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ITO'S FORMULA AND LEVY'S LAPLACIAN II

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Dedicated to Professor Takeyuki Hida

§1. Introduction

The white noise calculus was initiated by T. Hida in 1970 in his Princeton University Mathematical Notes [3]. Recent development of the theory shows that the Laplacian plays an essential role in the analysis in question. Indeed, several kinds of Laplacians should be introduced depending on the choice of the class of white noise functionals to be analysed, as can be seen in [4], [13], [18] and so forth. Among others, we should like to emphasize the importance of the infinite dimensional Laplace-Beltrami operator, Volterra's Laplacian and Lévy's Laplacian (See [13], [18] and [20]).

In this paper, we shall discuss characteristic properties of Lévy's Laplacian Δ_L and some of related topics as well as its applications, in particular, to form explicit solutions of a Schrödinger equation, where the Laplacian naturally appears.

Following [6] and [15], we shall first introduce, in Section 2, the space $(E)^*$ of generalized white noise functionals and the usual tools of the analysis like the S-transform, the U-functional, and the $\mathscr{T}$-transform on $(E)^*$. It is noted that Δ_L acts effectively on a certain subspace of $(E)^*$ and it does annihilate ordinary white noise functionals. We then come to the calculus of $(E)^*$-functionals in terms of the white noise $\dot{B}(t)$, $t \in T$, (T is an interval) which is now thought of as a member of the variables of white noise functionals.

We establish eigenfunctionals of Δ_L and deal with the heat equation satisfied by the expectation functional of the delta functional in Section 3. Lévy's group and an algebra generated by infinitesimal generators of the infinite dimensional rotation group are dealt with to some extent

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in Section 4. With these backgrounds characteristic properties of Lévy's Laplacian will be given there.

There are many applications of the Lévy's Laplacian. The most interesting one seems to be the Feynman path integral to find the Schrödinger equation satisfied by the propagator. These topics will be discussed in Section 5, however we now pause to show an explicit form of the Schrödinger equation:

$$\frac{1}{i}\cdot\frac{\partial\Phi(\dot{b},t)}{\partial t}=\frac{\hbar}{2m}(\Delta_L+\Delta_V)\Phi(\dot{b},t)+\frac{1}{\hbar}S^{-1}V[\dot{b}]\diamond\Phi(\dot{b},t),$$

where Δ_V is Volterra's Laplacian, S is the S-transform ([15]) and $\diamond$ denotes the Wick product ([23]).

The last section is devoted to the investigation of Itô's formula for generalized white noise functionals, where Lévy's Laplacian appears in the drift term:

$$d\Phi(B_{\cdot}(t))=\int_T \partial_{t-,x}\Phi(B_{\cdot}(t))\diamond dB_x(t)dx+\frac{1}{2}\Delta_L\Phi(B_{\cdot}(t))dt,$$

where $\partial_{t-,x}$ is the $W(t-,x)$-differentiation (See Sections 4 and 6).

We have a hope, as is mentioned in the concluding remark, that more general Itô's formula can be found in line with this approach.

§2. Generalized white noise functionals

In this section, we introduce the space of generalized white noise functionals, following [6], [15] and [23] (See also, [4], [5], [16], etc). Let T be an interval in $\mathbf{R}^d$ $(d\geq 1)$ and let $L^2(T^n)$ be the Hilbert space of real square-integrable functions on T^n with inner product $(\cdot,\cdot)_n$. Start with a Gel'fand triple

$$E(T^n)\subset L^2(T^n)\subset E^*(T^n),$$

associated with a countable system of consistent Hilbertian norms $\|\cdot\|_{n,p}=\sqrt{(\cdot,\cdot)_{n,p}}$, $p\in Z$. We can assume without loss of generality that there exists a positive number ρ less than 1 such that

$$\rho\|\xi\|_{n,p+1}\geq\|\xi\|_{n,p}\qquad\text{for any } \xi\in E(T^n),\ p\in Z, \text{ (See [15])}.$$

The measure μ on $E^*(T)$ of Gaussian white noise is given by the characteristic functional

$$C(\xi) \equiv \int_{E^*(T)} \exp\{i\langle x, \xi\rangle\} d\mu(x) = \exp\left\{-\frac{1}{2}\|\xi\|^2\right\}, \qquad \xi \in E(T),$$

$$\|\cdot\|: \text{ the } L^2(T)\text{-norm}.$$

Set $(L^2) = L^2(E^*(T), \mu)$ and define the *S-transform* by

$$S\varphi(\xi) = C(\xi)\int_{E^*(T)} \exp\{\langle x, \xi\rangle\}\varphi(x)d\mu(x), \qquad \varphi \in (L^2).$$

The Hilbert space (L^2) admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals φ of order n.

