm{sh}} \to A_x^{\mathrm{sh}}$ is the norm map associated to the Galois extension $E_{y_i}^{\mathrm{sh}}/F_x^{\mathrm{sh}}$. So, it suffices to show the surjectivity of each $N_{i,j}$ which, however, follows from that the cohomological dimension of $F_x^{\mathrm{sh}} \le 1$ ([Se], II-3.1, 4.3).

§4. Nisnevich topology and class sets

The aim of this section is to introduce a certain Grothendieck topology, called Nisnevich topology, and state a theorem which connects a class set of an affine group scheme and etale cohomology. For the details of proofs in the following, we refer to [N2].

Let X be any noetherian scheme. We define a Grothendieck topology

X_{Nis} on X by the following: As a category, X_{Nis} is the same as the small etale site on X. A family of morphisms $(\phi_i : U_i \to U)$ in X_{Nis} is a covering if and only if for any $x \in U$, there exists an index i and $y \in U_i$ such that $\phi_i(y) = x$ and $\kappa(y) \simeq \kappa(x)$, where $\kappa(x)$ and $\kappa(y)$ are the residue fields of U and U_i at x and y respectively. The cohomology $H_{\mathrm{Nis}}^*(X, \mathscr{F})$ for any sheaf of group $\mathscr{F}$ on X_{Nis} are defined in the usual way (cf. [M.A]).

Let $X = \operatorname{Spec} R$, R is a Dedekind domain and let G be an affine group scheme of finite type over X with the smooth generic fibre. Then we can define the class set $C(G)$ in the obvious way. (If X is ours in § 1, it is $C_s(G)$ in § 1.)

The next theorem is the main property of Nisnevich cohomology for our purpose.

THEOREM ([N1], [N2])   *We have a canonical bijection*

(4.1) $$H_{\mathrm{Nis}}^1(X, G) \simeq C(G).$$

Furthermore we have an exact sequence of pointed sets

(4.2) $$1 \longrightarrow C(G) \longrightarrow H_{\mathrm{et}}^1(X, G) \longrightarrow H_{\mathrm{Nis}}^0(X, R^1 u_* G) \longrightarrow 1$$

and the stalk $(R^1 u_* G)_x$ at $x \in X$ can be computed by the formula

(4.3) $$(R^1 u_* G)_x \simeq H_{\mathrm{et}}^1(R_x^h, G)$$

where $u : X_{\mathrm{et}} \to X_{\mathrm{Nis}}$ is the morphism of sites and R_x^h is the henselization of R at x. If G is commutative, then all maps are homomorphisms of groups.

Remark 1. (4.2) is an immediate consequence of (a non-abelian analogue of) the Cartan-Leray spectral sequence for $u : X_{\mathrm{et}} \to X_{\mathrm{Nis}}$, (4.1) and the fact that the cohomological dimension of $X_{\mathrm{Nis}} \leq 1$.

Remark 2. By (4.3), $H_{\mathrm{Nis}}^0(X, R^1 u_* G)$ can be computed by the following way. Let X° be the set of all closed points of X. For $x \in X^\circ$, let

$$\alpha_x : H^1(R_x^h, G) \longrightarrow H^1(L_x^h, G)$$
$$\beta_x : H^1(L, G) \longrightarrow H^1(L_x^h, G)$$

be the natural restrictions where L_x^h is the field of fractions of R_x^h. Then we have

(4.4) $H_{\mathrm{Nis}}^0(X, R^1 u_* G) = \{a \in H^1(L, G) \mid \beta_x(a) \in \operatorname{Im}(\alpha_x), \text{ for each } x \in X^\circ\}.$

In the above, we can replace L_x^h by $L_{v(x)}$ where $L_{v(x)}$ is the completion of L_x^h with respect to x, because Greenberg approximation theorem [G] tells

us the injectivity $H^1(L_x^h, G) \overset{\subset}{\longrightarrow} H^1(L_{v(x)}, G)$.

