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LATTICE PATH PROOF OF THE RIBBON DETERMINANT FORMULA FOR SCHUR FUNCTIONS

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In this note we give a lattice path proof of the ribbon determinant formula for Schur functions ((1) below) which was originally formulated and proved in [2].

We make use of the terminology and notation of [2]. In particular, we use the French notation of partitions and diagrams, and identify a partition with its diagram. The ribbon determinant formula for a Schur function reads:

$$S_J = \det\,(S_{\Theta_i^+ \& \Theta_j^-})_{1 \le i, j \le p}\,, \tag{1}$$

where J is a partition, $(\Theta_p, \cdots, \Theta_1)$ is the ribbon decomposition of J with Θ_i^+ resp. Θ_i^- the upper resp. lower part of Θ_i, and S_J is the Schur function for J.

EXAMPLE 1. A ribbon decomposition with $p = 3$.

(2)
```
      # # #
      $ $ &
J =   % & # # #            ,
      & $ $ $ # #
```

& = diagonal box,

```
                          # # #                                # # #
Θ3+ = # # #,   Θ3- =          # #,   Θ3 = Θ3+&Θ3- =              &
                                                                 # # #      ,
                                                                     # #

                                                     $ $
Θ2+ = $ $,   Θ2- = $ $ $,   Θ2 = Θ2+&Θ2- =           &        ,
                                                     $ $ $

                                                   %
Θ1+ = %,   Θ1- = empty,   Θ1 = Θ1+&Θ1- =           &  .
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Take the outermost ribbon Θ_p. We start from the leftmost and top-

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most box. Assign letter a to the first box. To a box other than the first one, if the box is on the right of the preceding one, then assign letter a; if the box is below the preceding one, then assign letter b. We thus obtain a sequence of letters a and b, which we call the *assigning sequence* for J.

EXAMPLE 2. To the ribbon Θ_3 of Example 1 corresponds the assigning sequence

$$a\ a\ a\ b\ b\ a\ a\ b\ a\,.$$

Note that an outermost ribbon determines a partition J uniquely. For example, the ribbon Θ_3 of Example 1 gives the partition (2) and its *assigning diagram* defined as

$$\begin{array}{ccccccccc} a & a & a & b & b & a & a & b & a \\ & a & a & b & b & a & a & & \\ & & a & b & & & & & \end{array}$$

in which the second resp. third row corresponds to the second resp. third outer ribbon. In a partition, the boxes on a particular line parallel to the diagonal assign the same letter; for instance, the diagonal boxes of (2) all assign letter b, and the boxes just above the diagonal all assign letter a. In the assigning diagram, the letters corresponding to the boxes on a particular line parallel to the diagonal are defined to be placed in the same column so that in a particular column we have all a's or all b's. We see that giving an outermost ribbon completely determines a partition and its assigning diagram.

We work with lattice paths in $\mathbb{Z}\times\mathbb{N}$ taking up-vertical, down-vertical, horizontal, and south-east steps which are as vectors (0, 1), (0, -1), (1, 0) and (1, -1) respectively. An up- or down-vertical step has weight 1, and both a horizontal step of height k and a south-east step at height k have weight u_k, which is an indeterminate.

Let θ_i^+ resp. θ_i^- be the number of boxes in Θ_i^+ resp. Θ_i^-. We take as starting points $\alpha_i := (-\theta_i^+ - 1, 1)$ $(i = 1, \cdots, p)$ and as ending points $\beta_i := (\theta_i^-, 1)$ $(i = 1, \cdots, p)$. We consider the lattice paths whose steps are subject to the following *conditions*:

(i) Let c_j be the jth letter of the assigning sequence for J. If $c_j = a$ resp. b, then a step starting on the line $x = -\theta_p^+ - 2 + j$ and ending on the line $x = -\theta_p^+ - 1 + j$ (x being the first coordinate) must be horizontal resp. south-east. (cf. definition of assigning diagram)

(ii) A down- resp. up-vertical step must not preceed a horizontal resp. south-east step.

We call the lattice paths under these conditions simply *paths*.

Let P_π be the set of all p-tuples of paths $s=(s_1, \cdots, s_p)$ with s_i a path from α_i to $\beta_{\pi(i)}$, where π is a permutation of $\{1, 2, \cdots, p\}$, and let $P := \bigcup_{\pi\in G} P_\pi$, where G is the symmetric group on $\{1, 2, \cdots, p\}$.

We first show that