Let $E_p(T^n)$ be the completion of $E(T^n)$ with respect to the inner product $(\cdot, \cdot)_{n,p}$. Then for any positive integer p, $E_{-p}(T^n)$ is the dual space of $E_p(T^n)$. Define

$$H_n^{(p)} = \{\varphi \mid \varphi \text{ is in } H_n \text{ with kernel in } \hat{E}_p(T^n)\},$$

$$\hat{E}_p(T^n) = \{f \in E_p(T^n) \mid f \text{ is symmetric}\}$$

for $p > 0$, and construct the space

$$(E_p) = \bigoplus_{n=0}^{\infty} H_n^{(p)}.$$

Let (E_{-p}) be the dual space of (E_p) for $p > 0$. Denote the projective limit space and the inductive limit space of the (E_p), $p \in Z$, by (E) and $(E)^*$, respectively. Then (E) is a nuclear space and $(E)^*$ is nothing but the dual space of (E). The space $(E)^*$ is said to be the space of *generalized (white noise) functionals.*

Since $\exp\langle\cdot, \xi\rangle \in (E)$, the S-transform is extended to an operator U defined on $(E)^*$:

$$U\Phi(\xi) = C(\xi)\langle\langle \Phi, \exp\langle\cdot, \xi\rangle\rangle\rangle, \qquad \xi \in E(T),$$

where $\langle\langle\cdot, \cdot\rangle\rangle$ is the canonical pairing of (E) and $(E)^*$. We call $U\Phi$ the *U-functional* of Φ. Moreover, we have

$$UH_n^{(-p)} = \{\langle F, \xi^{\otimes n}\rangle \mid F \in \hat{E}_{-p}(T^n)\}.$$

Using the U-functional, the $\mathscr{T}$-transform is defined on $(E)^*$ by

$$\mathscr{T}\Phi(\xi) = C(\xi)^{-1}U\Phi(i\xi), \qquad \xi \in E(T).$$

§ 3. Lévy's Laplacian

As there have been many equivalent definitions for the Lévy's Laplacian, we adopt one of them as follows. In order to make the present note be self-contained, we begin with definitions of Laplacian operators of Lévy and Volterra. Let U be a function defined on $E=E(T)$. The function U is said to be functional differentiable at $\xi \in E$ if there exists a generalized function $U'(\xi;\cdot) \in E^*$ such that the first variation $\delta U(\xi;\eta)=(d/d\lambda)U(\xi+\lambda\eta)|_{\lambda=0}$ is expressed in the form $\delta U(\xi;\eta)=\int_T U'(\xi;t)\eta(t)dt$ for every $\eta \in E$. We call $U'(\xi;t)$ functional derivative of U at $\xi \in E$, and denote it by $(\delta U(\xi))/\delta\xi(t)$. The second variation, if exists, is expressed in the form

$$\delta^2 U(\xi;\eta,\zeta)=\langle U''(\xi),\ \eta\otimes\zeta\rangle\,,$$

where U'' is the second functional derivative.

Definition. (I) A C^2-function U on E (See [17].) is called *LV-functional* if the second functional derivative U'' is the sum of U_s'' and U_r'' such that

$$\langle U_s''(\xi),\ \eta\otimes\zeta\rangle=\int_T U_s''(\xi;t)\eta(t)\zeta(t)dt\,,\qquad U_s''(\xi;\cdot)\in L^1(T)\,,$$

$$\langle U_r''(\xi),\ \eta\otimes\zeta\rangle=\int_{T\times T} U_r''(\xi;s,t)\eta(s)\zeta(t)dsdt\,,$$

$$U_r''(\xi;\cdot,\cdot)\text{: trace class kernel}\,.$$

The functionals U_s'' and U_r'' are called the *singular part* and the *regular part* of U'', respectively.

