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THE STATIONARY PHASE METHOD WITH AN ESTIMATE OF THE REMAINDER TERM ON A SPACE OF LARGE DIMENSION

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Dedicated to the memory of Professor Kôsaku Yosida

§ 1. Introduction

In discussing convergence of Feynman path integrals [2], we need a stationary phase method of oscillatory integrals over a space of large dimension. More precisely, we have to know how the remainder term behaves when the dimension of the space goes to ∞ (cf. [2], [3] and [5]). The aim of the present note is to give answer to this question under rather mild assumptions. Application to the Feynman path integrals is discussed in [3] and [5].

Oscillatory integral of a function $f(x)$, $x \in \mathbb{R}^k$, is defined by the equality

$$\tilde{\int}_{\mathbb{R}^k} f(x)dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^k} e^{-\varepsilon |x|^2} f(x)dx .$$

We consider the following oscillatory integrals

$$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \prod_{j=1}^{L} \left(\frac{\nu i}{2\pi t_j}\right)^{d/2} \tilde{\int}_{\mathbb{R}^{d(L-1)}} e^{-i\nu S(x_L, \cdots, x_0)} a(x_L, \cdots, x_0) \prod_{j=1}^{L-1} dx_j .$$

Here each x_j, $j = 0, \cdots, L$, runs in $\mathbb{R}^d$, $\nu > 1$ is a constant and t_j, $j = 1, 2, \cdots, L$, are positive constants.

Our assumption for the phase function $S(x_L, \cdots, x_0)$ is the following:

(H.1) $S(x_L, \cdots, x_0)$ is a real valued function of the form

$$S(x_L, x_{L-1}, \cdots, x_0) = \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1}) ,$$

where

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$$S_j(t_j, x_j, x_{j-1}) = \frac{1}{2t_j}|x_j - x_{j-1}|^2 + t_j\omega_j(t_j, x_j, x_{j-1}), \qquad j = 1, 2, \cdots, L.$$

For any $m \geqq 2$ there exists a constant $\kappa_m > 0$ independent of j such that

$$\max_{2\leqq|\alpha|+|\beta|\leqq m} \sup_{x,y\in\mathbf{R}^d} |\partial_x^\alpha \partial_y^\beta \omega_j(t_j, x, y)| \leqq \kappa_m.$$

(Since both x and y have d components, both α and β are multi-indices with d components.)

We need a little more notations to write down our assumptions about the amplitude function. If $T_L = t_1 + t_2 + \cdots + t_L$ is small enough, the critical point $(x^*_{L-1}, x^*_{L-2}, \cdots, x^*_1)$ of the phase is the unique solution of

$$\partial_{x_j} S_{j+1}(t_{j+1}, x^*_{j+1}, x^*_j) + \partial_{x_j} S_j(t_j, x^*_j, x^*_{j-1}) = 0, \qquad j = 1, 2, \cdots, L-1,$$

where $x^*_L = x_L$ and $x^*_0 = x_0$ (See §2 for the proof). We use the following notation

$$a(\overbrace{x_L, x_0}) = a(x_L, x^*_{L-1}, \cdots, x^*_1, x_0).$$

Similarly, for any pair of integers k, m with $k + 1 < m$ let $x^*_{k+1}, \cdots, x^*_{m-1}$ be the partial critical point, i.e.,

$$\partial_{x_j} S_{j+1}(t_{j+1}, x^*_{j+1}, x^*_j) + \partial_{x_j} S_j(t_j, x^*_j, x^*_{j-1}) = 0,$$

for $j = k+1, \cdots, m-1$, where $x^*_k = x_k$ and $x^*_m = x_m$. Then we set

$$a(x_L, \cdots, \overbrace{x_m, x_k}, \cdots, x_0) = a(x_L, \cdots, x_m, x^*_{m-1}, \cdots, x^*_{k+1}, x_k, \cdots, x_0).$$

If $m = k + 1$, we define

$$a(x_L, x_{L-1}, \cdots, \overbrace{x_{k+1}, x_k}, \cdots, x_0) = a(x_L, \cdots, x_{k+1}, x_k, \cdots, x_0).$$

Our assumption for the amplitude function $a(x_L, \cdots, x_0)$ is the following:

(H.2) For any positive integer K there exist positive constants A_K and X_K with the following properties:

(i) If $|\alpha_j| \leqq K$ for $j = 0, 1, \cdots, L$, then

$$\sup_{(x_L,\cdots,x_0)\in\mathbf{R}^{L+1}} \left| \left(\prod_{j=0}^{L} \partial_{x_j}^{\alpha_j} \right) a(x_L, \cdots, x_0) \right| \leqq A_K X_K^L.$$

(ii) For any sequence of positive integers

$$j_0 = 0 < j_1 - 1 < j_1 < j_2 - 1 < \cdots < j_s < L, \qquad s = 1, \cdots, L-1,$$

we have the estimate

$$|\partial_{x_0}^{\alpha_0}\partial_{x_{j_1-1}}^{\alpha_{j_1-1}}\cdots\partial_{x_{j_s}}^{\alpha_{j_s}}\partial_{x_L}^{\alpha_L}a(\overline{x_L, x_{j_s}}, \overline{x_{j_s-1}, x_{j_{s-1}}}, \cdots, \overline{x_{j_1-1}, x_0})| \leqq A_K X_K^s ,$$

as far as $|\alpha_j| \leqq K$, $j = 0, j_1 - 1, j_1, \cdots, j_s, L$.

Our main result is

THEOREM 1. *Under the assumptions* (H.1) *and* (H.2) *above there exists a positive constant* δ *independent of* a *and* L *such that if* $T_L = t_1 + t_2 + \cdots + t_L < \delta$ *then*

$$I(\{t_j\}, S, a, \nu)(x_L, x_0)$$
$$= \left(\frac{\nu i}{2\pi T_L}\right)^{d/2} \exp\{-i\nu S(\overline{x_L, x_0})\} \det(I + H^{-1}W)^{-1/2}(a(\overline{x_L, x_0}) + r(x_L, x_0)) ,$$

where $r(x_L, x_0)$ *satisfies the estimate: For any* $K \geqq 0$ *there exist positive constants* C_K *and* $M(K)$ *such that if* $|\alpha_0|$, $|\alpha_L| \leqq K$,

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} r(x_L, x_0)| \leqq A_{M(K)}\left(\prod_{j=1}^{L}(1 + C_K X_{M(K)}\nu^{-1}t_j) - 1\right).$$

Constants δ, C_K *are independent of* a, L, $\{t_j\}$, (x_L, x_0) *and of* ν *but depend on the dimensionality* d *of space* $\mathbb{R}^d$ *and* $\{\kappa_m\}$. $M(K)$ *depends only on* K *and* d. H *is the* $d(L-1) \times d(L-1)$ *matrix*

$$H = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{t_2}, & -\frac{1}{t_2}, & 0, & 0, & \cdots\cdots 0 \\ -\frac{1}{t_2}, & \frac{1}{t_2} + \frac{1}{t_3}, & -\frac{1}{t_3}, & 0, & \cdots\cdots 0 \\ 0, & -\frac{1}{t_3}, & \frac{1}{t_3} + \frac{1}{t_4}, & -\frac{1}{t_4},\ 0, \cdots & 0 \\ 0, & 0, & -\frac{1}{t_4}, & \frac{1}{t_4} + \frac{1}{t_5}, \cdots & \end{pmatrix}$$

and w *is the Hessian matrix of* $\sum_j t_j\omega_j(t_j, x_j, x_{j-1})$ *at the critical point* $(x_{L-1}^*, \cdots, x_1^*)$.

In case $a = 1$, we can prove a sharper estimate of the remainder term.

THEOREM 2. *We assume that* $a = 1$ *and* (H.1). *If* $T_L < \delta$, *then for any* K *there exists a constant* C_K' *such that if* $|\alpha_0|$ *and* $|\alpha_L| \leqq K$,

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} r(x_L, x_0)| \leqq \prod_{j=1}^{L}(1 + C_K'\nu^{-1}t_j T_L^2) - 1 .$$

The constant δ is the same as in Theorem 1.

Remark. 1) We can easily see from the proof that the phase function and amplitude function are not necessarily infinitely differentiable. Both Theorems 1 and 2 are still valid with obvious modification if both functions are of class C^k with sufficiently large k.

2) In the previous work [4], less sharp result was obtained.

The plan of the paper is as follows: Theorems 1 and 2 will be proved in the case $d=1$ in order to avoid excessive complexity of notations. In § 2 we collect basic properties of both the critical point and the critical value of the phase function. In § 3 we prove a key lemma which plays a fundamental role in this paper. This lemma, Lemma 3.1, may be of independent interest. In § 4 Theorem 1 is proved. Theorem 2 is proved in § 5.

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§ 2. Phase functions

In this section we discuss the followings:

(i) Unique existence of the critical point of S.

(ii) $S(\overline{x_L, x_0})$ is of the form $\dfrac{|x_L - x_0|^2}{2T_L} + T_L\omega^{\#}(x_L, x_0)$.

(iii) Some elementary facts related to the Hessian of S.

The critical point $x^* = (x^*_{L-1}, \cdots, x^*_1)$ of the phase function is given by the system of equations

$$
\begin{aligned}
(2.1)\qquad & \frac{1}{t_2}(x_1^* - x_2^*) + \frac{1}{t_1}(x_1^* - x_0) + t_2\partial_1\omega_2(t_2, x_2^*, x_1^*) + t_1\partial_1\omega_1(t_1, x_1^*, x_0) = 0\,, \\
& \frac{1}{t_3}(x_2^* - x_3^*) + \frac{1}{t_2}(x_2^* - x_1^*) + t_3\partial_2\omega_3(t_3, x_3^*, x_2^*) + t_2\partial_2\omega_2(t_2, x_2^*, x_1^*) = 0\,, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \frac{1}{t_L}(x_{L-1}^* - x_L) + \frac{1}{t_{L-1}}(x_{L-1}^* - x_{L-2}^*) + t_L\partial_{L-1}\omega_L(t_L, x_L, x_{L-1}^*) \\
& \qquad + t_{L-1}\partial_{L-1}\omega_{L-1}(t_{L-1}, x_{L-1}^*, x_{L-2}^*) = 0\,.
\end{aligned}
$$

Here and hereafter ∂_k is the abbreviation of $\partial_{x_k} = \partial/\partial x_k$.

The Hessian matrix of S is equal to $H(L, 1) + W(L, 1; x)$, where

$$(2.2)\qquad H(L,1)=\begin{pmatrix} \frac{1}{t_1}+\frac{1}{t_2}, & -\frac{1}{t_2}, & 0, & 0, & & \\ -\frac{1}{t_2}, & \frac{1}{t_2}+\frac{1}{t_3}, & -\frac{1}{t_3}, & 0, & 0, \cdots\cdots & \\ 0, \cdots\cdots & & & & & \\ 0, \cdots\cdots & & & , & -\frac{1}{t_{L-1}}, & \frac{1}{t_{L-1}}+\frac{1}{t_L} \end{pmatrix}$$

and

$$(2.3)\qquad W(L,1;x)=\begin{pmatrix} t_2\partial_1^2\omega_2+t_1\partial_1^2\omega_1, & t_2\partial_1\partial_2\omega_2, & 0, \cdots\cdots, 0 & \\ t_2\partial_2\partial_1\omega_2, & t_3\partial_2^2\omega_3+t_2\partial_2^2\omega_2, & t_3\partial_2\partial_3\omega_3, & 0, \cdots, 0 \\ \cdots\cdots & & & \end{pmatrix}.$$

We have

PROPOSITION 2.1.

$$(2.4)\qquad \det H(L,1)=\frac{T_L}{t_1t_2\cdots t_L}.$$

Let $G(L, 1)$ be the inverse of $H(L, 1)$. Then its (ij) entry is

$$(2.5)\qquad g_{ij}=\begin{cases}\dfrac{(t_1+\cdots+t_i)(t_{j+1}+\cdots+t_L)}{T_L}, & \text{if } 1\leqq i\leqq j\leqq L-1,\\[2mm] \dfrac{(t_1+\cdots+t_j)(t_{i+1}+\cdots+t_L)}{T_L}, & \text{if } 1\leqq j\leqq i\leqq L-1.\end{cases}$$

We use two norms $\|x\|_\infty=\max_{1\leqq k\leqq L-1}|x_k|$ and $\|x\|_1=\sum_{j=1}^{L-1}|x_j|$ for any $x\in\mathbb{R}^{L-1}$. The next proposition is clear.

PROPOSITION 2.2. *For any $u\in\mathbb{R}^{L-1}$ we have*

$$(2.6)\qquad \|W(L,1;x)u\|_1\leqq 4T_L\kappa_2\|u\|_\infty,$$

$$(2.7)\qquad \|G(L,1)u\|_\infty\leqq\frac{T_L}{4}\|u\|_1,$$

and

$$(2.8)\qquad \|G(L,1)W(L1;x)u\|_\infty\leqq T_L^2\kappa_2\|u\|_\infty.$$

Unique existence of the critical point of S is given by

PROPOSITION 2.3. *Assume that*

$$4\kappa_2 T_L^2 < 2^{-1}. \tag{2.9}$$

Then the critical point exists uniquely and satisfies the estimate:

$$\|x^* - x^0\|_\infty \leqq \frac{T_L}{2}\sum_{k=1}^{L-1}|t_{k+1}\partial_k\omega_{k+1}(t_{k+1}, x_{k+1}^0, x_k^0) + t_k\partial_k\omega_k(t_k, x_k^0, x_{k-1}^0)|, \tag{2.10}$$

where

$$x_j^0 = \frac{t_{j+1} + \cdots + t_L}{T_L}x_0 + \frac{t_1 + \cdots + t_j}{T_L}x_L, \qquad j = 1, \cdots, L-1.$$

Proof. The critical point $x^* = (x_{L-1}^*, \cdots, x_1^*)$ is the fixed point of the map $(x_{L-1}, \cdots, x_1) = x \to \Phi(x) = (y_{L-1}, \cdots, y_1)$, where

$$y_j = -\sum_{k=1}^{L-1} g_{jk}\{t_{k+1}\partial_k\omega_{k+1}(t_{k+1}, x_{k+1}, x_k) + t_k\partial_k\omega_k(t_k, x_k, x_{k-1})\} \tag{2.11}$$
$$+ g_{j1}\frac{1}{t_1}x_0 + g_{jL-1}\frac{1}{t_L}x_L.$$

The norm of the differential map $D\Phi(x) = G(L, 1)W(L, 1; x)$ is less than $\kappa_2 T_L^2 \leqq 1/8$ with respect to the norm $\|\ \|_\infty$ because of (2.8) and (2.9). Therefore, the map $\Phi(x)$ is a contraction map, which guarantees unique existence of the fixed point. Usual construction by iteration of the fixed point gives that

$$\|x^* - x^0\|_\infty < 2\|\Phi(x^0) - x^0\|_\infty. \tag{2.12}$$

We have $\Phi(x^0) - x^0 = G(L, 1)\Omega(x^0)$, where $\Omega(x) = (\Omega_1(x), \cdots, \Omega_{L-1}(x))$ and

$$\Omega_j(x) = t_j\partial_j\omega_j(t_j, x_j, x_{j-1}) + t_{j+1}\partial_j\omega_{j+1}(t_{j+1}, x_{j+1}, x_j), \qquad j = 1, \cdots, L-1.$$

This and (2.7) yield that $\|\Phi(x^0) - x^0\|_\infty < (T_L/4)\|\Omega\|_1$. This together with (2.12) proves the estimate (2.10). Proposition 2.3 has been proved.

Let y and z be points in $\mathbb{R}^{L-1}$ such that

$$y_j = g_{j1}(t_1^{-1} - t_1\partial_0\partial_1\omega_1(t_1, x_1^*, x_0)), \qquad j = 1, \cdots, L-1, \tag{2.13}$$

$$z_j = g_{jL-1}(t_L^{-1} - t_{L-1}\partial_{L-1}\partial_L\omega_L(t_{L-1}, x_L, x_{L-1}^*)), \qquad j = 1, \cdots, L-1. \tag{2.14}$$

Then

$$\|y\|_\infty, \qquad \|z\|_\infty \leqq 1 + \kappa_2 T_L^2 < \frac{9}{8}. \tag{2.15}$$

We consider the critical point as a function of (x_L, x_0). Let $X = \partial_0 x^*$ and $Y = \partial_L x^*$; let $D_X W(L, 1; x^*)$ and $D_Y W(L, 1; x^*)$ be derivatives at x^* of matrix valued function $W(L, 1; x)$ in the direction X and Y, respectively.