§5. Proof of theorem

First, we shall prove the formula (2.1). (2.2) is obtained in the similar way. In the following, notations will be as in §1 and §2. All cohomology groups are etale ones except Nisnevich ones.

By (1.1), we have a long exact sequence;

$$(5.1) \quad \cdots \longrightarrow H^0(X, \textstyle\prod_{X_K/X} \mathbf{G}_{m,X_K}) \overset{N}{\longrightarrow} H^0(X, \mathbf{G}_{m,x}) \longrightarrow H^1(X, T)$$
$$\longrightarrow H^1(X, \textstyle\prod_{X_K/X} \mathbf{G}_{m,X_K}) \overset{N}{\longrightarrow} H^1(X, \mathbf{G}_{m,x}) \longrightarrow \cdots$$

where it is easy to see the followings:

$$(5.2) \quad \begin{aligned} &H^0(X, \mathbf{G}_{m,x}) = \mathcal{O}_S^\times, \\ &H^0(X, \textstyle\prod_{X_K/X} \mathbf{G}_{m,X_K}) = H^0(X_K, \mathbf{G}_{m,X_K}) = \mathcal{O}_{S_K}^\times, \\ &H^1(X, \mathbf{G}_{m,x}) = \mathrm{Pic}(X) = C_S(\mathbf{G}_{m,x}), \\ &H^1(X, \textstyle\prod_{X_K/X} \mathbf{G}_{m,X_K}) = H^1(X_K, \mathbf{G}_{m,X_K}) = \mathrm{Pic}(X_K) = C_S(\textstyle\prod_{X_K/X} \mathbf{G}_{m,X_K}). \end{aligned}$$

For $H^1(X, T)$, by (4.1) and (4.2), we have an exact sequence

$$(5.3) \qquad 0 \longrightarrow C_S(T) \longrightarrow H^1(X, T) \longrightarrow H^0_{\mathrm{Nis}}(X, R^1 u_* T) \longrightarrow 0$$

where $u : X_{\mathrm{et}} \to X_{\mathrm{Nis}}$ is the morphism of sites.

From (1.3) and (5.1)–(5.3), we have

$$(5.4) \qquad E_s(K/k) = \frac{[H^0_{\mathrm{Nis}}(X, R^1 u_* T)]}{(\mathcal{O}_S^\times : N\mathcal{O}_{S_K}^\times)[\mathrm{Coker}(\mathrm{Pic}(X_K) \overset{N}{\longrightarrow} \mathrm{Pic}(X))]}.$$

For simplicity, we set

$$C: = \mathrm{Coker}(\mathrm{Pic}(X_K) \overset{N}{\longrightarrow} \mathrm{Pic}(X))$$
$$\Gamma: = H^0_{\mathrm{Nis}}(X, R^1 u_* T).$$

For C, in terms of idele, it is easy to see that

$$(5.5) \qquad\qquad C \simeq k_A^\times / k^\times J_s N K_A^\times$$

where $J_s: = \prod_{v \in S} k_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$ and N means the norm map of ideles in the obvious sense. On the other hand, the Artin reciprocity tells us a canonical isomorphism

$$(5.6) \qquad\qquad k_A^\times / k^\times N K_A^\times \simeq \mathrm{Gal}(K'/k).$$

Since the norm map of an unramified local extension is surjective, we have

(5.7) $\mathrm{Ker}(k_A^\times/k^\times NK_A^\times \longrightarrow k_A^\times/k^\times J_S NK_A^\times) \simeq J_S/J_S \cap k^\times NK_A^\times$

$$\simeq J_r/J_r \cap \varDelta \prod_{v \in S \cup S_r} N_v K_w^\times$$

where S_r denotes the set of all finite places of k which ramify in K/k,

$$J_r := \prod_{v \in S} k_v^\times \times \prod_{v \in S_r \setminus S} \mathcal{O}_v^\times \quad \text{and} \quad \varDelta := \{x \in k^\times \,|\, x \in N_v K_w^\times, \ v \notin S \cup S_r\}\,.$$

From (5.5)–(5.7), we have

(5.8) $$[C] = \frac{[K';k]}{[D]}$$

where we put $D := J_r/J_r \cap \varDelta \prod_{v \in S \cup S_r} N_v K_w^\times$.