(II) For the LV-functional U,

$$\tilde{\Delta}_L U=\int_T U_s''(\xi;t)dt\,,$$

and

$$\tilde{\Delta}_V U=\int_T U_r''(\xi;t,t)dt$$

are defined which acting on the space of functional of ξ.

We are now ready to define Lévy's Laplacian Δ_L and Volterra's Laplacian Δ_V. Let Φ be a generalized white noise functional whose U-functional is an LV-functional. As $[U\Phi]_s''(\xi;\cdot)$ is integrable and $[U\Phi]_r''(\xi;\cdot,\cdot)$ is of trace class, Δ_L and Δ_V are defined by

$$\Delta_L \Phi = U^{-1}\left[\int_T [U\Phi]''_s(\xi; t)dt\right],$$

and

$$\Delta_V \Phi = U^{-1}\left[\int_T [U\Phi]''_r(\xi; t, t)dt\right].$$

DEFINITION. The operators Δ_L and Δ_V are called *Lévy's Laplacian* and *Volterra's Laplacian*, respectively.

The domains of the above Laplacians Δ_L and Δ_V are both rich enough in the space $(E)^*$ of generalized white noise functionals as illustrated by the following fact. We introduce the following classes $\mathfrak{N}$ and $\mathscr{A}$ of functionals on E:

The class $\mathfrak{N}$ is the collection of (finite) linear combinations of

$$\int_{T^n} f(t_1, \cdots, t_n)\xi(t_1)^{p_1} \cdots \xi(t_n)^{p_n} dt_1 \cdots dt_n \,; \qquad f \in L^1(T^n)\,,$$
$$p_1, \cdots, p_n \in \mathbf{N} \cup \{0\}\,, \qquad n = 0, 1, 2, \cdots,$$

and the class $\mathscr{A}$ is the collection of (finite) linear combinations of

$$\exp\left(\int_T f(t)\xi(t)^p dt\right); \quad f \in L^1(T)\,, \quad p = 0, 1, 2\,.$$

A member of $\mathfrak{N}$ is called a *normal functional.*

Remark. The kernel f in the element of $\mathfrak{N}$ is not symmetric in general.

The characterization of $(S)^*$ due to Streit and Potthoff [23] proves that both $\mathfrak{N}$ and $\mathscr{A}$ are subspaces of U-functionals. Furthermore, it is proved that all the members of $\mathfrak{N}$ and $\mathscr{A}$ are LV-functionals and that they are in the domain of $\tilde{\Delta}_L$. We therefore see that $U^{-1}\mathfrak{N}$ and $U^{-1}\mathscr{A}$ are in the domain of Δ_L.

We will see profound properties of Lévy's Laplacian in what follows.

PROPOSITION 3-1. *Let T be* $[0, 1]$. *Then the exponential functional*

$$\varphi_c =: \exp\left\{\frac{c}{2(1+c)}\int_0^1 x(t)^2 dt\right\}:$$

is positive and is an eigenfunctional of Δ_L with eigenvalues c for any real number $c \neq -1$.

Proof. It is obvious that φ_c is an eigenfunctional with eigenvalue c.

The positivity of the functional φ_c follows from the fact that its $\mathscr{T}$-transform

$$\exp\left\{\frac{1-c}{2}\int_0^1 \xi(t)^2 dt\right\}$$

is a positive definite function of $\xi \in E[0, 1]$ (See [29]). (Q.E.D.)

Remark. A generalized functional Φ in $(E)^*$ is said to be positive if $\langle\langle \Phi, \varphi\rangle\rangle \geq 0$ for all φ in (E) such that its continuous version $\tilde{\varphi}$ is positive on E^*. It is well-known that the positivity of Φ is equivalent to the fact that its $\mathscr{T}$-transform $\mathscr{T}\Phi(\xi)$ is positive-definite (also, see [29]).