PROPOSITION 2.4. *We assume* (2.9). *Then we have*

$$\|\partial_0 x^* - y\|_\infty, \qquad \|\partial_{L-1} x^* - z\|_\infty < 4\kappa_2 T_L^2 < \frac{1}{2} \tag{2.16}$$

and

$$\|\partial_0 x^*\|_\infty, \qquad \|\partial_L x^*\|_\infty < 1 + 4\kappa_2 T_L^2 < \frac{3}{2}. \tag{2.17}$$

For any integers α and β there exists a positive constant $C_{\alpha\beta}$ such that

$$\|D_X^\alpha D_Y^\beta W(L, 1; x^*) v\|_1 < C_{\alpha\beta} T_L \|v\|_\infty. \tag{2.18}$$

And for any α and β with $\alpha + \beta \geqq 2$ we have, with some constant $C_{\alpha\beta}$,

$$\|\partial_0^\alpha \partial_{L-1}^\beta x^*\|_\infty \leqq C_{\alpha\beta} T_L^2. \tag{2.19}$$

The constants $C_{\alpha\beta}$ in (2.18) *and* (2.19) *may depend on T_L but are bounded if T_L is bounded.*

Proof. Since x^* is the fixed point of $\Phi(x)$, we have that

$$\partial_0 x^* = (D\Phi(x^*))\partial_0 x^* + y. \tag{2.20}$$

Since the norm of $D\Phi(x^*)$ is less than $\kappa_2 T_L^2 < 1/8$, we have

$$\|\partial_0 x^* - y\|_\infty < 2\kappa_2 T_L^2 \|y\|_\infty \leqq 3\kappa_2 T_L^2 < \frac{3}{8}.$$

Therefore we have

$$\|\partial_0 x^*\|_\infty \leqq 1 + 4\kappa_2 T_L^2 < \frac{3}{2}.$$

Similarly we can prove estimate for $\partial_L x^*$. (2.16) and (2.17) are proved.

Next we prove (2.18). It follows from (2.16) and (2.17) that for each pair of indices α and β there exist $\mathbb{R}^{L-1}$ valued functions $a^{\alpha\beta}(x_L, x_0)$, $b^{\alpha\beta}(x_L, x_0)$ and $c^{\alpha\beta}(x_L, x_0)$ satisfying

$$\begin{aligned}
(D_X^\alpha D_Y^\beta W(L, 1; x^*))_{jk} &= 0, && \text{for } |k - j| > 1, \\
&= t_j a_j^{\alpha\beta}(x_L, x_0), && \text{for } k = j - 1, \\
&= t_j b_j^{\alpha\beta}(x_L, x_0) + t_{j+1} c_j^{\alpha\beta}(x_L, x_0), && \text{for } k = j, \\
&= t_{j+1} a_{j+1}^{\alpha\beta}(x_L, x_0), && \text{for } k = j + 1.
\end{aligned}$$

These functions $a^{\alpha\beta}(x_L, x_0)$, $b^{\alpha\beta}(x_L, x_0)$ and $c^{\alpha\beta}(x_L, x_0)$ may depend also on $t_1, \cdots, t_L$ but remain uniformly bounded in the space $\mathscr{B}(\mathbb{R}_{x_L}\times\mathbb{R}_{x_0})$ as far as (2.9) holds. This proves (2.18).

We shall prove the estimate for $\partial_0\partial_L x^*$. It satisfies

$$\partial_L\partial_0 x^* = D\Phi(x^*)\partial_L\partial_0 x^* + G(L, 1)D_Y W(L, 1; x^*)\partial_0 x^* + \partial_L y .$$

Thus using (2.18), we have

$$\begin{aligned}\|\partial_L\partial_0 x^*\|_\infty &\leqq 2(\|G(L, 1)D_Y W(L, 1; x^*)\partial_0 x^*\|_\infty + \|\partial_L y\|_\infty) .\\ &\leqq CT_L^2\|\partial_0 x^*\|_\infty + CT_L^2 \leqq CT_L^2 .\end{aligned}$$

Here and hereafter we denote simply by C various constants which may be different from one occasion to another. Other higher derivatives of x^* will be estimated similarly. Proposition is proved.

Since $\partial_0 x^*$ satisfies (2.20) and $D\Phi(x^*) = G(L, 1)W(L, 1; x^*)$,

$$\partial_0 x^* = G(L, 1)Z(x_L, x_0) + y , \tag{2.21}$$

where $Z(x_L, x_0) = W(L, 1; x^*)\partial_0 x^*$. Using (2.6), we have

$$\|Z\|_1 = \|W(L, 1; x^*)\partial_0 x^*\|_1 \leqq 4\kappa_2 T_L\|\partial_0 x^*\|_\infty < 6\kappa_2 T_L . \tag{2.22}$$

Moreover, the j-th component of $Z(x_L, x_0)$ is of the form

$$Z_j(x_L, x_0) = t_j\xi_j(x_L, x_0) + t_{j+1}\eta_j(x_L, x_0), \qquad j = 1, \cdots, L-1 , \tag{2.23}$$

where $\{\xi_j\}$ and $\{\eta_j\}$ are functions, which may depend on $t_1, \cdots, t_L$ but bounded in $\mathscr{B}(\mathbb{R}_{x_L}\times\mathbb{R}_{x_0})$. It follows from this that

$$T_L^{-1}\sum_{j=1}^{L-1} Z_j(x_L, x_0) \text{ remains bounded in } \mathscr{B}(\mathbb{R}_{x_L}\times\mathbb{R}_{x_0}) .$$

Next we consider the second derivatives of the critical value $S(\overline{x_L, x_0})$.

PROPOSITION 2.5. *We assume* (2.9). *Then* $S(\overline{x_L, x_0})$ *is of the following form*:

$$S(\overline{x_L, x_0}) = \frac{|x_L - x_0|^2}{2T_L} + T_L\omega^\#(x_L, x_0) . \tag{2.24}$$

Here $\omega^\#(x_L, x_0)$ *is a function, which may depend also on* $t_1, \cdots, t_L$ *but remains bounded in* $\mathscr{B}(\mathbb{R}_{x_L}\times\mathbb{R}_{x_0})$, *satisfying the estimate*

$$\max_{2\leqq|\alpha|+|\beta|\leqq m}\ \sup_{x_L, x_0} |\partial_{x_0}^\alpha\partial_{x_L}^\beta\omega^\#(x_L, x_0)| \leqq \kappa_m^\# , \tag{2.25}$$

where $\kappa_m^\#$ is a constant depending only on κ_j, $2 \leqq j \leqq m$. We can choose

$$\kappa_2^\# = 10\kappa_2 . \tag{2.26}$$

Proof. Since x^* is the critical point of S, we have

$$\begin{aligned}\partial_0 S(\overline{x_L, x_0}) &= \partial_0 S(x_L, x_{L-1}, \cdots, x_1, x_0)|_{x_{L-1}=x^*_{L-1},\cdots,x_1=x^*_1} \\ &= (\partial_0 S_1)(x_1^*, x_0) ,\end{aligned}$$

where we abbreviated $S_1(t_1, x_1, x_0)$ simply by $S_1(x_1, x_0)$. This implies that

$$\partial_0^2 S(\overline{x_L, x_0}) = (\partial_0^2 S_1)(x_1^*, x_0) + (\partial_1\partial_0 S_1)(x_1^*, x_0)\partial_0 x_1^* . \tag{2.27}$$

We have from (2.21) and (2.13) that

$$\partial_0 x_1^* - \frac{t_2 + \cdots + t_L}{T_L} = t_1 T_L h(x_L, x_0) ,$$

where

$$\begin{aligned}h(x_L, x_0) = &-T_L^{-2} t_1(t_2 + \cdots + t_L)\partial_0\partial_1\omega_1(t_1, x_1^*, x_0) \\ &+ T_L^{-2}\sum_{j=1}^{L-1}(t_{j+1} + \cdots + t_L)Z_j(x_L, x_0)\end{aligned}$$

depends also on $t_1, \cdots, t_L$ but we used abbreviation. Since $T_L^{-1}\sum_{j=1}^{L-1} Z_j(x_L, x_0)$ is bounded in $\mathscr{B}(\mathbf{R}_{x_L} \times \mathbf{R}_{x_0})$, $h(x_L, x_0)$ remains bounded in $\mathscr{B}(\mathbf{R}_{x_L} \times \mathbf{R}_{x_0})$ uniformly with respect to $t_1, \cdots, t_L$. In particular, (2.22) yields that

$$|h(x_L, x_0)| \leqq T_L^{-1}\|Z\|_1 + \kappa_2 < 7\kappa_2 . \tag{2.28}$$

Hence we have

$$\partial_{x_0}^2 S(\overline{x_L, x_0}) = \frac{1}{T_L} + T_L\psi(x_L, x_0) ,$$

where

$$\begin{aligned}\psi(x_L, x_0) = &-h(x_L, x_0) + \frac{t_1}{T_L}\partial_0^2\omega_1(x_1^*, x_0) \\ &+ \left(\frac{t_1(t_2 + \cdots + t_L)}{T_L^2} + t_1^2 h(x_L, x_0)\right)\partial_0\partial_1\omega_1(x_1^*, x_0)\end{aligned}$$

remains bounded in the space $\mathscr{B}(\mathbf{R}_{x_L} \times \mathbf{R}_{x_0})$. Moreover by definition $|\partial_0^2\omega_1(x_1^*, x_0)| \leqq \kappa_2$ and $|\partial_0\partial_1\omega_1(x_1^*, x_0)| \leqq \kappa_2$. And (2.28) and (2.9) imply that $t_1^2|h(x_L, x_0)| < 1$. Therefore

$$|\psi(x_L, x_0)| \leqq 10\kappa_2 .$$

Similar discussions hold for other derivatives of $S(\overline{x_L, x_0})$. Therefore, we have proved Proposition 2.5.

Finally we discuss Hessian determinant of S.

PROPOSITION 2.6. *Let $\phi(x, y)$ be a real valued C^∞-function of $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Let $y^\#$: $\mathbb{R}^m \ni x \to y^\#(x) \in \mathbb{R}^n$ be a C^∞-map such that*

$$\partial_y \phi(x, y^\#(x)) = 0 \qquad \text{for any } x \in \mathbb{R}^m . \tag{2.29}$$

We assume that

$$\det \operatorname{Hess}_y \phi(x, y)|_{y=y^\#(x)} \neq 0 \tag{2.30}$$

and that the map $\phi^\#$: $\mathbb{R}^m \ni x \to \phi(x, y^\#(x)) \in \mathbb{R}$ has a critical point x^*, i.e.,

$$\partial_x \phi^\#(x^*) = 0 . \tag{2.31}$$

Then $(x^*, y^*) = (x^*, y^\#(x^*))$ is a critical point of $\phi(x, y)$. Moreover we have the product formula of Hessian determinant:

$$\det \operatorname{Hess}_{(x^*, y^*)} \phi = (\det \operatorname{Hess}_{x^*} \phi^\#)(\det \operatorname{Hess}_y \phi(x, y))|_{(x,y)=(x^*,y^*)} . \tag{2.32}$$

Proof. We have, from (2.29),

$$\partial_y \phi(x^*, y^*) = 0 . \tag{2.33}$$

On the other hand we have, from (2.31), that

$$\partial_x \phi(x^*, y^*) + \partial_y \phi(x^*, y^*) \partial_x y^\#(x^*) = \partial_x \phi^\#(x^*) = 0 .$$

This and (2.33) gives that $\partial_x \phi(x^*, y^*) = 0$. Therefore, (x^*, y^*) is a critical point of ϕ.

We have

$$\operatorname{Hess}_{(x^*, y^*)} \phi = \begin{pmatrix} \partial_x^2 \phi, & \partial_y \partial_x \phi \\ \partial_x \partial_y \phi, & \partial_y^2 \phi \end{pmatrix} \Bigg|_{(x,y)=(x^*,y^*)} .$$

On the other hand we have

$$\begin{aligned} \operatorname{Hess}_x \phi^\# &= \partial_x^2 \phi(x^*, y^*) + \partial_y \partial_x \phi(x^*, y^*) \partial_x y^\#(x^*) \\ &= \partial_x^2 \phi(x^*, y^*) - \partial_y \partial_x \phi(x^*, y^*) \partial_y^2 \phi(x^*, y^*)^{-1} \partial_x \partial_y \phi(x^*, y^*) , \end{aligned}$$

because

$$\partial_y^2 \phi(x, y^\#(x)) \partial_x y^\#(x) + \partial_x \partial_y \phi(x, y^\#(x)) = 0 .$$

Therefore the next Lemma proves (2.32). Proposition 2.6 is proved.

LEMMA 2.7. *Let A be a $(m+n)\times(m+n)$ matrix. We write*

$$A=\begin{pmatrix} B, & C \\ {}^tC, & D\end{pmatrix},$$

where B is an $m\times m$ matrix, C is an $m\times n$ matrix and D is an $n\times n$ regular matrix. Then we have

$$\det(A)=\det(D)\det(B-CD^{-1}\,{}^tC)\,.$$

Proof. Take the determinant of the following matrix identity:

$$\begin{pmatrix} B, & C \\ {}^tC, & D\end{pmatrix}\begin{pmatrix} I, & 0 \\ -D^{-1}\,{}^tC, & I\end{pmatrix}=\begin{pmatrix} B-CD^{-1}\,{}^tC, & C \\ 0, & D\end{pmatrix}.$$

Let x_1^* be the critical point with respect to x_1 of $S_2(x_2, x_1)+S_1(x_1, x_0)$. We define a function $D(S_2+S_1; x_2, x_1)$ from the hessian determinant at x_1^* in the following way:

$$\text{(2.34)} \qquad \det \mathrm{Hess}_{x^*}(S_2+S_1)=\frac{t_1+t_2}{t_1t_2}D(S_2+S_1; x_2, x_0)\,.$$

Let $k<m$ be two positive integers. Then we define $(x_{m-1}^*, \cdots, x_{k+1}^*)$ as the partial critical point, i.e.,

$$\partial_j S_{j+1}(x_{j+1}^*, x_j^*)+\partial_j S(x_j^*, x_{j-1}^*)=0, \qquad j=k+1, \cdots, m-1\,.$$

Here $x_m^*=x_m$ and $x_k^*=x_k$. We denote the critical level by $S^\#_{m,k+1}(x_m, x_k)$, i.e.,

$$\text{(2.35)} \qquad S^\#_{m,k+1}(x_m, x_k)=S_m(t_m, x_m, x_{m-1}^*)+\cdots+S_{k+1}(t_{k+1}, x_{k+1}^*, x_k)\,.$$

As a consequence of Proposition 2.5 we can write

$$\text{(2.36)} \quad S^\#_{m,k+1}(x_m, x_k)=\frac{(x_m-x_k)^2}{2(t_{k+1}+\cdots+t_m)}+(t_{k+1}+\cdots+t_m)\omega^\#_{m,k+1}(x_m, x_k)\,.$$

We define $D(x_m, x_k)$ by

$$\text{(2.37)} \quad \det(\mathrm{Hess}_{(x_{m-1}^*,\cdots,x_{k+1}^*)}(S_m+\cdots+S_{k+1}))=\frac{t_{k+1}+\cdots+t_m}{t_{k+1}\cdots t_m}D(x_m, x_k)\,.$$

If $m=1$ and $k=1$ then we set $S^\#_{1,1}(x_1, x_0)=S_1(t_1, x_1, x_0)$.

PROPOSITION 2.8. *We have*

$$\text{(2.38)} \qquad D(x_L, x_0)=\left(\prod_{k=2}^{L} D(S_k+S^\#_{k-1,1}; x_k, x_0)\right)\Bigg|_{(x_{L-1},\cdots,x_1)=(x_{L-1}^*,\cdots,x_1^*)}\,.$$

Proof. We prove (2.38) by induction on L. The case $L=2$ is clear.