Finally, let us analyze $\varGamma$ by the Remark 2 in §4. We identify the closed points of X and the places of k outside S. Let

$$\alpha_v : H^1(\mathcal{O}_v^h, T) \longrightarrow H^1(k_v, T)$$
$$\beta_v : H^1(k, T) \longrightarrow H^1(k_v, T)$$

be the natural restrictions for $v \notin S$ where $\mathcal{O}_v^h$ is the henselization of $\mathcal{O}_S$ at v. Here, from (1.1), we have $H^1(\mathcal{O}_v^h, T) \simeq \mathcal{O}_v^{h\times}/N\mathcal{O}_w^{h\times}$ and $H^1(k_v, T) \simeq k_v^\times/NK_w^\times$ because $\mathrm{Pic}(\mathcal{O}_w^h) = 0$ and Hilbert 90 respectively, where $\mathcal{O}_w^h$ is the henselization of $\mathcal{O}_{S_K}$ at w. Hence, by the density of $\mathcal{O}_v^h$ in $\mathcal{O}_v$ and the openness of $N\mathcal{O}_w^\times$ in $\mathcal{O}_v^\times$, we have

(5.9) $$\mathrm{Im}(\alpha_v) \simeq \mathcal{O}_v^\times/N\mathcal{O}_w^\times\,.$$

Since $(\mathcal{O}_v^\times : N\mathcal{O}_w^\times) = e(K_w'/k_v) :=$ the ramification index of K_w'/k_v by local class field theory, from (4.4) and (5.9), we have

(5.10) $\varGamma = \mathrm{Ker}(H^1(k, T) \xrightarrow{\ m\ } \prod_{v \notin S \cup S_r} H^1(k_v, T) \times \prod_{v \in S_r \setminus S} H^1(k_v, T)/(\mathrm{Im}\,\alpha_v))$

where the m is the natural restriction.

So, we are led to look at the following commutative exact diagram.

$$0 \longrightarrow \prod_{v \in S} H^1(k_v, T) \times \prod_{v \in S_r \setminus S} \mathrm{Im}\,\alpha_v$$
$$\wr|$$
$$\prod_{v \in S} k_v^\times/N_v K_w^\times \times \prod_{v \in S_r \setminus S} \mathcal{O}_v^\times/N_v\mathcal{O}_w^\times$$

$$\quad k^\times/NK^\times \qquad\qquad\qquad k^\times/NK^\times$$
$$\quad \wr| \qquad\qquad\qquad\qquad\qquad \wr|$$
$$\quad H^1(k, T) \qquad = \qquad H^1(k, T)$$
$$\quad \imath\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow m$$
$$\longrightarrow \prod_v H^1(k_v, T) \longrightarrow \prod_{v \notin S_r \cup S} H^1(k_v, T) \times \prod_{v \in S_r \setminus S} H^1(k_v, T)/\mathrm{Im}\,\alpha_v \longrightarrow 0$$
$$\wr| \qquad\qquad\qquad\qquad\qquad \wr|$$
$$\prod_v k_v^\times/N_v K_w^\times \qquad \prod_{v \notin S_r \cup S} k_v^\times/N_v K_w^\times \times \prod_{v \in S_r \setminus S} k_v^\times/\mathcal{O}_v^\times N_v K_w^\times$$

where the maps in the bottom are componentwise and each isomorphism follows from (1.1) and Hilbert 90, or (5.9). Here, $\mathrm{Ker}(m) = \Gamma$ by (5.10), $\mathrm{Ker}(l) = \mathrm{III}_k(T)$ by the definition, and $k_v^\times/NK_w^\times \simeq G(K_w'/k_v)$ by local class field theory. To apply snake lemma to the above diagram, let us see