PROPOSITION 3-2 (cf. [20]). *Let U be an LV-functional and ψ be a C^2-function on* $\mathbf{R}$. *Then it follows that*

$$\tilde{\Delta}_L \psi(U) = \psi'(U)\tilde{\Delta}_L U .$$

Proof. By definition, we have

$$\frac{\delta}{\delta\xi(x)}\psi(U(\xi)) = \psi'(U(\xi))\frac{\delta}{\delta\xi(x)}U(\xi) ,$$

and hence

$$\begin{aligned}\frac{\delta^2}{\delta\xi(x)\delta\xi(y)}\psi(U(\xi)) &= \psi''(U(\xi))\frac{\delta}{\delta\xi(x)}U(\xi)\frac{\delta}{\delta\xi(y)}U(\xi)\\ &\quad + \psi'(U(\xi))(U_s''(\xi; x)\delta(x-y) + U_r''(\xi; x, y)) .\end{aligned}$$

Consequently,

$$\tilde{\Delta}_L \psi(U(\xi)) = \psi'(U(\xi))\int_T U_s''(\xi; x)dx = \psi'(U(\xi))\tilde{\Delta}_L U(\xi) . \qquad \text{(Q.E.D.)}$$

As an easy consequence of Proposition 3-2 we claim the following: If U is a polynomial in ξ of degree 2 and if ψ is the exponential function, then $\psi(U)$ is an eigenfunctional of $\tilde{\Delta}_L$ belonging to eigenvalue $\tilde{\Delta}_L U$.

§ 4. Some properties of Lévy's Laplacian

In order to define the operator $\gamma_{s,t}$ following [11] and [13], we prepare some basic notions. Let Φ be in $(E)^*$. If, for all t, there exists Φ_t in $(E)^*$ such that $U\Phi_t(\xi) = (\delta U\Phi(\xi))/\delta\xi(t)$, then the Φ_t is denoted by $\partial_t\Phi$. In the case of Streit-Potthoff's space $(S)^*$, the following result has been obtained in [7].

PROPOSITION 4-1. *The U-functional $U\Phi$ is functional differentiable for each Φ in $(S)^*$.*

Proof. The proof follows from a characterization of Hida distributions in [23]. (Q.E.D.)

The adjoint operator ∂_t^* of ∂_t is defined as

$$\langle\langle \partial_t^* \Phi, \varphi \rangle\rangle = \langle\langle \Phi, \partial_t \varphi \rangle\rangle\,, \quad \Phi \in (E)^*\,, \quad \varphi \in (E)\,.$$

It is also known that $\partial_t^* + \partial_t$ is the multiplication operator by the coordinate function $x(t)$ (See [6], [16]). Hence,

$$\gamma_{s,t} = x(s)\partial_t - x(t)\partial_s$$

is viewed as an analogue of an infinitesimal generator of two dimensional rotations on the $(x(s), x(t))$-plane. The operator $\gamma_{s,t}$ can be expressed in the form

$$\gamma_{s,t} = \partial_s^* \partial_t - \partial_t^* \partial_s\,.$$

The Volterra's Laplacian Δ_V and the number operator Δ_∞ have been characterized by using the operators $\gamma_{s,t}$, $s, t \in T$ ([11], [13]). In fact, they are uniquely determined (up to constant) in such a way that Δ_V is a quadratic form of ∂_t's which commutes with all the $\gamma_{s,t}$ and that Δ_∞ is a bilinear form of ∂_t^*'s and ∂_t's which also commutes with all the $\gamma_{s,t}$.

Similar, but somewhat weaker characteristic properties of Lévy's Laplacian can be given in terms of the subgroup, called Lévy's group, of an infinite dimensional rotation group. On the other hand, it has been shown that Lévy's Laplacian commutes with Lévy's group in [12]. Set

$$\begin{aligned} O(E) = \{g \,|\, g &\text{ is a linear homeomorphism of } E \text{ and} \\ &\|g\xi\| = \|\xi\| \text{ for every } \xi \in E\}\,. \end{aligned}$$

Then, $O(E)$ forms a group under the usual product, and it is called the infinite dimensional rotation group. Let $\{\xi_n\} \subset E = E(T)$ be an orthonormal basis for $L^2(T)$ and let π be a permutation of $\{0, 1, 2, \cdots\}$. Define an operator g_π on E by

$$g_\pi \xi = \sum_{n=0}^{\infty} a_n \xi_{\pi(n)} \qquad \text{for } \xi = \sum_{n=0}^{\infty} a_n \xi_n \in E\,.$$

Lévy's group $\mathscr{G}$ is defined as follows:

$$\mathscr{G} = \left\{ g_\pi \,\middle|\, \lim_{N\to\infty} \frac{1}{N} |\{1 \le n \le N;\ \pi(n) > N\}| = 0 \right\}.$$

Let T be a finite interval, and let Ψ be given by

$$\Psi = \sum_{n=0}^{\infty} :\langle x, \xi_n\rangle^2: .$$

This Ψ is regarded as a generalization of the Euclidian metric in a finite dimensional space.