By induction hypothesis the Hessian determinant of $S_{L-1}+\cdots+S_1$ at the critical point $x^\# = (x^\#_{L-2}(x_{L-1}, x_0), \cdots, x^\#_1(x_{L-1}, x_0))$ with respect to $x_{L-2}, \cdots, x_1$ equals

$$\frac{t_1+\cdots+t_{L-1}}{t_1\cdots t_{L-1}}\left(\prod_{k=2}^{L-1} D(S_k + S^\#_{k-1,1};\, x_k, x_0)\right)\Bigg|_{x_j = x^\#_j(x_{L-1}, x_0),\, j=1,\cdots,L-2}.$$

So Proposition 2.6 gives that

$$\begin{aligned}
&\frac{t_1+\cdots+t_L}{t_1\cdots t_L}D(x_L, x_0)\\
&= \det \mathrm{Hess}_{x^*_{L-1}}(S_L + S^\#_{L-1,1}) \det \mathrm{Hess}_{(x^\#_{L-2},\cdots,x^\#_1)}(S_{L-1}+\cdots+S_1)|_{x_{L-1}=x^*_{L-1}}\\
&= \frac{t_L+\cdots+t_1}{t_L(t_{L-1}+\cdots+t_1)}D(S_L + S^\#_{L-1,1};\, x_{L-1}, x_0)\\
&\qquad\qquad \times \frac{t_1+\cdots+t_{L-1}}{t_1\cdots t_{L-1}}\left(\prod_{k=2}^{L-1} D(S_k + S^\#_{k-1,1};\, x_k, x_0)\right)\Bigg|_{x_j=x^*_j}.
\end{aligned}$$

We have proved (2.38) for L and Proposition is proved.

PROPOSITION 2.9. *We have*

$$D(S_2+S_1;\, x_2, x_0) = 1 + t_1 t_2 g(x_2, x_0).$$

Here $g(x_2, x_0)$ remains bounded in $\mathscr{B}(\mathbb{R}_{x_2}\times\mathbb{R}_{x_0})$.

Proof.

$$\mathrm{Hess}_{x_1}(S_2+S_1) = \frac{t_1+t_2}{t_1t_2}\left(1 + \frac{t_1t_2}{t_1+t_2}(t_1\partial_1^2\omega_1 + t_2\partial_1^2\omega_2)\right).$$

This proves Proposition.

PROPOSITION 2.10. *Assume that* (2.9) *holds. We write*

$$\det \mathrm{Hess}_{x^*} S(x_L, x_0) = \frac{T_L}{t_1t_2\cdots t_L}D(x_L, x_0).$$

Then

$$D(x_L, x_0) = 1 + T_L^2 q(x_L, x_0), \tag{2.39}$$

where $q(x_L, x_0)$ may depend also on $t_1, \cdots, t_L$ but remains bounded in the space $\mathscr{B}(\mathbb{R}\times\mathbb{R})$ uniformly with respect to $t_1, \cdots, t_L$.

Proof. We have from Proposition 2.8 that

$$D(x_L, x_0) = \left(\prod_{k=2}^{L} D(S_k + S^{\#}_{k+1,1};\, x_k, x_0)\right)\Big|_{(x_{L-1},\cdots,x_1)=(x^*_{L-1},\cdots,x^*_1)}.$$

We can apply the previous proposition. Then we have

$$D(x_L, x_0) = \prod_{k=2}^{L} (1 + t_k T_{k-1} g_k(x_L, x_0)),$$

where $g_k(x_L, x_0)$ are bounded in the space $\mathscr{B}(\mathbb{R}\times\mathbb{R})$. Proposition is proved.

§ 3. Key Lemma

The aim of the section is to prove the following Lemma 3.1, which plays an important role in the proof of the main results. In the present section assumption (H.2) is not needed. Instead we require the following assumption about the amplitude function:

(H.3) For any $K \geqq 0$ there exists a positive constant A_K such that

$$\max_{\alpha} \sup_{x_{L-1},\cdots,x_1} |\partial_{x_L}^{\alpha_L} \partial_{x_{L-1}}^{\alpha_{L-1}} \cdots \partial_{x_0}^{\alpha_0} a(x_L, \cdots, x_0)| < A_K,$$

where max is taken with respect to multi-indices $(\alpha_L, \cdots, \alpha_0)$ satisfying $|\alpha_j| \leqq K$, $j = 0, 1, \cdots, L$.

LEMMA 3.1. *We assume the hypothesis* (H.1) *for the phase function and hypothesis* (H.3) *above for the amplitude function. Then there exists a positive constant* $\delta > 0$ *such that* $I(\{t_j\}, S, a, \nu)(x_L, x_0)$ *can be written as*

$$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} \exp\{-i\nu S(\overline{x_L, x_0})\}\, b(x_L, x_0),$$

as far as $T_L = t_1 + t_2 + \cdots + t_L < \delta$. *For any* $m \geqq 0$ *there exist constants* C_m *and* $K(m)$ *such that if* $|\alpha_0| \leqq m$, $|\alpha_L| \leqq m$,

$$|\partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} b(x_L, x_0)| \leqq C_m^L \max_{\beta} \sup_{x_{L-1},\cdots,x_1} |\partial_{x_L}^{\beta_L} \partial_{x_{L-1}}^{\beta_{L-1}} \cdots \partial_{x_0}^{\beta_0} a(x_L, \cdots, x_0)|,$$

where max *is taken with respect to all* $(\beta_L, \cdots, \beta_0)$ *satisfying* $\beta_0 \leqq \alpha_0$, $\beta_L \leqq \alpha_L$ *and* $|\beta_j| \leqq K(m)$. *Here constants* $K(m)$ *and* C_m *do not depend on* L, ν *and* a. *We can choose* $K(m) = 4m + 17 + 6$.

Proof of basic Lemma 3.1 will be given after Lemma 3.6. Before that, we collect preparatory facts. Most of them are well known but we will write them down for the convenience of the reader.

From now on we let $E = (i\nu/2\pi)$ for the sake of brevity of notation.

The following Lemma is found in Kumanogo [6],

LEMMA 3.2 (Kumanogo [6]). *Let*

$$J(\mu, a)\psi(x_L) = \left(\frac{\mu}{2\pi}\right)^L \tilde{\int}_{\mathbf{R}^{2L}} \exp\left\{-i\mu\sum_{j=1}^{L}\xi_j(x_{j+1}-x_j)\right\} \times a(x_{L+1}, \xi_L, \cdots, \xi_1, x_1)\psi(x_1)\prod_{j=1}^{L} d\xi_j\, dx_j\,.$$

Then there exists a function $U(a)(x_{L+1}, \xi_1, x_1)$ *such that*

$$J(\mu, a)\psi(x_{L+1}) = \left(\frac{\mu}{2\pi}\right)\tilde{\int}_{\mathbf{R}^2} \exp\{-i\mu(x_{L+1}-x_1)\xi_1\}U(a)(x_{L+1}, \xi_1, x_1)\psi(x_1)dx_1\, d\xi_1\,.$$

We have

$$\partial_{x_{L+1}}U(a)(x_{L+1}, \xi_1, x_1) = U\left(\sum_{j=1}^{L+1}\partial_{x_j}a\right)(x_{L+1}, \xi_1, x_1)\,,$$

$$\partial_{\xi_1}U(a)(x_{L+1}, \xi_1, x_1) = U\left(\sum_{j=1}^{L+1}\partial_{\xi_j}a\right)(x_{L+1}, \xi_1, x_1)\,,$$

and

$$\partial_{x_1}U(a)(x_{L+1}, \xi_1, x_1) = U(\partial_{x_1}a)(x_{L+1}, \xi_1, x_1)\,.$$

Moreover, there exists a constant C_0 *independent of* μ, L *and of* a *such that*

$$\sup|U(a)(x_{L+1}, \xi_1, x_1)| \leqq C_0^L\|a\|_3$$

where

$$\|a\|_k = \max_{|\alpha_j|, |\beta_j| \leqq k} \sup\left|\left(\prod_{j=1}^{L}\partial_{x_j}^{\alpha_j}\partial_{\xi_j}^{\beta_j}\right)a(x_{L+1}, \xi_L, x_L, \cdots, \xi_1\, x_1)\right|\,.$$

Proof is found in Kumanogo [6]. A simple corollary is

COROLLARY 3.3. *For any* $m \geqq 0$ *there are constants* C_m *and* $K_1(m)$ *such that if* $|\alpha_{L+1}|$, $|\beta_1|$, $|\alpha_1| \leqq m$,

$$\sup|\partial_{x_{L+1}}^{\alpha_{L+1}}\partial_{\xi_1}^{\beta_1}\partial_{x_1}^{\alpha_1}U(a)(x_{L+1}, \xi_1, x_1)| \leqq C_m(L+1)^{2m}C_1^{L+1}\|a\|_{K_1(m)}\,.$$

We can choose $K_1(m) = 2m + 3$.

Let $S(t, x, y) = (1/2t)|x-y|^2 + t\omega(t, x, y)$ and let $a(x, y)$ be in the space $\mathscr{B}(\mathbb{R}_x \times \mathbb{R}_y)$, then for any $f(y) \in C_0^\infty(\mathbb{R})$ we set

$$\mathrm{Op}(t, S, a, \nu)f(x) = \left(\frac{E}{t}\right)^{1/2}\tilde{\int}_{\mathbf{R}} e^{-i\nu S(t,x,y)}\, a(x, y)f(y)dy\,.$$

We assume as in § 1 that

$$\max_{2\leqq|\alpha|+|\beta|\leqq m}\ \sup_{x,y\in\mathbb{R}} |\partial_x^\alpha\partial_y^\beta\omega(x, y)| \leqq \kappa_m .$$

If $8|t|^2\kappa_2 < 1$ then $\mathrm{Op}(t, S, a, \nu)$ defines a bounded linear operator on $L^2(\mathbb{R})$ (cf. [1]). Its adjoint $\mathrm{Op}(t, S, a, \nu)^*$ is of the form

$$\mathrm{Op}(t, S, a, \nu)^*f(x) = \left(\frac{E}{t}\right)^{1/2} \tilde{\int}_{\mathbb{R}} e^{i\nu S(t,z,x)}\overline{a(z, x)}f(z)dz .$$

LEMMA 3.4. *Assume that $S(t, x, y)$ is as above. Then there exists a positive constant $\delta_1 = \delta_1(\{\kappa_m\})$ depending only on dimensionality of the space and $\{\kappa_m\}_m$ such that if $|t|\leqq\delta_1$ then $\mathrm{Op}(t, S, 1, \nu)^{-1}$ exists and is of the form*

$$\mathrm{Op}(t, S, 1, \nu)^{-1} = \mathrm{Op}(t, S, 1 + tp, \nu)^* ,$$

where $p(t, x, y)$ satisfies the estimate: For any multi-indices α and β there exists a positive constant $C_{\alpha\beta}$ independent of t, ν such that

$$|\partial_x^\alpha\partial_y^\beta p(t, x, y)| \leqq C_{\alpha\beta} .$$

Proof is given in [3].

Let $S_i(t_i, x, y) = (1/(2t_i))|x-y|^2 + t_i\,\omega_i(t_i, x, y)$, $i = 1, 2$, be phase functions and $a(x, y, z)$ be an amplitude function as in § 1. Then we consider

$$I(\{t_j\}, S_2 + S_1, a, \nu)(x, y) = \left(\frac{E}{t_1}\right)^{1/2}\left(\frac{E}{t_2}\right)^{1/2} \tilde{\int}_{\mathbb{R}} e^{-i\nu(S_1(t_1,x,z)+S_2(t_2,z,y))}a(x, z, y)dz ,$$

here $a(x, z, y) \in \mathscr{B}(\mathbb{R}_x\times\mathbb{R}_z\times\mathbb{R}_y)$. We employ the notations $D(S_2 + S_1; x, y)$ of (2.34) in § 2 and denote $(t_1t_2)/(t_1 + t_2)$ by τ. Applying the stationary phase method [1], we easily obtain

LEMMA 3.5. *Assume that $8(t_1 + t_2)^2\kappa_2^2 < 1$. Then*

$$\begin{aligned}&\left(\frac{E}{t_1}\right)^{1/2}\left(\frac{E}{t_2}\right)^{1/2} \tilde{\int}_{\mathbb{R}} e^{-i\nu(S_1(t_1,x,z)+S_2(t_2,z,y))}a(x, z, y)dz\\ &\quad = \left(\frac{E}{t_1+t_2}\right)^{1/2} e^{-i\nu S^{\#}_{2,1}}D(S_2 + S_1; x, y)^{-1/2}b(x, y) ,\end{aligned}$$

where $b(x, y)$ is of the following form:

$$\begin{aligned}b(x, y) = \Big(&a(x, z^*, y) + \left(\frac{\tau}{i\nu}\right)D(S_2 + S_1; x, y)^{-1}\Big\{\frac{1}{2}\Delta_z a(x, z, y)|_{z=z^*}\\ &+ \tau D(S_2 + S_1; x, y)^{-1}r_1(x, y)\Big\} + \left(\frac{\tau}{i\nu}\right)^2 D(S_2 + S_1; x, y)^{-2}r_2(x, y)\Big) ,\end{aligned}$$

where Δ_k is the laplacian with respect to z. For each $m \geqq 0$ there exist $K(m)$ and C_m such that for any α and β with $|\alpha|, |\beta| \leqq m$

$$|\partial_x^\alpha \partial_y^\beta r_1(x, y)| + |\partial_x^\alpha \partial_y^\beta r_2(x, y)| \leqq C_m \max \sup |\partial_x^{\alpha'} \partial_y^{\beta'} \partial_z^{\gamma'} a(x, z, y)|,$$

where max *is taken for such α', β' and γ' as $\alpha' \leqq \alpha$, $\beta' \leqq \beta$, $\gamma' \leqq K(m)$. $K(m)$ can be chosen as $2m + 4 + 2$.*

Proof. We have only to apply stationary phase method (cf. Theorem 4.1 of [1]).

Let $S_j(t_j, x_j, x_{j-1})$, $j = 1, 2, \cdots, L$, be the phase functions as in Lemma 3.1. We employ the notation $S^\#_{m,k+1}(x_m, x_k)$ of (2.35) in § 2 if $m > k$.

LEMMA 3.6. *There exists a positive constant $\delta_2 = \delta_2(\{\kappa_m\})$ such that if $T_k = t_k + t_{k+1} + \cdots + t_1 < \delta_2$, then*

$$\left(\frac{E}{t_k}\right)^{1/2} e^{-i\nu S_k(t_k, x_k, x_{k-1})} = \left(\frac{E}{T_k}\right)^{1/2}\left(\frac{-E}{T_{k-1}}\right)^{1/2} \times \tilde{\int}_{\mathbb{R}} e^{-i\nu(S^\#_{k,1}(x_k, y_{k-1}) - S^\#_{k-1,1}(x_{k-1}, y_{k-1}))} b_k(x_k, y_{k-1}, x_{k-1}) dy_{k-1}.$$

Here the function $b_k(x_k, y_{k-1}, x_{k-1})$ satisfies the following estimate: For any α, β and γ, there is a constant $C_{\alpha\beta\gamma}$ such that

$$|\partial_{x_k}^\alpha \partial_{y_{k-1}}^\beta \partial_{x_{k-1}}^\gamma b(x_k, y_{k-1}, x_{k-1})| \leqq C_{\alpha\beta\gamma}.$$

Proof. Let δ_2 be so small that $8\delta_2^2\kappa_2 < 1$, $8\delta_2^2\kappa_2^\# < 1$, $\delta_2 < \delta_1(\{\kappa_m\})$ and $\delta_2 < \delta_1(\{\kappa_2^\#\})$. Assume $T_k < \delta_2$. Then $8(T_{k-1} + t_k)^2\kappa_2 < 1$ and $8(T_{k-1} + t_k)^2\kappa_2^\# < 1$. So we apply Lemma 3.5 to the kernel function of the operator

$$\mathrm{Op}(t_k, S_k, 1, \nu)\mathrm{Op}(T_{k-1}, S^\#_{k-1,1}, 1, \nu).$$

We obtain

$$\mathrm{Op}(t_k, S_k, 1, \nu)\mathrm{Op}(T_{k-1}, S^\#_{k-1,1}, 1, \nu) = \mathrm{Op}(T_k, S^\#_{k,1}, 1 + \tau_k p_k, \nu),$$

where $\tau_k = t_k T_{k-1}(t_k + T_{k-1})^{-1}$ and $p_k = p_k(x_k, x_0) \in \mathscr{B}(\mathbb{R} \times \mathbb{R})$. Since $T_k < \delta_1(\{\kappa_m^\#\})$, we can apply Lemma 3.4 to $\mathrm{Op}(T_{k-1}, S^\#_{k-1,1}, 1, \nu)$. Therefore, we have

$$\begin{aligned}\mathrm{Op}(t_k, S_k, 1, \nu) &= \mathrm{Op}(T_k, S^\#_{k,1}, 1 + \tau_k p_k, \nu)\mathrm{Op}(T_{k-1}, S^\#_{k-1,1}, 1, \nu)^{-1} \\ &= \mathrm{Op}(T_k, S^\#_{k,1}, 1 + \tau_k p_k, \nu)\mathrm{Op}(T_{k-1}, S^\#_{k-1,1}, 1 + T_{k-1}q_k, \nu)^*.\end{aligned}$$

This means that

$$\left(\frac{E}{t_k}\right)^{1/2} e^{-i\nu S_k(t_k, x_k, x_{k-1})} = \left(\frac{E}{T_k}\right)^{1/2}\left(\frac{-E}{T_{k-1}}\right)^{1/2}$$
$$\times \tilde{\int}_{\mathbb{R}} e^{-i\nu(S^{\#}_{k,1}(x_k, y_{k-1}) - S^{\#}_{k-1,1}(x_{k-1}, y_{k-1}))} b_k(x_k, y_{k-1}, x_{k-1}) dy_{k-1}$$

with

$$b_k(x_k, y_{k-1}, x_{k-1}) = (1 + \tau_k p_k(x_k, y_{k-1}))(1 + T_{k-1} q_k(x_{k-1}, y_{k-1})) .$$

Lemma 3.6 is proved.