(5.11) $\mathrm{Ker}(\mathrm{Coker}(l) \longrightarrow \mathrm{Coker}(m))$

$$\simeq \mathrm{Ker}(\prod_v k_v^\times/k^\times \prod_v N_v K_w^\times \longrightarrow \prod_{v \notin S} k_v^\times/k^\times (\prod_{v \notin S \cup S_r} N_v K_w^\times \times \prod_{v \in S_r \setminus S} \mathcal{O}_v^\times N_v K_w^\times))$$

$$\simeq J_r/J_r \cap \Delta \prod_{v \in S \cup S_r} N_v K_w^\times = D((5.8))$$

and so we have

(5.12)
$$\Gamma = \frac{[\mathrm{III}_k(T)] \prod_{v \in S} [K_w' : k_v] \prod_{v \notin S} e(K_w'/k_v)}{[D]}.$$

Together with (5.4), (5.8) and (5.12), we get (2.1).

Next, let us turn to the proof of (2.2). As in the above case, by the long exact sequence associated to (1.2), we have

$$E_S(K_1, K_2/k) = \frac{[\Gamma']}{(\mathcal{O}_S^\times : N_1 \mathcal{O}_{S_1}^\times N_2 \mathcal{O}_{S_2}^\times)[C']}$$

where

$$C' := \mathrm{Coker}(\mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2) \xrightarrow{N_1 N_2} \mathrm{Pic}(X))$$
$$\Gamma' := H^0_{\mathrm{Nis}}(X, R^1 u_* T').$$

In terms of idele, $C' \simeq k_A^\times/k^\times J_S N_1 K_{1A}^\times N_2 K_{2A}^\times$, however, by class field theory, the subgroup $k^\times N_1 K_{1A}^\times N_2 K_{2A}^\times$ of $k_A^\times$ corresponds to the maximal abelian subextension of $K_1 \cap K_2$ over k, which is k itself by our assumption. Hence $C' = \{1\}$.

To analyze Γ', note that

$$H^1(k_v, T') \simeq k_v^\times/N_v M_v^\times$$

which follows from (1.2). Hilbert 90 and local class field theory. The rest of argument is quit same as in the case of Γ. So we left the details to the reader. The reason that the term I appears in (2.2) may be clear by (5.11).

Concerning the corollaries, corollaries 1, 2 are immediate consequences of theorem. (2.7) and (2.8) i.e., the vanishing of $\mathrm{III}_k(T)$ and $\mathrm{III}_k(T')$, follow from the Tchebotareff density theorem and [H], Proposition 3.3.

§ 6. Examples

For some examples of $E_{S_\infty}(K/k)$ for the number field case, we refer to [O2], § 5.

EXAMPLE 1. (Gauss' genus formula for the function field case)   Suppose that $k = \mathbf{F}_q(T)$, the rational function field over a finite field with q elements, and K is a quadratic extension of k and suppose that $S = \{\infty\}$, where ∞ denotes the place of k corresponding to the pole of T. For simplicity, we assume that the characteristic of k is different from 2.

According to E. Artin [E.A], let us say that K is real when ∞ is decomposed into two distinct places in K, and K is imaginary, otherwise. By S-unit theorem or [E.A], § 14,

$$\mathscr{O}_S^\times = \mathbf{F}_q^\times, \quad \mathscr{O}_{S_K}^\times \simeq \begin{cases} \mathbf{F}_q^\times \times \mathbf{Z} & K \text{ is real.} \\ F_q^\times & K \text{ is imaginary.} \end{cases}$$

and so

$$(\mathscr{O}_S^\times : N\mathscr{O}_{S_K}^\times) = \begin{cases} 1 & K \text{ is real and } N\mathscr{O}_{S_K}^\times = \mathbf{F}_q^\times. \\ 2 & K \text{ is real and } N\mathscr{O}_{S_K}^\times = (\mathbf{F}_q^\times)^2 \text{ or } K \text{ is imaginary.} \end{cases}$$

Therefore (2.7) yields

$$E_S(K/k) = \begin{cases} 2^{t_K - 2} & K \text{ is real and } N\mathscr{O}_{S_K}^\times = (\mathbf{F}_q^\times)^2. \\ 2^{t_K - 1} & K \text{ is real and } N\mathscr{O}_{S_K}^\times = \mathbf{F}_q^\times \text{ or } K \text{ is imaginary.} \end{cases}$$

where t_K is the number of places of $k \neq \infty$ which ramify in K/k. This is the exact analogue of the number field case; [O2], § 5, Example 2. See also [E.A] § 11.