$$\det(S_{\theta_i^+ \& \theta_j^-})_{1\le i,j\le p} = \sum_{s\in P} \mathrm{wt}(s)\,, \tag{3}$$

where $\mathrm{wt}(s) = \mathrm{sgn}(\pi)\,\mathrm{wt}(s_1)\cdots\mathrm{wt}(s_p)$ with $s=(s_1, \cdots, s_p)\in P_\pi$, and $\mathrm{wt}(s_i)$ is the product of the weights of all the steps appearing in s_i.

Proof of (3). The left-hand side of (3) is equal to

$$\sum_{\pi\in G} \mathrm{sgn}(\pi) S_{\theta_1^+ \& \theta_{\pi(1)}^-} \cdots S_{\theta_p^+ \& \theta_{\pi(p)}^-}\,.$$

It suffices to show that

$$S_{\theta_i^+ \& \theta_{\pi(i)}^-} = \sum_{s_i\in P_{\pi(i)}} \mathrm{wt}(s_i) \qquad (i=1, \cdots, p) \tag{4}$$

where $P_{\pi(i)}$ is the set of all paths from α_i to $\beta_{\pi(i)}$. Let T_i be the set of all column-strict tableaux with shape $\Theta_i^+ \& \Theta_{\pi(i)}^-$. Then the left-hand side of (4) is equal to $\sum_{t\in T_i} \mathrm{WT}(t)$, where WT is the usual indeterminate weighting for tableaux [3, 4], so that we have only to give a weight-preserving bijection between $P_{\pi(i)}$ and T_i. Let $s_i \in P_{\pi(i)}$. Read the 2nd coordinates of the ending points of all the horizontal and south-east steps appearing in s_i in order from left to right. The number of such 2nd coordinates is $\theta_{\pi(i)}^- + \theta_i^+ + 1$, which is equal to the number of boxes in $\Theta_i^+ \& \Theta_{\pi(i)}^-$. Write down these 2nd coordinates one by one in the boxes in order from the leftmost and topmost. The condition (i) corresponds to the condition that in a particular column of the assigning diagram for J we have all a's or all b's, and the latter describes the ribbon decomposition of J. The condition (ii) corresponds to the condition that the array of integers on $\Theta_i^+ \& \Theta_{\pi(i)}^-$ gives a column-strict tableaux with shape $\Theta_i^+ \& \Theta_{\pi(i)}^-$. Hence the integer sequence read off from s_i fits into $\Theta_i^+ \& \Theta_{\pi(i)}^-$ and yields a tableau $t\in T_i$.

Conversely, let $t \in T_i$. Read the integers in the boxes in order from the leftmost and topmost. If the first box carries integer k, then we draw a horizontal step from $(-\theta_i^+ - 1, k)$ to $(-\theta_i^+, k)$. For $j=2, \cdots, \theta_{\pi(i)}^- + \theta_i^+$

$+1$, if the jth box is on the right of the preceding one and carries integer k, then we draw a horizontal step from $(-\theta_i^+ - 2 + j, k)$ to $(-\theta_i^+ - 1 + j, k)$, or if the jth box is under the preceding one and carries integer k, then we draw a south-east step from $(-\theta_i^+ - 2 + j,\ k + 1)$ to $(-\theta_i^+ - 1 + j, k)$. Adding the necessary down- or up-vertical steps, we obtain a path $s_i \in P_{\pi(i)}$; the condition (i) is automatically satisfied and the condition (ii) corresponds to the assumption that t is a ribbon column-strict tableau. (See the last part of the reverse implication.)

We next show that

$$S_J = \sum_{s \in \mathrm{NP}} \mathrm{wt}(s)\,, \tag{5}$$

where NP denotes the set of all nonintersecting p-tuples of paths $s = (s_1, \cdots, s_p)$ with s_i a path from α_i to β_i $(i = 1, \cdots, p)$.

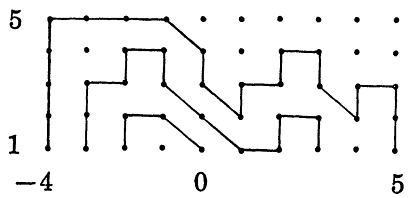
Proof of (5). Let T be the set of all column-strict tableau with shape J. Then the left-hand side of (5) is equal to $\sum_{t \in T} \mathrm{WT}(t)$ (see the proof of (3)), so that we have only to construct a weight-preserving bijection between NP and T. Let $s = (s_1, \cdots, s_p) \in \mathrm{NP}$. The proof of (3) with $\pi = \mathrm{id}$ gives a ribbon column-strict tableau t_i with shape $\Theta_i = \Theta_i^+ \& \Theta_i^-$ corresponding to s_i $(i = 1, \cdots, p)$. We compose an array t of integers with shape J from t_i $(i = 1, \cdots, p)$ according to the ribbon decomposition $(\Theta_p, \cdots, \Theta_1)$ of J. Since s is nonintersecting, t is in fact a column-strict tableau, i.e. $t \in T$. (See Example 3 below.)

Conversely, let $t \in T$. We can reverse the above procedure to obtain $s \in \mathrm{NP}$ corresponding to the tableau t.

EXAMPLE 3. To the tableau

5	5	5			
3	4	4			
2	2	2	3	4	
1	1	1	2	2	3

with shape (2) corresponds the nonintersecting 3-tuple of paths:



Finally we give:

Proof of (1). In view of (3) and (5), it suffices to show that

$$\sum_{s\in P} \mathrm{wt}(s) = \sum_{s\in \mathrm{NP}} \mathrm{wt}(s)\,, \tag{6}$$

which we see using the Gessel-Viennot method [1, 5]; in fact we can apply [1, Corollary 2] or [5, Theorem 1.2] to obtain (6) by noting that, if $s \in P_\pi$ is nonintersecting, then π must be the identity permutation.

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