We can now prove Lemma 3.1. The proof is a modification of the discussion in [6], [7], [8] and [4]. Let δ be so small that $\delta < \delta_1(\{\kappa_m\})$ and $\delta < \delta_2(\{\kappa_m\})$ and $8\delta^2\kappa_2 < 1$. Assume that $T_L < \delta$. Using the function $I(\{t_j\}, S, a, \nu)(x_L, x_0)$, we define integral transform

$$\mathrm{Op}(\{t_j\}, S, a, \nu) f(x_L) = \tilde{\int}_{\mathbb{R}} I(\{t_j\}, S, a, \nu)(x_L, x_0) f(x_0) dx_0 .$$

Since $8T_L^2\kappa_2 < 1$, $\mathrm{Op}(\{t_j\}, S, a, \nu)$ is a bounded operator on $L^2(\mathbb{R})$. (cf. [1]).

Since $T_L < \delta_2(\{\kappa_m\})$, we can apply Lemma 3.6 to $e^{-i\nu S_k(t, x_k, x_{k-1})}$ for any $k = 2, 3, \cdots, L$. Thus

$$\mathrm{Op}(\{t_j\}, S, a, \nu) f(x_L) = \prod_{k=1}^{L}\left(\frac{E}{t_k}\right)^{1/2} \tilde{\int}_{\mathbb{R}^L} e^{-i\nu(\sum S_j(t_j, x_j, x_{j-1}))} a(x_L, \cdots, x_0) f(x_0) \prod_{j=0}^{L-1} dx_j$$
$$= \left(\frac{E}{t_1}\right)^{1/2} \prod_{j=2}^{L}\left(\frac{E}{T_j}\right)^{1/2}\left(\frac{-E}{T_{j-1}}\right)^{1/2} \tilde{\int}_{\mathbb{R}^{2(L-1)+1}} e^{-i\nu\Phi(x_L, y_{L-1}, x_{L-1}, \cdots, y_1, x_1, x_0)}$$
$$\times a(x_L, \cdots, x_0) \prod_{j=2}^{L} b_j(x_j, y_{j-1}, x_{j-1}) f(x_0) dx_0 \prod_{k=1}^{L-1} dx_j dy_j ,$$

where the phase function Φ equals

$$\Phi(x_L, y_{L-1}, x_{L-1}, \cdots, y_1, x_1, x_0)$$
$$= \sum_{j=2}^{L} \{S^{\#}_{j,1}(x_j, y_{j-1}) - S^{\#}_{j-1,1}(x_{j-1}, y_{j-1})\} + S_1(t_1, x_1, x_0)$$
$$= S^{\#}_{L,1}(x_L, y_{L-1}) + \sum_{j=2}^{L-1} \{S^{\#}_{j,1}(x_j, y_{j-1}) - S^{\#}_{j,1}(x_j, y_j)\}$$
$$+ S_1(t_1, x_1, x_0) - S_1(t_1, x_1, y_1) .$$

We next employ Kuranishi's technique. We rewrite

$$S^{\#}_{j,1}(x_j, y_{j-1}) - S^{\#}_{j,1}(x_j, y_j) = \frac{y_{j-1} - y_j}{T_j} \xi_j ,$$

where

$$\xi_j = T_j \int_0^1 \partial_y S^{\#}_{j,1}(x_j, sy_{j-1} + (1-s)y_j)ds$$
$$= -x_j + \frac{1}{2}(y_{j-1} + y_j) + T_j \int_0^1 \partial_y \omega^{\#}_{j,1}(x_j, sy_{j-1} + (1-s)y_j)ds .$$

Therefore, the jacobian of the correspondence $x_j \to \xi_j$ is

$$\frac{\partial \xi_j}{\partial x_j} = -I + T_j p(y_j, x_j, y_{j-1}) ,$$

where

$$p_j(y_j, x_j, y_{j-1}) = \int_0^1 \partial_{x_j} \partial_y \omega^{\#}_{j,1}(x_j, sy_{j-1} + (1-s)y_j)ds$$

satisfies the estimate: For any α, β and γ, with $|\alpha|$, $|\beta|$ and $|\gamma| < m$,

$$|\partial^{\alpha}_{y_j} \partial^{\beta}_{x_j} \partial^{\gamma}_{y_{j-1}} p(y_j, x_j, y_{j-1})| \leqq \kappa^{\#}_{m+2} .$$

Since $8T_L^2 \kappa_2^{\#} < 1$, we have $|\partial \xi_j / \partial x_j| < 2^{-1}$. The correspondence $\mathbb{R} \ni x_j \to \xi_j \in \mathbb{R}$ is one to one and onto. We may consider x_j as a function $x_j(y_j, \xi_j, y_{j-1})$.

This diffeomorphism has the following property: Let $f(y_j, x_j, y_{j-1})$ be an arbitrary function in $\mathscr{B}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Then for any multi-indices α, β and γ there exists a positive constant $C_{\alpha\beta\gamma}$ such that

$$|\partial^{\alpha}_{y_j} \partial^{\beta}_{\xi_j} \partial^{\gamma}_{y_{j-1}} f(y_j, x_j(y_j, \xi_j, y_{j-1}), y_{j-1})| < C_{\alpha\beta\gamma} \max \sup |\partial^{\alpha'}_{y_j} \partial^{\beta'}_{x_j} \partial^{\gamma'}_{y_{j-1}} f(y_j, x_j, y_{j-1})| ,$$

where maximum is taken with respect to those multi-indices $\alpha' \leqq \alpha$, $\beta' \leqq \beta$ and $\gamma' \leqq \gamma$.

Let $\eta_j = T_j^{-1} \xi_j(y_j, x_j, y_{j-1})$. Then for any α, β and γ there exists a constant $C_{\alpha\beta\gamma}$ independent of $\{t_j\}_j$ such that

$$|\partial^{\alpha}_{y_j} \partial^{\beta}_{\eta_j} \partial^{\gamma}_{y_{j-1}} f(y_j, x_j(y_j, T_j \eta_j, y_{j-1}), y_{j-1})|$$
$$\leqq C_{\alpha\beta\gamma} T_j^{|\beta|} \max \sup_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} |\partial^{\alpha'}_{y_j} \partial^{\beta'}_{x_j} \partial^{\gamma'}_{y_{j-1}} f(y_j, x_j, y_{j-1})| ,$$

where max is taken with respect to multi-indices with $\alpha' \leqq \alpha$, $\beta' \leqq \beta$ and $\gamma' \leqq \gamma$.

Similarly, we make change of variables from x_1 to η_1.

$$S_1(t_1, x_1, x_0) - S_1(t_1, x_1, y_1) = \eta_1(x_0 - y_1) ,$$

here

$$\eta_1 = \frac{-1}{t_1}\left(\left(x_1 - \frac{1}{2}(x_0 + y_1)\right) + \int_0^1 \partial_{x_0}\omega_1(t_1, x_1, sx_0 + (1-s)y_1)ds\right).$$

After these change of variables the phase function becomes

$$\Phi = S^\#_{L,1}(x_L, y_{L-1}) + \sum_{j=2}^{L-1} \eta_j(y_{j-1} - y_j) + \eta_1(x_0 - y_1),$$

where $y_L = x_L$ and $y_0 = x_0$. Therefore we have

$$\text{(3.1)}\quad \mathrm{Op}(\{t_j\}, S, a, \nu)f(y_L) = \left(\frac{E}{T_L}\right)^{1/2} \tilde{\int}_{\mathbf{R}} e^{-i\nu S^\#_{L,1}(y_L, y_{L-1})} J(f)(y_L, y_{L-1})dy_{L-1},$$

where

$$J(f)(y_L, y_{L-1}) = \left(\frac{\nu}{2\pi}\right)^{L-1} \int_{\mathbf{R}^{2(L-1)}} \exp\left\{-i\nu\left(\sum_{j=1}^{L-1}(y_j - y_{j-1})\eta_j\right)\right\}$$
$$\times a_1(y_L, y_{L-1}, \eta_{L-1} \cdots, y_1, \eta_1, y_0)f(y_0) \prod_{j=1}^{L-1} dy_{j-1}d\eta_j$$

with

$$\text{(3.2)}\quad \begin{aligned} &a_1(y_L, y_{L-1}, \eta_{L-1}, \cdots, y_1, \eta_1, y_0) \\ &= a(x_L, \cdots, x_0) \prod_{k=2}^{L} b_k(x_k, y_k, x_{k-1}) \prod_{j=1}^{L-1} \left|\frac{\partial x_j}{\partial \xi_j}\right|. \end{aligned}$$

We consider y_L as a parameter and apply Lemma 3.2 to $J(f)(y_L, y_{L-1})$. Then we obtain

$$\text{(3.3)}\quad J(f)(y_L, y_{L-1}) = \left(\frac{\nu}{2\pi}\right) \tilde{\int}_{\mathbf{R}^2} e^{-i\nu(y_{L-1}-y_0)\eta} U(a_1)(y_L, y_{L-1}, \eta, y_0)f(y_0)dy_0 d\eta.$$

For any multi-indices α_L, α_{L-1}, α_0 and β with $|\alpha_L|$, $|\alpha_{L-1}|$, $|\alpha_0|$, $|\beta| \leqq m_1$, we have the following estimate:

$$\begin{aligned} &|\partial_{y_L}^{\alpha_L}\partial_{y_{L-1}}^{\alpha_{L-1}}\partial_\eta^\beta\partial_{y_0}^{\alpha_0} U(a_1)(y_L, y_{L-1}, \eta, y_0)| \\ &\leqq C_1 C_2^{L-1} \max\sup\left|\partial_{y_L}^{\alpha_L}\partial_{y_{L-1}}^{\alpha_{L-1}} \prod_{k=2}^{L-1}(\partial_{y_{k-1}}^{\alpha'_{k-1}}\partial_{\eta_k}^{\beta'_k})\partial_{\eta_1}^{\beta'_1}\partial_{y_0}^{\alpha'_0} a_1(y_L, y_{L-1}, \eta_{L-1}, \cdots, y_0)\right|. \end{aligned}$$

Here C_1 and C_2 are positive constants depending on m_1, max is taken with respect to multi-indices with $|\alpha_{L-1}|$, $|\alpha'_k|$, $|\beta'_k| \leqq K(m_1) \leqq 2m_1 + 3$ and sup is taken with respect to $y_j \in \mathbb{R}$, $\eta_j \in \mathbb{R}$, $j = 1, \cdots, L-1$. Since relationship $a_1(x_L, \cdots, x_0)$ with $a(x_L, \cdots, x_0)$ is given by (3.2), we have

(3.4) $$|\partial_{y_L}^{\alpha_L}\partial_{y_{L-1}}^{\alpha_{L-1}}\partial_\eta^\beta\partial_{y_0}^{\alpha_0}U(a_1)(y_L, y_{L-1}, \eta, y_0)| \leqq C_1C_3^{L-1}\max\sup_{x_{L-1}\cdots x_1}|\partial_{x_L}^{\alpha_L''}\partial_{x_{L-1}}^{\alpha_{L-1}''}\cdots\partial_{x_0}^{\alpha_0''}a(x_L, x_{L-1}, \cdots, x_0)|,$$

where C_3 is another constant and max is taken with respect to multi-indices satisfying $|\alpha_L''|$, $|\alpha_{L-1}''|$, $\cdots$, $|\alpha_0''| \leqq 2m_1 + 3$.

We replace $J(f)(y_L, y_{L-1})$ in (3.1) by the right hand side of (3.3). Then we have the expression

(3.5) $$\mathrm{Op}(\{t_j\}, S, a, \nu)f(x_L) = \left(\frac{E}{T_L}\right)^{1/2}\left(\frac{\nu}{2\pi}\right) \times \tilde{\int}_{\mathbf{R}^3} e^{-i\nu(S_{L,1}^\#(x_L, y_{L-1}) + (y_{L-1}-y_0)\eta)}U(a_1)(x_L, y_{L-1}, \eta, y_0)f(y_0)dy_0d\eta dy_{L-1}\,.$$

Using stationary phase method with respect to y_{L-1} and η, we can find $b(x_L, x_0)$ such that

$$\mathrm{Op}(\{t_j\}), S, a, \nu)f(x_L) = \left(\frac{E}{T_L}\right)^{1/2}\tilde{\int}_{\mathbf{R}} e^{-i\nu S_{L,1}^\#(x_L, x_0)}b(x_L, x_0)f(x_0)dx_0\,.$$

This means that

(3.6) $$I(\{t_j\}, S, a, \nu)(x_L, x_0) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S_{L,1}^\#(x_L, x_0)}b(x_L, x_0)\,.$$

Here b satisfies the following estimate: For any $m \geqq 0$ there exists a positive constant $C(m)$ such that if $|\alpha_L|$, $|\alpha_0| \leqq m$

(3.7) $$|\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0}b(x_L, x_0)| \leqq C(m)\max\sup_{y_{L-1},\eta}|\partial_{x_L}^{\alpha_L'}\partial_{y_{L-1}}^{\alpha_{L-1}'}\partial_\eta^{\beta'}\partial_{y_0}^{\alpha_0}U(a_1)(x_L, y_{L-1}, \eta, y_0)|,$$

where max is taken with respect to multi-indices satisfying $\alpha_L' \leqq \alpha_L$, $\alpha_0' \leqq \alpha_0$ and $|\alpha_{L-1}'|$, $|\beta'| < 2m + 10$. Combining (3.7) with (3.4), we have

(3.8) $$|\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0}b(x_L, x_0)| \leqq C_4C_5^{L-1}\max\sup_{x_{L-1},\cdots,x_1}|\partial_{x_L}^{\alpha_L''}\partial_{x_{L-1}}^{\alpha_{L-1}''}\cdots\partial_{x_0}^{\alpha_0''}a(x_L, x_{L-1}, \cdots, x_0)|,$$

where C_4, C_5 are positive constants depending on m and max is taken with respect to multi-indices satisfying $|\alpha_L''|$, $|\alpha_{L-1}''|$, $\cdots$, $|\alpha_0''| \leqq 2(2m + 10) + 3 = 4m + 23$. This together with (3.6) above proves Lemma 3.1 with $K(m) = 4m + 23$.

§ 4. Proof of Theorem 1

For any $k > j$, we denote $t_k + \cdots + t_j$ by $T(k, j)$. Let δ be as in Lemma 3.1. We have to treat the oscillatory integral

$$(4.1)\quad I(\{t_j\}, S, a, \nu)(x_L, x_0)$$
$$= \prod_{j=1}^{L}\left(\frac{E}{t_j}\right)^{1/2} \tilde{\int}_{\mathbf{R}^{(L-1)}} \exp\left\{-i\nu \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1})\right\} a(x_L, \cdots, x_0) \prod_{j=1}^{L-1} dx_j\,,$$

when $T_L < \delta$.

First we perform integration over x_1 space. Using stationary phase method, we have

$$(4.2)\quad \left(\frac{E}{t_2}\right)^{1/2}\left(\frac{E}{t_1}\right)^{1/2} \tilde{\int}_{\mathbf{R}} e^{-i\nu(S_2(t_2, x_2, x_1) + S_1(t_1, x_1, x_0))} a(x_L, \cdots, x_0)\, dx_1$$
$$= \left(\frac{E}{T(2, 1)}\right)^{1/2} e^{-i\nu S^{\#}_{2,1}(x_2, x_0)}((S_1 a)(x_L, \cdots, x_2, x_0) + (R_1 a)(x_L, \cdots, x_2, x_0)).$$

The amplitude of the main term of the right hand side equals

$$(4.3)\qquad (S_1 a)(x_L, \cdots, x_2, x_0) = a(x_L, \cdots, \overline{x_2, x_0}) D(S_2 + S_1; x_2, x_0)^{-1/2}\,,$$

and $R_1 a(x_L, \cdots, x_2, x_0)$ is the remainder term.