EXAMPLE 2. (Cyclotomic function fields)   We refer to [G-R] for some properties about cyclotomic function fields, which we shall use below.

Suppose that $k = \mathbf{F}_q(T)$ and $K = k(\Lambda_f)$, the f-th cyclotomic function field, where f is an irreducible polynomial of degree d in $R := \mathbf{F}_p[T]$. Suppose $S = \{\infty\}$. It is known that K/k is a cyclic extension whose Galois group is isomorphic to $(R/(f))^\times$, which is the cyclic group of order q^{d-1}. Let K^+ be the fixed field of $\mathbf{F}_q^\times = R^\times \subset (R/(f))^\times$. For the ramifications and units, the following analogies of cyclotomic number fields are known.

(1) Every place except (f) and ∞ is unramified in K/k. (f) is totally ramified in K/k. ∞ splits totally in K^+/k and each of these places of K^+

is totally ramified in K/K^+.

(2) $\mathcal{O}_{S_K}^{\times} = \mathcal{O}_{S_{K+}}^{\times}$.

Since $\mathcal{O}_{S_{K+}}^{\times} \simeq \mathbf{F}_q^{\times} \times \mathbf{Z}^{d/(q-1)}$ by (1) and S-unit theorem, we have, by (2),

$$(\mathcal{O}_{S_{K+}}^{\times} : N_{K/k}\mathcal{O}_{S_K}^{\times}) = (\mathcal{O}_{S_{K+}}^{\times} : (\mathcal{O}_{S_{K+}}^{\times})^{q-1}) = (q-1)^{d/(q-1)+1}$$

and

$$(\mathcal{O}_S^{\times} : N_{K/k}\mathcal{O}_{S_K}^{\times}) = (\mathbf{F}_q^{\times} : (N_{K+/k}\mathcal{O}_{S_{K+}}^{\times})^{q-1}) = q-1 .$$

Therefore, by (1) again, (2.7) yields

$$E_S(K/k) = E_S(K/K^+) = 1 .$$

These are exact analogues of the cyclotomic number field case [O2], § 5, Examples 4, 5. Moreover the formula tells us that in the case of K/K^+, $h_{T,S}$ can be regarded as the "$-$ part" of $h_{K,S}$.

EXAMPLE 3. Let p_1 and p_2 be distinct prime numbers and let n_1 and n_2 be integers such that $p_1^{n_1}$ and $p_2^{n_2} \geq 3$. Suppose that $k = \mathbf{Q}$, $K_1 = \mathbf{Q}(\zeta_{p_1^{n_1}})$ and $K_2 = \mathbf{Q}(\zeta_{p_2^{n_2}})$ where $\zeta_{p_1^{n_1}}$ and $\zeta_{p_2^{n_2}}$ denote primitive $p_1^{n_1}$-th and $p_2^{n_2}$-th roots of 1, and suppose $S = S_\infty = \{\infty\}$. It is easy to see that $M_\infty = \mathbf{C}$, $M_v = \mathbf{Q}_v$ for $v \in S_r'$ and $-1 \in \Delta'$. So, $I = \{1\}$. Furthermore, $\mathcal{O}_S^{\times} = \{\pm 1\}$ and $N_1\mathcal{O}_{S_1}^{\times} = N_2\mathcal{O}_{S_2}^{\times} = \{1\}$. Therefore, by (2.8), we have

$$E_{S_\infty}(K_1, K_2/k) = 1 .$$

In other word, $h_{K_1} \cdot h_{K_2} = h_{T'}$ where h_{K_1}, h_{K_2} and $h_{T'}$ are the class numbers of K_1, K_2 and T' respectively.

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