Similarly, integrating $S_1 a$ over x_2 space and applying the stationary phase method, we obtain

$$\left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2, 1)}\right)^{1/2} \tilde{\int}_{\mathbf{R}} e^{-i\nu(S_3(t_3, x_3, x_2) + S^{\#}_{2,1}(x_2, x_0))} S_1 a(x_L, \cdots, x_2, x_0)\, dx_2$$
$$= \left(\frac{E}{T(3, 1)}\right)^{1/2} e^{-i\nu S^{\#}_{3,1}(x_3, x_0)}(S_2 S_1 a(x_L, \cdots, x_3, x_0) + R_2 S_1 a(x_L, \cdots, x_3, x_0))\,,$$

where $S_2 S_1 a$ is the main term and $R_2 S_1 a$ is the remainder term. We have

$$(4.4)\quad S_2 S_1 a(x_L, \cdots, x_0) = D(S_3 + S^{\#}_{2,1}; x_3, x_0)^{-1/2}(S_1 a)(x_L, \cdots, x_3, x_2^*, x_1)\,,$$

where x_2^* is the critical point of $S_3 + S^{\#}_{2,1}$ with respect to x_2.

When we integrate the term including $S_2 S_1 a(x_L, \cdots, x_3, x_0)$ over x_3 space, we use the stationary phase method:

$$\left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{T(3, 1)}\right)^{1/2} \tilde{\int}_{\mathbf{R}} e^{-i\nu(S_4(t_4, x_4, x_3) + S^{\#}_{3,1}(x_3, x_0))} S_2 S_1 a(x_L, \cdots, x_3, x_0)\, dx_3$$
$$= \left(\frac{E}{T(4, 1)}\right)^{1/2} e^{-i\nu S^{\#}_{4,1}(x_4, x_0)}(S_3 S_2 S_1 a(x_L, \cdots, x_4, x_0) + R_3 S_2 S_1 a(x_L, \cdots, x_4, x_0)).$$

$S_3 S_2 S_1 a$ is the main term and $R_3 S_2 S_1 a$ is the remainder.

Repeating this process $L - 1$ times, finally we obtain, among other terms,

$$\left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)} S_{L-1} S_{L-2} \cdots S_1(a)(x_L, x_0) .$$

Since we have

$$S_{L-1} S_{L-2} \cdots S_1 a(x_L, x_0) = \prod_{k=2}^{L} D(S_k + S^{\#}_{k-1,1}; x_k, x_0)^{-1/2} a(x)|_{x^*_{L-1}, \cdots x^*_1} ,$$

Proposition 2.8 yields that

$$S_{L-1} S_{L-2} \cdots S_1 a(x_L, x_0) = D(x_L, x_0)^{-1/2} a(\overline{x_L, x_0}) .$$

Here $D(x_L, x_0) = (t_1 t_2 \cdots t_L/(t_1 + \cdots + t_L))$ det Hess_{x^*} $(S_L + \cdots + S_1)$ as in § 2. Therefore,

$$\begin{aligned} (4.5) \qquad & \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)} S_{L-1} S_{L-2} \cdots S_1(a)(x_L, x_0) \\ & = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)} D(x_L, x_0)^{-1/2} a(\overline{x_L, x_0}) . \end{aligned}$$

This is nothing but the main term of Theorem 1. The remainder term consists of others.

Now we treat the remainder term. Since $(R_1 a)(x_L, \cdots, x_2, x_0)$ has complicated structure as a function of x_2, we postpone integration over x_2 space of the term including $(R_1 a)(x_L, \cdots, x_2, x_0)$ until later stage of the proof. We do perform integration over x_3 space beforehand, because the structure of $R_1 a(x_L, \cdots, x_3, x_2, x_0)$ as a function of x_3 is much simpler. The stationary phase method gives

$$\begin{aligned} & \left(\frac{E}{t_4}\right)^{1/2} \left(\frac{E}{t_3}\right)^{1/2} \left(\frac{E}{T(2, 1)}\right)^{1/2} \tilde{\int}_{\mathbf{R}} e^{-i\nu(S_4(t_4, x_4, x_3) + S_3(t_3, x_3, x_2) + S^{\#}_{2,1}(x_2, x_0))} R_1 a(x_L, \cdots, x_2, x_0) dx_3 \\ & = \left(\frac{E}{T(4, 3)}\right)^{1/2} \left(\frac{E}{T(2, 1)}\right)^{1/2} e^{-i\nu(S^{\#}_{4,3}(x_4, x_2) + S^{\#}_{2,1}(x_2, x_0))} \\ & \quad \times (S_3 R_1 a(x_L, \cdots, x_4, x_2, x_0) + R_3 R_1 a(x_L, \cdots, x_4, x_2, x_0)) . \end{aligned}$$

Again $S_3 R_1 a(x_L, \cdots, x_4, x_2, x_0) = (R_1 a)(x_L, \cdots, \overline{x_4, x_2}, x_0) D(S_4 + S_3; x_4, x_2)^{-1/2}$ is the main term and $R_3 R_1 a(x_L, \cdots, x_4, x_2, x_0)$ is the remainder.

We skip integration over x_3 space of the term including $R_2 S_1 a(x_L, \cdots, x_3, x_1)$, because this is complicated as a function of x_3. By virtue of the stationary phase method,

$$\left(\frac{E}{t_5}\right)^{1/2}\left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{T(3,1)}\right)^{1/2} e^{-i\nu S^\#_{3,1}(x_3,x_0)}$$
$$\times \tilde{\int}_{\mathbf{R}} e^{-i\nu(S_5(t_5,x_5,x_4)+S_4(t_4,x_4,x_3))} R_2 S_1 a(x_L, \cdots, x_4, x_3, x_0)dx_4$$
$$= \left(\frac{E}{T(5,4)}\right)^{1/2}\left(\frac{E}{T(3,1)}\right)^{1/2} e^{-i\nu(S^\#_{5,3}(x_5,x_3)+S^\#_{3,1}(x_3,x_0))}$$
$$\times (S_4 R_2 S_1 a(x_L, \cdots, x_5, x_3, x_0) + R_4 R_2 S_1 a(x_L, \cdots, x_5, x_3, x_0)).$$

Here

$$S_4 R_2 S_1 a(x_L, \cdots, x_5, x_3, x_0) = (R_2 S_1 a)(x_L, \cdots, \overline{x_5, x_3}, x_0) D(S_5 + S_4; x_5, x_3)^{-1/2}$$

is the main term and $R_4 R_2 S_1 a$ is the remainder.

Similarly, we perform integration over x_4 space of the term including $S_3 R_1 a$. But we skip integration over x_4 space of the term including $R_3 R_1 a$.

We continue this process; the rule is that if R_k appears we skip integration over x_{k+1} space. Then, finally, we get the expression

$$(4.6) \qquad I(\{t_j\}, S, a, \nu)(x_L, x_0) = A_0(x_L, x_0) + \textstyle\sum' A_{j_s j_{s-1} j_{s-2} \cdots j_1}(x_L, x_0).$$

Here the main term is

$$(4.7) \qquad A_0(x_L, x_0) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^\#_{L,1}(x_L,x_0)} S_{L-1} S_{L-2} \cdots S_1 a(x_L, x_0)$$
$$= \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^\#_{L,1}(x_L,x_0)} D(x_L, x_0)^{-1/2} a(\overline{x_L, x_0}).$$

$\sum'$ stands for the summation with respect to sequence of integers $(j_s, j_{s-1}, \cdots, j_1)$ with the property

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < j_2 < j_3 - 1 < \cdots < j_{s-1} < j_s - 1 < j_s$$
$$\leqq L - 1 < j_{s+1} = L.$$

The summand is

$$(4.8) \qquad A_{j_s j_{s-1} \cdots j_1}(x_L, x_0) = \prod_{m=1}^{s}\left(\frac{E}{T(j_m, j_{m-1}+1)}\right)^{1/2}$$
$$\times \tilde{\int}_{\mathbf{R}^s} \exp\{-i\nu S^\#_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0)\}$$
$$\times b_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0) \prod_{u=1}^{s} dx_{j_u}.$$

The amplitude function of this is

$$(4.9)\qquad \begin{aligned} &b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0)\\ &\quad = (Q_{L-1}Q_{L-2}\cdots Q_1 a)(x_L, x_{j_s}, \cdots, x_{j_1}, x_0)\,, \end{aligned}$$

where
$$\begin{aligned} Q_j &= \mathrm{Id}, && \text{if } j = j_s, j_{s-1}, \cdots, j_1;\\ &= R_j, && \text{if } j = j_s - 1,\ j_{s-1} - 1, \cdots, j_1 - 1;\\ &= S_j, && \text{otherwise.} \end{aligned}$$

The phase function is

$$(4.10)\qquad S^{\#}_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0) = \sum_{k=1}^{s+1} S^{\#}_{j_k, j_{k-1}+1}(x_{j_k}, x_{j_{k-1}})\,.$$

We can apply Lemma 3.1 to $A_{j_s j_{s-1}\cdots j_1}$ and obtain

$$(4.11)\qquad A_{j_s j_{s-1}\cdots j_1}(x_L, x_0) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)} a_{j_s j_{s-1}\cdots j_1}(x_L, x_0)\,.$$

For any $m \geqq 0$ Lemma 3.1 gives positive constants C_m and $K(m)$ such that if $|\alpha_L|$ and $|\alpha_0| \leqq m$

$$(4.12)\qquad \begin{aligned} &|\partial_{x_L}^{\alpha_L}\partial_{x_0}^{\alpha_0} a_{j_s j_{s-1}\cdots j_1}(x_L, x_0)|\\ &\quad \leqq C_m^s \max \sup_{x_{j_u}} |\partial_{x_L}^{\beta_L}\partial_{x_{j_s}}^{\beta_{j_s}}\cdots\partial_{x_{j_1}}^{\beta_{j_1}}\partial_{x_0}^{\beta_0} b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0)|\,. \end{aligned}$$

Here max is taken over those indices β's which satisfy $\beta_L \leqq \alpha_L$, $|\beta_0| \leqq m$, $|\beta_{j_k}| \leqq K(m)$ for $k = 1, 2, \cdots, s$ and sup is taken with respect to $x_{j_u} \in \mathbb{R}$, $u = 1, \cdots, s$. This implies that

$$(4.13)\qquad \begin{aligned} &I(\{t_j\}, S, a, \nu)(x_L, x_0)\\ &\quad = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)}(D(x_L, x_0)^{-1/2}(a(\overline{x_L, x_0}) + r(x_L, x_0)))\,, \end{aligned}$$

$$(4.14)\qquad r(x_L, x_0) = D(x_L, x_0)^{1/2} \textstyle\sum' a_{j_s j_{s-1}\cdots j_1}(x_L, x_0)\,.$$

Therefore, we have only to obtain estimate of $\sum' a_{j_s}\cdots{}_{j_1}(x_L, x_0)$ for the proof of Theorem 1.

In order to prove the estimate of $a_{j_s j_{s-1}\cdots j_1}(x_L, x_0)$ we can use estimate of $b_{j_s j_{s-1}\cdots j_1}$, because (4.12) holds.

LEMMA 4.1. *Assume* (H.1) *for the phase function and* (H.2) *for the amplitude function. Let δ be as in Lemma* 3.1. *Then for any $m \geqq 0$ there exist a constant $C_{m,1}$ and an integer $M(m)$ such that for any α_0, α_L, α_{j_k}, $0 \leqq k \leqq s$, with $|\alpha_{j_k}| \leqq m$, $|\alpha_L| \leqq m$, $|\alpha_0| \leqq m$,*

$$(4.15) \quad \left| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} \prod_{k=1}^{s} \partial_{x_k}^{\alpha_{j_k}} b_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0) \right| \leqq C_{m,1}^{s} \left(\prod_{k=1}^{s} \nu^{-1} t_{j_k} \right) \| a \|_{M(m), \{j_u\}},$$

where

$$\| a \|_{M(m), \{j_u\}} = \operatorname{Max} \sup \left| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} \prod_{k=1}^{s} \partial_{x_{j_k}}^{\beta_{j_k}} \partial_{x_{j_k-1}}^{\beta_{j_k-1}} a(\overline{x_L, x_{j_s}}), \overline{x_{j_s-1}, x_{j_{s-1}}}, \cdots, \overline{x_{j_1-1}, x_0}) \right|,$$

where Max *is taken over indices satisfying* $|\beta_{j_k}|$, $|\beta_{j_k-1}| \leqq M(m)$ *and* sup *is taken for* $x_{j_u-1} \in \mathbb{R}$, $u = 1, \cdots, s$. *We can choose* $M(m) = 2m + 4 + 2$.

We assume Lemma 4.1 for the time being. Then we can prove Theorem 1. In fact, combining (4.7) and Lemma 4.1, we have

$$|\partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} a_{j_s j_{s-1} \cdots j_1}(x_L, x_0)| \leqq C_m^s C_{m',1}^s \left(\prod_{r=1}^{s} \nu^{-1} t_{j_r} \right) \| a \|_{M(m'), \{j_u\}},$$

where $m' = K(m)$. On the other hand (H.2) implies that

$$\| a \|_{M(m'), \{j_u\}} \leqq A_{M(m')} X_{M(m')}^s .$$

Therefore, we obtain from (4.14) that

$$(4.16) \quad |\partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} r(x_L, x_0)| \leqq \left| \left(\sum{}' C_m^s C_{m',1}^s X_{M(m')}^s \prod_{r=1}^{s} (\nu^{-1} t_{j_r}) \right) \right| A_{M(m')} \leqq \left[\prod_{j=1}^{L} (1 + C_m C_{m',1} X_{M(m')} \nu^{-1} t_j) - 1 \right] A_{M(m')} .$$

We have proved our Theorem 1 up to the proof of Lemma 4.1. (Since we can choose $m' = K(m) = 10m + 10 + 20$, we choose $M(m') = 50(m + 1 + 1)$).

Lemma 4.1 follows immediately from the next

LEMMA 4.2. *We assume* (H.1) *for the phase function. Let* $a(x_L, x_{L-1}, \cdots, x_1, x_0)$ *be a function of* $L + 1$ *variables satisfying assumption* (H.2). *Then for any sequence of integers* $0 = k_0 < k_1 - 1 < k_1 < k_2 - 1 < \cdots < k_r - 1 < k_r < k_{r+1} = L$ *we introduce the function*

$$(4.17) \quad p_{k_r k_{r-1} \cdots k_1}(x_L, x_{L-1}, \cdots, x_{k_r+1}, x_{k_r}, x_{k_{r-1}}, \cdots, x_{k_1}, x_0) = (Q_{k_r} Q_{k_r-1} \cdots Q_1 a)(x_L, \cdots, x_{k_r+1}, x_{k_r}, x_{k_r-1}, \cdots, x_{k_1}, x_0),$$

where $Q_j = \mathrm{Id}$ *for* $j = k_r, k_{r-1}, \cdots, k_1$,

$= R_j$ *for* $j = k_r - 1, k_{r-1} - 1, \cdots, k_1 - 1$,

$= S_j$ *otherwise.*

This function enjoys the following estimate: For any $m \geqq 0$, *there exist constants* $C_{m,2}$ *and* $M(m)$ *such that if* $|\alpha_L|$, $|\alpha_0|$ *and* $|\alpha_{k_j}| \leqq m$, $(j = 1, 2, \cdots, r)$,

$$(4.18) \qquad \Big|\Big(\prod_{j=0}^{r+1} \partial_{x_{k_j}}^{\alpha_{k_j}}\Big) p_{k_r k_{r-1} \cdots k_1}(\overline{x_L, x_{k_r}}, x_{k_{r-1}} \dots, x_{k_1}, x_0)\Big|$$
$$\leqq C_{m,2}^r \prod_{j=1}^{r} \Big(\frac{t_{k_j} T(k_j - 1, k_{j-1} + 1)}{\nu T(k_j, k_{j-1} + 1)}\Big)$$
$$\times \max \sup \Big| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\beta_0} \Big(\prod_{j=1}^{r} \partial_{x_{k_j}}^{\beta_{k_j}} \partial_{x_{k_j-1}}^{\beta_{k_j-1}}\Big) a(\overline{x_L, x_{k_r}}, \overline{x_{k_r-1}, x_{k_{r-1}}}, \cdots, \overline{x_{k_1-1}, x_0})\Big|,$$

where max *is taken over those indices* $\beta_{k_j} \leqq \alpha_{k_j}$, $|\beta_{k_j-1}| \leqq M(m)$, $j = 0, 1, \cdots, r$ *and* sup *is taken with respect to* $x_{k_j-1} \in \mathbb{R}$, $j = 1, \cdots, r$. *Moreover, for any integers* $l_1, \cdots, l_q$ *with* $k_r < l_1 - 1 < l_1 < l_2 - 1 < \cdots < l_q \leqq L - 1$, *for arbitrary multi-indices* $\alpha_{l_u}, \alpha_{l_u-1}$ $(1 \leqq u \leqq q)$ *and for multi-indices* α_{k_j} *with* $|\alpha_{k_j}| \leqq m$ $(0 \leqq j \leqq r + 1)$, *we have*

$$(4.19) \qquad \Big| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} \prod_{u=1}^{q} (\partial_{x_{l_u}}^{\alpha_{l_u}} \partial_{x_{l_u-1}}^{\alpha_{l_u-1}}) \prod_{j=1}^{r} (\partial_{x_{k_j}}^{\alpha_{k_j}})$$
$$\times p_{k_r k_{r-1} \cdots k_1}(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, \cdots}, \overline{x_{l_1-1}, x_{k_r}}, x_{k_{r-1}}, \cdots, x_0)\Big|$$
$$\leqq C_{m,2}^r \prod_{u=1}^{r} \Big(\frac{t_{k_u} T(k_u - 1, k_{u-1} + 1)}{\nu T(k_u, k_{u-1} + 1)}\Big)$$
$$\times \max \sup \Big| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\beta_0} \prod_{u=1}^{q} (\partial_{x_{l_u}}^{\alpha_{l_u}} \partial_{x_{l_u-1}}^{\alpha_{l_u-1}}) \prod_{u=1}^{r} (\partial_{x_{k_u}}^{\beta_{k_u}} \partial_{x_{k_u-1}}^{\beta_{k_u-1}})$$
$$\times a(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overline{x_{l_1-1}, x_{k_r}}, \cdots, \overline{x_{k_1-1}, x_0})\Big|,$$

where max *is taken over those indices which satisfy,* $\beta_{k_u} \leqq \alpha_{k_u}$, $|\beta_{k_u-1}| \leqq M(m)$, $(u = 1, 2, \cdots, r)$ *and* $\beta_0 \leqq \alpha_0$; sup *is taken with respect to* $x_{k_u-1} \in \mathbb{R}$, $u = 1, \cdots, r$. *Constants* $C_{m,2}$ *and* $M(m)$ *depend only on* m. *We can choose* $M(m) = 2m + 4 + 2$.

Proof. We prove by induction on r. The case of $r = 1$. We abbreviate k_1 as k.

If $k \geqq 3$, then $p_k(x_L, \cdots, x_{k+1}, x_k, x_0) = R_{k-1} S_{k-2}, \cdots, S_1 a(x_L, \cdots, x_k, x_0)$.

If $k = 2$, then $p_k(x_L, \cdots, x_{k+1}, x_k, x_0) = R_1 a(x_L, \cdots, x_2, x_0)$.

We set

$$(4.20) \qquad q(x_L, \cdots, x_k, x_{k-1}, x_0) = S_{k-2}, \cdots, S_1 a(x_L, \cdots, x_k, x_{k-1}, x_0), \text{ if } k \geqq 3,$$
$$= a(x_L, \cdots, x_2, x_1, x_0), \quad \text{if } k = 2.$$

Let $S_{1,1}^{\#}(x_1, x_0) = S_1(t_1, x_1, x_0)$. Then p_k is defined by the equality:

$$\left(\frac{E}{t_k}\right)^{1/2}\left(\frac{E}{T(k-1,1)}\right)^{1/2}\tilde{\int}_{\mathbf{R}} e^{-i\nu(S_k(t_k,x_k,x_{k-1})+S^{\#}_{k-1,1}(x_{k-1},x_0))}$$

$$\times\, q(x_L,\cdots,x_k,x_{k-1},x_0)dx_{k-1}$$

$$=\left(\frac{E}{T(k,1)}\right)^{1/2} e^{-i\nu S^{\#}_{k,1}(x_k,x_0)}$$

$$\times\,[D(x_k,x_0)^{-1/2}q(x_L,\cdots,\overline{x_k,x_0})+p_k(x_L,\cdots,x_{k+1},x_k,x_0)]\,.$$

Therefore,

$$(4.21)\quad \left(\frac{E}{t_k}\right)^{1/2}\left(\frac{E}{T(k-1,1)}\right)^{1/2}\tilde{\int}_{\mathbf{R}} e^{-i\nu(S_k(t_k,x_k,x_{k-1})+S^{\#}_{k-1,1}(x_{k-1},x_0))}$$

$$\times\, q(\overline{x_L,x_k},x_{k-1},x_0)dx_{k-1}$$

$$=\left(\frac{E}{T(k,1)}\right)^{1/2} e^{-i\nu S^{\#}_{k,1}(x_k,x_0)}(D(x_k,x_0)^{-1/2}q(\overline{x_L,x_k}\overline{,x_0})+p_k(\overline{x_L,x_k},x_0))\,.$$

Similarly, if $k<l_1-1<l_1<l_2-1<\cdots<l_q\leqq L-1$, then

$$(4.22)\quad \left(\frac{E}{t_k}\right)^{1/2}\left(\frac{E}{T(k-1,1)}\right)^{1/2}\tilde{\int}_{\mathbf{R}} e^{-i\nu(S_k(t_k,x_k,x_{k-1})+S^{\#}_{k-1,1}(x_{k-1},x_0))}$$

$$\times\, q(\overline{x_L,x_{l_q}},\overline{x_{l_q-1},x_{l_{q-1}}},\cdots,\overline{x_{l_1-1},x_k},x_{k-1},x_0)dx_{k-1}$$

$$=\left(\frac{E}{T(k,1)}\right)^{1/2} e^{-i\nu S^{\#}_{k,1}(x_k,x_0)}$$

$$\times\,(D(x_k,x_0)^{-1/2}q(\overline{x_L,x_{l_q}},\overline{x_{l_q-1},x_{l_{q-1}}},\cdots,\overline{x_{l_1-1},x_k}\overline{,x_0})$$

$$+\,p_k(\overline{x_L,x_{l_q}},\overline{x_{l_q-1},x_{l_{q-1}}},\cdots,\overline{x_{l_1-1},x_k},x_0))\,.$$

We prove (4.18) for $r=1$. Differentiating both sides of (4.21) with respect to x_L and applying the stationary phase method Lemma 3.5 to (4.21), we obtain the estimate for p_k: For any m there exists $C_{m,0}$ such that if $|\alpha_k|$, $|\alpha_0|\leqq m$ and α_L is arbitrary,

$$(4.23)\quad |\partial^{\alpha_L}_{x_L}\partial^{\alpha_k}_{x_k}\partial^{\alpha_0}_{x_0}p_k(x_L,x_k,x_0)|\leqq C_{m,0}\frac{t_kT(k-1,1)}{T(k,1)\nu}$$

$$\times\,\max\sup_{x_{k-1}}|\partial^{\alpha_L}_{x_L}\partial^{\beta_k}_{x_k}\partial^{\beta_{k-1}}_{x_{k-1}}\partial^{\beta_0}_{x_0}q(\overline{x_L,x_k},x_{k-1},x_0)|\,,$$

here max is taken over those indices for which $\beta_k\leqq\alpha_k$, $|\beta_{k-1}|\leqq 2m+4+2$ and $\beta_0\leqq\alpha_0$. Since (4.20) implies

$$(4.24)\quad q(x_L,\cdots,x_k,x_{k-1},x_0)=D(x_{k-1},x_0)^{-1/2}a(x_L,\cdots,x_k,\overline{x_{k-1},x_0})\quad \text{if } k\geqq 3$$

$$=a(x_L,\cdots,x_2,x_1,x_0)\qquad \text{if } k=2\,.$$

Leibnitz' rule gives

$$
\begin{aligned}
&|\partial^{\alpha_L}_{x_L}\partial^{\alpha_k}_{x_k}\partial^{\alpha_0}_{x_0}p_k(\overbrace{x_L, x_k}, x_0)| \\
&\quad < C_{m,0}C_{m,3}\left(\frac{t_k T(k-1, 1)}{\nu T(k, 1)}\right) \max \sup_{x_{k-1}} |\partial^{\alpha_L}_{x_L}\partial^{\beta_k}_{x_k}\partial^{\beta_{k-1}}_{x_{k-1}}\partial^{\beta_0}_{k_0} a(\overbrace{x_L, x_k}, \overbrace{x_{k-1}, x_0})| .
\end{aligned}
$$

Here $C_{m,3} = 2^{(2m+4+2)}$, max is taken for $\beta_k \leqq \alpha_k$, $|\beta_{k-1}| \leqq 2m + 4 + 2$ and $\beta_0 \leqq \alpha_0$. Choose $M(m)$ and $C_{m,2}$ so that

$$
M(m) = 2m + 4 + 2 \quad \text{and} \quad C_{m,2} \geqq C_{m,0}C_{m,3} . \tag{4.25}
$$

Then this proves estimate (4.18) for $r = 1$.

We prove (4.19) for $r = 1$. Using the stationary phase method to (4.22) and using (4.24) again, we obtain the following estimate: For any $m \geqq 0$ if $|\alpha_k|$, $|\alpha_0| \leqq m$ and α_{l_u}, α_{l_u-1}, $(u = 1, \cdots, q)$, α_L are arbitrary multi-indices,

$$
\begin{aligned}
&\left|\partial^{\alpha_L}_{x_L}\partial^{\alpha_k}_{x_k}\partial^{\alpha_0}_{x_0}\prod_{u=1}^{q}(\partial^{\alpha_{l_u}}_{x_{l_u}}\partial^{\alpha_{l_u-1}}_{x_{l_u-1}})p_k(\overbrace{x_L, x_{l_q}}, \overbrace{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbrace{x_{l_1-1}, x_k}, x_0)\right| \\
&\quad \leqq C_{m,0}\left(\frac{t_k T(k-1, 1)}{\nu T(k, 1)}\right) \max \sup_{x_{k-1}} \Bigg|\partial^{\alpha_L}_{x_L}\partial^{\beta_k}_{x_k}\partial^{\beta_{k-1}}_{x_{k-1}}\partial^{\beta_0}_{x_0}\prod_{u=1}^{q}\partial^{\alpha_{l_u}}_{x_{l_u}}\partial^{\alpha_{l_u-1}}_{x_{l_u-1}} \\
&\qquad\qquad \times q(\overbrace{x_L, x_{l_q}}, \overbrace{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbrace{x_{l_1-1}, x_k}, x_{k-1}, x_0)\Bigg| \\
&\quad \leqq C_{m,0}C_{m,3}\left(\frac{t_k T(k-1, 1)}{\nu T(k, 1)}\right) \max \sup_{x_{k-1}} \Bigg|\partial^{\alpha_L}_{x_L}\partial^{\beta_k}_{x_k}\partial^{\beta'_{k-1}}_{x_{k-1}}\partial^{\beta'_0}_{x_0}\left(\prod_{u=1}^{q}\partial^{\alpha_{l_u}}_{x_{l_u}}\partial^{\alpha_{l_u-1}}_{x_{l_u-1}}\right) \\
&\qquad\qquad \times a(\overbrace{x_L, x_{l_q}}, \overbrace{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbrace{x_{l_1-1}, x_k}, \overbrace{x_{k-1}, x_0})\Bigg| .
\end{aligned}
$$

Max is taken for $\beta_k \leqq \alpha_k$, $\beta_0 \leqq \alpha_0$ and $|\beta_{k-1}| \leqq 2m + 4 + 2$ in the middle term. In the last term max is taken for $\beta_k \leqq \alpha_k$, $\beta'_0 \leqq \alpha_0$ and $|\beta'_{k-1}| \leqq 2m + 4 + 2$. We choose $M(m)$ and $C_{m,2}$ as in (4.25). Then (4.19) of the case $r = 1$ is proved.

Now assuming Lemma 4.2 for r, we prove (4.18) for $r + 1$.

Let k_{r+1} be any integer such that $L > k_{r+1} - 1 > k_r$ and we let

$$
\begin{aligned}
&p_{k_{r+1}k_r\cdots k_1}(x_L, \cdots, x_{k_{r+1}+1}, x_{k_{r+1}}, x_{k_r}, \cdots, x_{k_1}, x_0) \\
&\quad = R_{k_{r+1}-1}\cdots R_{k_r-1}\cdots R_{k_1-1}\cdots a(x_L, \cdots, x_{k_{r+1}+1}, x_{k_{r+1}}, x_{k_r}, \cdots, x_{k_1}, x_0) .
\end{aligned} \tag{4.26}
$$

Set

$$
\begin{aligned}
&q(x_L, \cdots, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \cdots, x_{k_1}, x_0) \\
&\quad = S_{k_{r+1}-2}\cdots S_{k_r+1}p_{k_r\cdots k_1}(x_L, \cdots, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \cdots, x_{k_1}, x_0) .
\end{aligned} \tag{4.27}
$$

Then $p_{k_{r+1}k_r\cdots k_1}$ is defined by the equality

$$
\begin{aligned}
(4.28)\quad & \left(\frac{E}{t_{k_{r+1}}}\right)^{1/2}\left(\frac{E}{T(k_{r+1}-1,k_r+1)}\right)^{1/2} \\
& \times \tilde{\int_{\mathbf{R}}} e^{-i\nu(S_{k_{r+1}}(t_{k_{r+1}},x_{k_{r+1}},x_{k_{r+1}-1}))+S^{\#}_{k_{r+1}-1,k_r+1}(x_{k_{r+1}-1},x_{k_r})} \\
& \times q(x_L,\cdots,x_{k_{r+1}},x_{k_{r+1}-1},x_{k_r},x_{k_{r-1}},\cdots,x_{k_1},x_0)dx_{k_{r+1}-1} \\
= & \left(\frac{E}{T(k_{r+1},k_r+1)}\right)^{1/2} e^{-i\nu S^{\#}_{k_{r+1},k_r+1}(x_{k_{r+1}},x_{k_r})} \\
& \times (D(x_{k_{r+1}},x_{k_r})^{-1/2}q(x_L,\cdots,\overline{x_{k_{r+1}}},x_{k_r},x_{k_{r-1}},\cdots,x_0) \\
& + p_{k_{r+1}\cdots k_1}(x_L,\cdots,x_{k_{r+1}+1},x_{k_{r+1}},x_{k_r},x_{k_{r-1}},\cdots,x_{k_1},x_0))\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(4.29)\quad & \left(\frac{E}{t_{k_{r+1}}}\right)^{1/2}\left(\frac{E}{T(k_{r+1}-1,k_r+1)}\right)^{1/2} \\
& \times \tilde{\int_{\mathbf{R}}} e^{-i\nu(S_{k_{r+1}}(t_{k_{r+1}},x_{k_{r+1}},x_{k_{r+1}-1})+S^{\#}_{k_{r+1}-1,k_r+1}(x_{k_{r+1}-1},x_{k_r}))} \\
& \times q(\overline{x_L},x_{k_{r+1}},x_{k_{r+1}-1},x_{k_r},\cdots,x_{k_1},x_0)dx_{k_{r+1}-1} \\
= & \left(\frac{E}{T(k_{r+1},k_r+1)}\right)^{1/2} e^{-i\nu S^{\#}_{k_{r+1},k_r+1}(x_{k_{r+1}},x_{k_r}} \\
& \times (D(x_{k_{r+1}},x_{k_r})^{-1/2}q(\overline{x_L},\overline{x_{k_{r+1}}},x_{k_r},x_{k_{r-1}},\cdots,x_{k_1},x_0) \\
& + p_{k_{r+1}k_r\cdots k_1}(\overline{x_L},x_{k_{r+1}},x_{k_r},x_{k_{r-1}},\cdots,x_{k_1},x_0))\,.
\end{aligned}
$$

Apply the stationary phase method Lemma 3.5 to (4.28). Then for any $m\geqq 0$ if $|\alpha_{k_r}|$ and $|\alpha_{k_{r+1}}|\leqq m$, we have with the same $C_{m,0}$ as in (4.23)

$$
\begin{aligned}
(4.30)\quad & \left|\partial^{\alpha_L}_{x_L}\partial^{\alpha_{k_{r+1}}}_{x_{k_{r+1}}}\partial^{\alpha_{k_r}}_{x_{k_r}}\partial^{\alpha_0}_{x_0}\left(\prod_{u=1}^{r-1}\partial^{\alpha_{k_u}}_{x_{k_u}}\right)p_{k_{r+1}\cdots k_1}(\overline{x_L,x_{k_{r+1}}},x_{k_r},\cdots,x_{k_1},x_0)\right| \\
& < C_{m,0}\left(\frac{t_{k_{r+1}}T(k_{r+1}-1,k_r+1)}{\nu T(k_{r+1},k_r+1)}\right) \\
& \qquad \times \max \sup_{x_{k_{r+1}-1}}\left|\partial^{\alpha_L}_{x_L}\partial^{\beta_{k_{r+1}}}_{x_{k_{r+1}}}\partial^{\beta_{k_{r+1}-1}}_{x_{k_{r+1}-1}}\partial^{\beta_{k_r}}_{x_{k_r}}\partial^{\alpha_0}_{x_0}\prod_{u=1}^{r-1}\partial^{\alpha_{k_u}}_{x_{k_u}}\right. \\
& \qquad\qquad \left.\times q(\overline{x_L,x_{k_{r+1}}},x_{k_{r+1}-1},x_{k_r},\cdots,x_{k_1},x_0)\right|,
\end{aligned}
$$

max is taken for $\beta_{k_{r+1}}\leqq\alpha_{k_{r+1}}$, $\beta_{k_r}\leqq\alpha_{k_r}$ and $|\beta_{k_{r+1}-1}|\leqq 2m+4+2$. Here α_0, α_{k_u} $(u=1,2,\cdots,r-1)$ and α_L are arbitrary multi-indices.

On the other hand, by definition (4.27) we have

$$(4.31)\quad q(x_L, \cdots, x_{k_{r+1}}, x_{k_{r+1}-1}, x_{k_r}, \cdots, x_{k_1}, x_0) = D(x_{k_{r+1}-1}, x_{k_r})^{-1/2} p_{k_r \cdots k_1}(x_L, \cdots, x_{k_{r+1}}, \overline{x_{k_{r+1}-1}, x_{k_r}}, \cdots, x_{k_1}, x_0)\,.$$

Therefore,

$$(4.32)\quad q(\overline{x_L, x_{k_{r+1}}}, x_{k_{r+1}-1}, x_{k_r}, \cdots, x_{k_1}, x_0) = D(x_{k_{r+1}-1}, x_{k_r})^{-1/2} p_{k_r \cdots k_1}(\overline{x_L, x_{k_{r+1}}}, \overline{x_{k_{r+1}-1}, x_{k_r}}, \cdots, x_{k_1}, x_0)\,.$$

And we have

$$(4.33)\quad \begin{aligned} &\left|\partial_{x_L}^{\alpha_L}\partial_{x_{k_{r+1}}}^{\alpha_{k_{r+1}}}\partial_{x_{k_r}}^{\alpha_{k_r}}\partial_{x_0}^{\alpha_0}\left(\prod_{u=1}^{r-1}\partial_{x_{k_u}}^{\alpha_{k_u}}\right)p_{k_{r+1}\cdots k_1}(\overline{x_L, x_{k_{r+1}}}, x_{k_r}, \cdots, x_{k_1}, x_0)\right| \\ &\leqq C_{m,0}C_{m,3}\left(\frac{t_{k_{r+1}}T(k_{r+1}-1, k_r+1)}{\nu T(k_{r+1}, k_r+1)}\right) \\ &\quad\times \max \sup_{x_{k_{r+1}-1}}\left|\partial_{x_L}^{\alpha_L}\partial_{x_{k_{r+1}}}^{\beta_{k_{r+1}}}\partial_{x_{k_{r+1}-1}}^{\beta'_{k_{r+1}-1}}\partial_{x_{k_r}}^{\beta'_{k_r}}\partial_{x_0}^{\alpha_0}\prod_{u=1}^{r-1}\partial_{x_{k_u}}^{\alpha_{k_u}}\right. \\ &\quad\left.\times\, p_{k_r\cdots k_1}(\overline{x_L, x_{k_{r+1}}}, \overline{x_{k_{r+1}-1}, x_{k_r}}, \cdots, x_{k_1}, x_0)\right|. \end{aligned}$$

Here max is taken for multi-indices $|\beta'_{k_{r+1}-1}| \leqq 2m+4+2$, and $\beta'_{k_r} \leqq \beta_{k_r}$. Now we restrict to the case $|\alpha_{k_j}| \leqq m$, $j = 1, \cdots, r$. We can apply induction hypothesis (4.19) for r with $q = 1$, $l_1 = k_{r+1}$, $\alpha_{l_1} = \beta_{k_{r+1}}$ and $\alpha_{l_1-1} = \beta_{k_{r+1}-1}$ to (4.33) and we get a majorization for the right hand side of (4.33). Consequently, if $|\alpha_k|$, $|\alpha_{k_j}|$, $|\alpha_0| \leqq m$, we have

$$\begin{aligned} &\left|\partial_{x_L}^{\alpha_L}\partial_{x_{k_{r+1}}}^{\alpha_{k_{r+1}}}\partial_{x_{k_r}}^{\alpha_{k_r}}\partial_{x_0}^{\alpha_0}\left(\prod_{u=1}^{r-1}\partial_{x_{k_u}}^{\alpha_{k_u}}\right)p_{k_{r+1}\cdots k_1}(\overline{x_L, x_{k_{r+1}}}, \overline{x_{k_{r+1}-1}, x_{k_r}}, \cdots, x_{k_1}, x_0)\right| \\ &\leqq C_{m,0}C_{m,3}C_{m,2}^{r}\left(\frac{t_{k_{r+1}}T(k_{r+1}-1, k_r+1)}{\nu T(k_{r+1}, k_{r-1}+1)}\right)\prod_{u=1}^{r}\left(\frac{t_{k_u}T(k_u-1, k_{u-1}+1)}{\nu T(k_u, k_{u-1}+1)}\right) \\ &\quad\times \max \sup\left|\partial_{x_L}^{\alpha_L}\partial_{x_{k_{r+1}}}^{\beta_{k_{r+1}}}\partial_{x_{k_{r+1}-1}}^{\beta_{k_{r+1}-1}}\partial_{x_0}^{\beta'_0}\prod_{u=1}^{r}\partial_{x_{k_u}}^{\beta'_{k_u}}\partial_{x_{k_u-1}}^{\beta'_{k_u-1}}\right. \\ &\quad\left.\times\, a(\overline{x_L, x_{k_{r+1}}}, \overline{x_{k_{r+1}-1}, x_{k_r}}, \cdots, \overline{x_{k_1-1}, x_0})\right|. \end{aligned}$$

Here max is taken for $\beta'_0 \leqq \alpha_0$, $\beta_{k_{r+1}} \leqq \alpha_{k_{r+1}}$, $|\beta_{k_{r+1}-1}| \leqq 2m+4d+2$, $\beta'_{k_u} \leqq \alpha_{k_u}$, $|\beta'_{k_u-1}| \leqq 2m+4+2$, $(u = 1, \cdots, r)$ and sup is taken with respect to $x_{k_u-1} \in \mathbb{R}$, $u = 1, \cdots, r+1$. We may choose $M(m)$ and $C_{m,2}$ as in (4.25). We have proved (4.18) for $r+1$.

We next prove (4.19) for $r+1$. Let $l_1, l_2, \cdots, l_q$ be a sequence of integers with the property $k_r < k_{r+1} - 1 < k_{r+1} < l_1 - 1 < l_1 < \cdots < l_q - 1 < l_q \leqq L - 1$. Then we have, from (4.28),

$$
\begin{aligned}
(4.34)\quad & \Big(\frac{E}{t_{k_{r+1}}}\Big)^{1/2}\Big(\frac{E}{T(k_{r+1}-1,\,k_r+1)}\Big)^{1/2} \\
& \times \tilde{\int}_{\mathbf{R}} e^{-i\nu(S_{k_{r+1}}(t_{k_{r+1}},x_{k_{r+1}},x_{k_{r+1}-1})+S^{\#}_{k_{r+1}-1,k_r+1}(x_{k_{r+1}-1},x_{k_r}))} \\
& \times q(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbracket{x_{l_1-1}, x_{k_{r+1}}}, x_{k_{r+1}-1}, \cdots, x_{k_1}, x_0)\,dx_{k_{r+1}-1} \\
= & \Big(\frac{E}{T(k_{r+1},\,k_r+1)}\Big)^{1/2} e^{-i\nu S^{\#}_{k_{r+1},k_r+1}(x_{k_{r+}},x_{k_r})} \\
& \times (D(x_{k_{r+1}}, x_{k_r}))^{-1/2} q(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \\
& \qquad \overbracket{x_{l_1-1}, x_{k_{r+1}}}, x_{k_r}, \cdots, x_{k_1}, x_0) \\
& \quad + p_{k_{r+1}\cdots k_1}(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \overbracket{x_{l_1-1}, x_{k_{r+1}}}, x_{k_r}, \cdots, x_{k_1}, x_0))\,.
\end{aligned}
$$

We apply the stationary phase method Lemma 3.5 to (4.34). For any $m \geqq 0$ let α_{k_r} and $\alpha_{k_{r+1}}$ be two multi-indices with $|\alpha_{k_r}|$, $|\alpha_{k_{r+1}}| \leqq m$; let α_{k_u} $(u = 1, 2, \cdots, r-1)$, α_L, α_0, and α_{l_u} $(u = 1, 2, \cdots, q)$ be arbitrary multi-indices. Then with the same constant $C_{m,0}$ as in (4.23) and (4.30), we have

$$
\begin{aligned}
(4.35)\quad & \Big|\partial^{\alpha_L}_{x_L}\partial^{\alpha_0}_{x_0}\Big(\prod_{u=1}^{q}\partial^{\alpha_{l_u}}_{x_{l_u}}\partial^{\alpha_{l_u-1}}_{x_{l_u-1}}\Big)\prod_{u=1}^{r+1}\partial^{\alpha_{k_u}}_{x_{k_u}} \\
& \times p_{k_{r+1}\cdots k_1}(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbracket{x_{l_1-1}, x_{k_{r+1}}}, x_{k_r}, \cdots, x_{k_1}, x_0)\Big| \\
\leqq\ & C_{m,0}\Big(\frac{t_{k_{r+1}}T(k_{r+1}-1,\,k_r+1)}{\nu T(k_{r+1},\,k_r+1)}\Big) \\
& \times \max \sup_{x_{k_{r+1}-1}} \Big|\partial^{\alpha_L}_{x_L}\partial^{\alpha_0}_{x_0}\partial^{\beta_{k_{r+1}}}_{x_{k_{r+1}}}\partial^{\beta_{k_{r+1}-1}}_{x_{k_{r+1}-1}}\partial^{\beta_{k_r}}_{x_{k_r}}\prod_{u=1}^{q}\partial^{\alpha_{l_u}}_{x_{l_u}}\partial^{\alpha_{l_u-1}}_{x_{l_u-1}}\prod_{u=1}^{r-1}\partial^{\alpha_{k_u}}_{x_{k_u}} \\
& \times q(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbracket{x_{l_1-1}, x_{k_{r+1}}}, x_{k_{r+1}-1}, x_{k_r}, \cdots, x_{k_1}, x_0)\Big|\,.
\end{aligned}
$$

We use the relationship (4.31) between q and $p_{k_r\cdots k_1}$. The right hand side of this inequality is majorized by

$$
\begin{aligned}
(4.36)\quad & C_{m,0}C_{m,3}\Big(\frac{t_{k_{r+1}}T(k_{r+1}-1,\,k_r+1)}{\nu T(k_{r+1},\,k_r+1)}\Big) \\
& \times \max \sup_{x_{k_{r+1}-1}} \Big|\partial^{\alpha_L}_{x_L}\partial^{\alpha_0}_{x_0}\partial^{\beta_{k_{r+1}}}_{x_{k_{r+1}}}\partial^{\beta'_{k_{r+1}-1}}_{x_{k_{r+1}-1}}\partial^{\beta'_{k_r}}_{x_{k_r}}\prod_{u=1}^{q}\partial^{\alpha_{l_u}}_{x_{l_u}}\partial^{\alpha_{l_u-1}}_{x_{l_u-1}}\prod_{u=1}^{r-1}\partial^{\alpha_{k_u}}_{x_{k_u}} \\
& \times p_{k_r\cdots k_1}(\overbracket{x_L, x_{l_q}}, \overbracket{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overbracket{x_{l_1-1}, x_{k_{r+1}}}, \overbracket{x_{k_{r+1}-1}, x_{k_r}}, \cdots, x_{k_1}, x_0)\Big|\,.
\end{aligned}
$$

Here max is taken for $\beta_{k_{r+1}} \leqq \alpha_{k_{r+1}}$, $|\beta'_{k_{r+1}-1}| \leqq 2m + 4 + 2$ and $\beta'_{k_r} \leqq \alpha_{k_r}$.

Now we assume that $|\alpha_{k_j}| \leqq m$, $j = 1, \cdots, r+1$, and $|\alpha_0|$, $|\alpha_L| \leqq m$ as in Lemma 4.2 but that α_{l_u}, $u = 1, \cdots, q$, are arbitrary. Then we use the induction hypothesis (4.19) for r where q is replaced by $q+1$ and $(l_1, \cdots, l_q)$ is replaced by $(k_{r+1}, l_1, \cdots, l_q)$ to majorize (4.36). Then we have

$$
\begin{aligned}
(4.37)\quad & \Big| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} \Big(\prod_{u=1}^{q} \partial_{x_{l_u}}^{\alpha_{l_u}} \partial_{x_{l_u-1}}^{\alpha_{l_u-1}} \Big) \prod_{u=1}^{r} \partial_{x_{k_u}}^{\alpha_{k_u}} \\
& \quad \times p_{k_{r+1}\cdots k_1}(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overline{x_{l_1-1}, x_{k_{r+1}}}, x_{k_r}, \cdots, x_{k_1}, x_0) \Big| \\
& \leqq C_{m,0} C_{m,3} C_{m,2}^{r} \prod_{u=1}^{r+1} \Big(\frac{t_{k_u} T(k_u - 1, k_{u-1} + 1)}{\nu T(k_u, k_{u-1} + 1)} \Big) \\
& \quad \times \max \sup \Big| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\beta_0} \partial_{x_{k_{r+1}}}^{\beta_{k_{r+1}}} \partial_{x_{k_{r+1}-1}}^{\beta'_{k_{r+1}-1}} \partial_{x_{k_r}}^{\beta''_{k_r}} \partial_{x_{k_r-1}}^{\beta''_{k_r-1}} \prod_{u=1}^{q} \partial_{x_{l_u}}^{\alpha_{l_u}} \partial_{x_{l_u-1}}^{\alpha_{l_u-1}} \prod_{u=1}^{r-1} \partial_{x_{k_u}}^{\alpha_{k_u}} \\
& \quad \times a(\overline{x_L, x_{l_q}}, \overline{x_{l_q-1}, x_{l_{q-1}}}, \cdots, \overline{x_{l_1-1}, x_{k_{r+1}}}, \overline{x_{k_{r+1}-1}, x_{k_r}}, \cdots, \overline{x_{k_1}, x_0}) \Big|
\end{aligned}
$$

max is taken for $\beta_{k_{r+1}} \leqq \alpha_{k_{r+1}}$, $|\beta'_{k_{r+1}-1}| \leqq 2m + 4d + 2$, $\beta''_{k_r} \leqq \alpha_{k_r}$, $|\beta''_{k_r-1}| < 2m + 4 + 2$, $\beta_0 \leqq \alpha_0$ and sup is taken with respect to $x_{k_u-1} \in \mathbb{R}$, $u = 1, 2, \cdots, r+1$. We can choose $M(m)$ and $C_{m,2}$ as in (4.25). Then the above inequality proves (4.19) for $r+1$. We have completed proof of Lemma 4.2.

Proof of Theorem 1 has been completed.

§ 5. Proof of Theorem 2

We can proceed just as in the proof of Theorem 1. And we have

$$
(5.1)\qquad I(\{t_j\}, S, 1, \nu)(x_L, x_0) = A_0(x_L, x_0) + \sum{}' A_{j_s j_{s-1} \cdots j_1}(x_L, x_0) .
$$

Here

$$
(5.2)\qquad A_0(x_L, x_0) = \Big(\frac{E}{T_L} \Big)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L, x_0)} D(x_L, x_0)^{-1/2}
$$

is the main term; $\sum'$ is the same as in § 4 and

$$
\begin{aligned}
(5.3)\quad A_{j_s j_{s-1} \cdots j_1}(x_L, x_0) &= \prod_{k=1}^{s+1} \Big(\frac{E}{T(j_k, j_{k-1} + 1)} \Big)^{1/2} \\
& \quad \times \tilde{\int}_{\mathbb{R}^s} e^{-i\nu S^{\#}_{j_s \cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0)} b_{j_s \cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0) \prod_{k=1}^{s} dx_{j_k} ,
\end{aligned}
$$

where

$$
(5.4)\qquad S^{\#}_{j_s j_{s-1} \cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0) = \sum_{k=1}^{s+1} S^{\#}_{j_k, j_{k-1}+1}(x_{j_k}, x_{j_{k-1}})
$$

and

(5.5) $$b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0) = (Q_{L-1}Q_{L-2}, \cdots, Q_1 1)(x_L, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0),$$

where $Q_j = \mathrm{Id}$ if $j = j_s, j_{s-1}, \cdots, j_1$,
$= R_j$ if $j = j_s - 1, j_{s-1} - 1, \cdots, j_1 - 1$,
$= S_j$ otherwise.

Applying Lemma 3.1 to $A_{j_s j_{s-1}\cdots j_1}(x_L, x_0)$, we have

$$A_{j_s\cdots j_1}(x_L, x_0) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L,x_0)} a_{j_s\cdots j_1}(x_L, x_0).$$

Thus setting, just as in (4.14),

(5.6) $$r(x_L, x_0) = D(x_L, x_0)^{1/2}\left(\textstyle\sum' a_{j_s\cdots j_1}(x_L, x_0)\right),$$

we have

(5.7) $$I(\{t_j\}, S, 1, \nu) = \left(\frac{E}{T_L}\right)^{1/2} e^{-i\nu S^{\#}_{L,1}(x_L,x_0)} D(x_L, x_0)^{-1/2}(1 + r(x_L, x_0)).$$

We have only to obtain estimate of $r(x_L, x_0)$ to prove Theorem 2. By virtue of Lemma 3.1, we get estimates of $a_{j_s\cdots j_1}(x_L, x_0)$: For any $m \geqq 0$ there exists C_m and $K(m)$ such that if $|\alpha_L|, |\alpha_0| \leqq m$

(5.8) $$|\partial^{\alpha_L}_{x_L}\partial^{\alpha_0}_{x_0} a_{j_s\cdots j_1}(x_L, x_0)| \leqq C^s_m \max\sup\left|\partial^{\alpha_L}_{x_L}\partial^{\beta_0}_{x_0}\prod_{r=1}^{s}\partial^{\beta_{j_r}}_{x_{j_r}} b_{j_s\cdots j_1}(x_L, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0)\right|$$

max is taken over $\beta_0 \leqq \alpha_0$ and $|\beta_{j_r}| \leqq K(m)$.

We wish to obtain estimate of $b_{j_s\cdots j_1}$. Consider $x_L, \cdots, x_{j_s+1}$ as parameters and set

(5.9) $$\tilde{b}_{j_s j_{s-1}\cdots j_1}(x_L, \cdots, x_{j_s+1}, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0) = (Q_{j_s}Q_{j_s-1}, \cdots, Q_1 1)(x_L, \cdots, x_{j_s+1}, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0),$$

where $Q_j = \mathrm{Id}$ if $j = j_s, j_{s-1}, \cdots, j_1$,
$= R_j$ if $j = j_s - 1, j_{s-1} - 1, \cdots, j_1 - 1$,
$= S_j$ otherwise.

The next Lemma gives estimate of $\tilde{b}_{j_s\cdots j_1}$.

LEMMA 5.1. *Let δ be as in Theorem 1 and $T_L < \delta$. Then $\tilde{b}_{j_s j_{s-1}\cdots j_1}$ is independent of $x_L, \cdots, x_{j_s+1}$. It is of the form*

(5.10) $$\tilde{b}_{j_s j_{s-1}\cdots j_1}(x_L, \cdots, x_{j_s+1}, x_{j_s}, x_{j_{s-1}}, \cdots, x_{j_1}, x_0) = \prod_{r=1}^{s} \frac{t_{j_r}}{\nu} T(j_r - 1, j_{r-1} + 1)^2 p_{j_r}(x_{j_r}, x_{j_{r-1}}),$$

where $p_{j_r}(x_{j_r}, x_{j_{r-1}})$ *satisfies the estimates*

(5.11) $$|\partial^{\alpha}_{x_{j_r}} \partial^{\beta}_{x_{j_{r-1}}} p_{j_r}(x_{j_r}, x_{j_{r-1}})| \leqq C_{\alpha\beta} \quad \text{for any } \alpha \text{ and } \beta.$$

Here constant $C_{\alpha\beta}$ *depends only on* α *and* β.

We assume Lemma 5.1 for the moment and continue proof of Theorem 2. Since $b_{j_s j_{s-1}\cdots j_1} = S_{L-1} S_{L-2} \cdots S_{j_s+1} \tilde{b}_{j_s j_{s-1}\cdots j_1}$, we can apply Lemma 5.1 to (5.5) and obtain

$$b_{j_s j_{s-1}\cdots j_1}(x_L, x_{j_s}, \cdots, x_{j_1}, x_0) = D(x_L, x_{j_s})^{-1/2} \prod_{r=1}^{s} \left(\frac{t_{j_r}}{\nu}\right) T(j_r - 1, j_{r-1} + 1)^2 p_{j_r}(x_{j_r}, x_{j_{r-1}}).$$

Combining this with (5.8), we have

$$|\partial^{\alpha_L}_{x_L} \partial^{\alpha_0}_{x_0} a_{j_s\cdots j_1}(x_L, x_0)| < C^s_m \max \sup \left| \partial^{\alpha_L}_{x_L} \partial^{\beta_0}_{x_0} \prod_{r=1}^{s} \partial^{\beta_{j_r}}_{x_{j_r}} D(x_L, x_{j_s})^{-1/2} \times \prod_{r=1}^{s} \left(\frac{t_{j_r}}{\nu} T(j_r - 1, j_{r-1} + 1)^2 p_{j_r}(x_{j_r}, x_{j_{r-1}})\right)\right|$$

Therefore, for any $m \geqq 0$ we can find a constant $C_{m,1}$ such that

$$|\partial^{\alpha_L}_{x_L} \partial^{\alpha_0}_{x_0} a_{j_s\cdots j_1}(x_L, x_0)| \leqq C^s_{m,1} \prod_{r=1}^{s} (\nu^{-1} t_{j_r} T(j_r - 1, j_{r-1} + 1)^2)$$

as far as $|\alpha_L|$ and $|\alpha_0| \leqq m$. This and (5.6) imply

$$|\partial^{\alpha_L}_{x_L} \partial^{\alpha_0}_{x_0} r(x_L, x_0)| < \sum_{(j_s\cdots j_1)}{}' C^s_{m,1} \prod_{r=1}^{s} \nu^{-1} t_{j_r} T(j_r - 1, j_{r-1} + 1)^2 < \prod_{j=1}^{L} (1 + C_{m,1} \nu^{-1} t_j T^2_L) - 1.$$

We have proved Theorem 2 upto the proof of Lemma 5.1.

Proof of Lemma 5.1. We prove Lemma by induction on s. The case $s = 1$. We abbreviate $j_1 = j$. Just as in the proof of Lemma 4.2, we let

(5.12) $$\tilde{b}_j(x_L, \cdots, x_{j+1}, x_j, x_0) = R_{j-1} S_{j-2} \cdots S_1(1).$$

Then this is defined by the equality

$$(5.13)\quad \left(\frac{E}{t_j}\right)^{1/2}\left(\frac{E}{T(j-1,1)}\right)^{1/2}\tilde{\int}_{\mathbb{R}} e^{-i\nu(S_j(t_j,x_j,x_{j-1})+S^{\#}_{j-1,1}(x_{j-1},x_0))}D(x_{j-1},x_0)^{-1/2}dx_{j-1}$$
$$=\left(\frac{E}{T(i,1)}\right)^{1/2}e^{-i\nu S^{\#}_{j,1}(x_j,x_0)}(D(x_j,x_0)^{-1/2}+\tilde{b}_j(x_L,\cdots,x_{j+1},x_j,x_0)).$$

This means that $\tilde{b}_j(x_L,\cdots,x_{j+1},x_j,x_0)$ is indepenent of $x_L,\cdots,x_{j+1}$. We can write, $\tilde{b}_j(x_L,\cdots,x_{j+1},x_j,x_0)$ by $\tilde{b}_j(x_j,x_0)$. Furthermore we wish to show that we can write

$$(5.14)\quad \tilde{b}_j(x_j,x_0)=\nu^{-1}t_jT(j-1,1)^2p(x_j,x_0).$$

To show (5.14), we need a closer look at the amplitude function of (5.13). By virtue of Proposition 2.10, there exists a function $q_j(x_{j-1},x_0)\in\mathscr{B}(\mathbb{R}\times\mathbb{R})$ such that we have

$$D(x_{j-1},x_0)^{-1/2}=1+T(j-1,1)^2q_{j-1}(x_{j-1},x_0).$$

This means that

$$(5.15)\quad \partial_{x_{j-1}}D(x_{j-1},x_0)^{-1/2}=T(j-1,1)^2\partial_{x_{j-1}}q_{j-1}(x_{j-1},x_0).$$

We apply Lemma 3.5 to (5.13). For any m there exists a constant C_m such that if $|\alpha_0|,\ |\alpha_j|\leqq m$ we have

$$(5.16)\quad |\partial_{x_j}^{\alpha_j}\partial_{x_0}^{\alpha_0}\tilde{b}_j(x_j,x_0)|\leqq C_m\left(\frac{t_jT(j-1,1)}{\nu T(j,1)}\right)\max\sup_{x_{j-1}}|\partial_{x_{j-1}}^{\beta_{j-1}}\partial_{x_0}^{\beta_0}(D(x_{j-1},x_0)^{-1/2})|$$
$$\leqq C_m\left(\frac{t_jT(j-1,1)^3}{\nu T(j,1)}\right)\max\sup_{x_{j-1}}|\partial_{x_{j-1}}^{\beta_{j-1}}\partial_{x_0}^{\beta_0}q_{j-1}(x_{j-1},x_0)|.$$

Here max is taken over β_j and β_0 with $1\leqq|\beta_{j-1}|\leqq 2m+4+2$, $|\beta_0|\leqq m$. This proves (5.14). Lemma 5.1 for $s=1$ is true.

Assuming Lemma 5.1 for s, we prove it for $s+1$. By induction hypothesis the function $\tilde{b}_{j_sj_{s-1}\cdots j_1}(x_L,\cdots,x_{j_s+1},x_{j_s},x_{j_{s-1}},\cdots,x_{j_1},x_0)$ does not depend on $x_L,\cdots,x_{j_s+1}$. So we may denote it by

$$\tilde{b}_{j_sj_{s-1}\cdots j_1}(x_{j_s},x_{j_{s-1}},\cdots,x_{j_1},x_0).$$

Let j_{s+1} be an arbitrary integer such that $j_s+1<j_{s+1}<L$. Then we have by definition (5.9).

$$(5.17)\quad \tilde{b}_{j_{s+1}j_s\cdots j_1}=R_{j_{s+1}-1}S_{j_{s+1}-2}\cdots S_{j_s+1}\tilde{b}_{j_sj_{s-1}\cdots j_1}.$$

We see that

$$(5.18)\qquad S_{j_{s+1}-2}\cdots S_{j_s+1}\tilde{b}_{j_s j_{s-1}\cdots j_1}(1)(x_L,\cdots,x_{j_{s+1}-1},x_{j_s},\cdots,x_{j_1},x_0)$$
$$=D(x_{j_{s+1}-1},x_{j_s})^{-1/2}\tilde{b}_{j_s j_{s-1}\cdots j_1}(x_{j_s},x_{j_{s-1}},\cdots,x_{j_1},x_0)\,.$$

Here we set $D(x_{j_{s+1}-1},x_{j_s})=1$ if $j_{s+1}-1=j_s+1$. Thus (5.17) and (5.18) imply that the function

$$\tilde{b}_{j_{s+1}j_s\cdots j_1}(x_L,\cdots,x_{j_{s+1}+1},x_{j_{s+1}},x_{j_s},\cdots,x_{j_1},x_0)$$

is defined by the equality:

$$(5.19)\qquad \Big(\frac{E}{t_{j_{s+1}}}\Big)^{1/2}\Big(\frac{E}{T(j_{s+1}-1,j_s+1)}\Big)^{1/2}$$
$$\times\tilde{\int}_{\mathbf{R}} e^{-i\nu(S_{j_{s+1}}(t_{j_{s+1}},x_{j_{s+1}},x_{j_{s+1}-1})+S^{\#}_{j_{s+1}-1,j_s+1}(x_{j_{s+1}-1},x_{j_s}))}$$
$$\times D(x_{j_{s+1}-1},x_{j_s})^{-1/2}\tilde{b}_{j_s j_{s-1}\cdots j_1}(x_{j_s},x_{j_{s-1}},\cdots,x_0)dx_{j_{s+1}-1}$$
$$=\Big(\frac{E}{T(j_{s+1},j_s+1)}\Big)^{1/2}e^{-i\nu S^{\#}_{j_{s+1}j_s+1}(x_{j_{s+1}},x_{j_s})}$$
$$\times\big(D(x_{j_{s+1}},x_{j_s})^{-1/2}\tilde{b}_{j_s j_{s-1}\cdots j_1}(x_{j_s},x_{j_{s-1}},\cdots,x_0)$$
$$+\tilde{b}_{j_{s+1}j_s\cdots j_1}(x_L,\cdots,x_{j_{s+1}+1},x_{j_{s+1}},x_{j_s},\cdots,x_{j_1},x_0)\big)\,.$$

The left hand side of this equals

$$\tilde{b}_{j_s j_{s-1}\cdots i_1}(x_{j_s},x_{j_{s-1}},\cdots,x_0)\Big(\frac{E}{t_{j_{s+1}}}\Big)^{1/2}\Big(\frac{E}{T(j_{s+1}-1,j_s+1)}\Big)^{1/2}$$
$$\times\tilde{\int}_{\mathbf{R}} e^{-i\nu(S_{j_{s+1}}(t_{j_{s+1}},x_{j_{s+1}},x_{j_{s+1}-1})+S^{\#}_{j_{s+1}-1,j_s+1}(x_{j_{s+1}-1},x_{j_s}))}D(x_{j_{s+1}-1},x_{j_s})^{-1/2}dx_{j_{s+1}-1}\,.$$

The last integral was treated earlier in (5.13). Using discussions there, we can prove that (5.19) equals

$$\Big(\frac{E}{T(j_{s+1}\ j_s+1)}\Big)^{1/2}e^{-i\nu S^{\#}_{j_{s+1},j_s+1}(x_{j_{s+1}},x_{j_s})}\tilde{b}_{j_s\cdots j_1}(x_{j_s},\cdots,x_0)$$
$$\times\big(D(x_{j_{s+1}},x_{j_s})^{-1/2}+\nu^{-1}t_{j_{s+1}}T(j_{s+1}-1,j_s+1)^2 p_{j_{s+1}}(x_{j_{s+1}},x_{j_s})\big)$$

with some $p_{j_{s+1}}(x_{j_{s+1}},x_{j_s})\in\mathscr{B}(\mathbf{R}\times\mathbf{R})$. It follows from this that

$$(5.20)\qquad \tilde{b}_{j_{s+1}j_s\cdots j_1}(x_L,\cdots,x_{j_{s+1}+1},x_{j_{s+1}},x_{j_s},\cdots,x_{j_1},x_0)$$
$$=\Big(\frac{t_{j_{s+1}}T(j_{s+1}-1,j_s+1)^2}{\nu}\Big)p_{j_{s+1}}(x_{j_{s+1}},x_{j_s})\tilde{b}_{j_s\cdots j_1}(x_{j_s},\cdots,x_{j_1},x_0)\,.$$

We can use induction hypothesis for $\tilde{b}_{j_s\cdots j_1}(x_{j_s},\cdots,x_{j_1},x_0)$, i.e., replace $\tilde{b}_{j_s\cdots j_1}(x_{j_s},\cdots,x_{j_1},x_0)$ in (5.20) by the right hand side of (5.10). Consequently, we obtain

$$\tilde{b}_{j_{s+1}\cdots j_1}(x_L, \cdots, x_{j_{s+1}+1}, x_{j_{s+1}}, x_{j_s}, \cdots, x_{j_1}, x_0)$$
$$= \left(\frac{t_{j_{s+1}}T(j_{s+1}-1, j_s+1)^2}{\nu}\right)\prod_{r=1}^{s}\left(\frac{t_{j_r}}{\nu}T(j_r-1, j_{r-1}+1)^2\right)$$
$$\times\, p_{j_{s+1}}(x_{j_{s+1}}, x_{j_s})\prod_{r=1}^{s} p_{j_r}(x_{j_r}, x_{j_{r-1}})\,.$$

This proves (5.10) for $s+1$. Lemma 5.1 has been proved.

We have completed proof of Theorem 2.

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