PROPOSITION 4-2. i) *Ψ is a member of $(E)^*$ and is invariant under $\mathscr{G}$,*

ii) *Ψ has a representation of the form*

$$\Psi(x) = \int_T :x(t)^2: dt = \int_T \partial_t^{*2} 1 dt ,$$

iii) *The operator $\bar{\Delta}_L = \lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} (\partial/\partial x_n)^2$ with $x_n = \langle x, \xi_n\rangle$, is applied to Ψ and*

$$\bar{\Delta}_L \Psi = \Delta_L \Psi$$

holds.

Proof. i) The S-transform of $\Psi_N = \sum_{n=0}^{N-1} :\langle x, \xi_n\rangle^2:$ is $\sum_{n=0}^{N-1} (\xi, \xi_n)^2$ which converges to $\|\xi\|^2$. We can further show that Ψ_N converges in $(E)^*$. The $\mathscr{G}$-invariance is almost obvious.

ii) The result follows from the fact that the S-transform of $\int_T :x(t)^2: dt$ is $\int_T \xi(t)^2 dt$.

iii) is obvious. (Q.E.D.)

It is noted that the functional Ψ satisfies the following trivial relation

$$\gamma_{s,t} \Delta_L \Psi = \Delta_L \gamma_{s,t} \Psi .$$

As for above assertion iii), we can also show that

$$\bar{\Delta}_L \Phi = \Delta_L \Phi$$

holds for any normal functional Φ.

§5. Lévy's Laplacian in the Schrödinger equation

One of the important applications of Lévy's Laplacian can be seen in the theory of path integrals.

Consider the Feynman path integral intuitively expressed in a formal form

$$U(\phi, t) = \int \exp\left[\frac{im}{2\hbar}\int_0^t\int_0^1\left(\frac{\partial q(u, x)}{\partial u}\right)^2 dxdu - \frac{i}{\hbar}\int_0^t V[q](u)du\right] \prod_{(u,x)=(0,0)}^{(t,1)} dq(u, x),$$

with a potential $V[q](\cdot)$, where q stands for a possible trajectory. We understand this integral as the following "expectation" with respect to the white noise measure μ on $E^*(T) = E^*(\mathbf{R} \times [0, 1])$

$$U(\phi, t) = E\left[:\mathcal{N} \exp\left\{\frac{i}{\hbar}\int_0^t L(q, \dot{q})(u)du\right\}:\right],$$

E denoting the expectation, where $\mathcal{N}$ is a normalizing factor and $L(q, \dot{q})$ is the Lagrangian:

$$L(q, \dot{q})(u) = \frac{m}{2}\int_0^1\left(\frac{\partial q(u, x)}{\partial u}\right)^2 dx - V[q](u).$$

Getting the expectation can be realized by taking q to be a trajectory interfered with by the fluctuation denoted by $B_x(u)$ such that $(d/dt)B_x(t) = W(t, x)$ is a white noise with parameter set $T = \mathbf{R} \times [0, 1]$.

Take a complete orthonormal system $\{e_n; e_n \in E[0, 1], n = 0, 1, 2, \cdots\}$ for $L^2[0, 1]$ and set $B^{(n)}(t) = \langle W, 1_{[0,t]} \otimes e_n\rangle$, $t \geq 0$, $n = 0, 1, 2, \cdots$. Then $\{B^{(n)}(t)\}$ is an infinite sequence of independent one-dimensional Brownian motions, where $1_{[0,t]}$ is the indicator function of $[0, t]$. The $B_x(t)$ is expanded into the series

$$B_x(t) = \sum_{n=0}^{\infty} B^{(n)}(t)e_n(x), \quad x \in [0, 1], \quad t \geq 0.$$

We are now ready to introduce trajectories q consisting a sure path ϕ in $E[0, 1]$ plus Gaussian fluctuation:

(5-1)
$$q(u, x) = \frac{u}{t}\phi(x) + \left(\frac{m}{\hbar}\right)^{1/2}\left(B_x(u) - \frac{u}{t}B_x(t)\right),$$
$$0 \leq u \leq t, \quad 0 \leq x \leq 1.$$

Thus the expectation $U(\phi, t)$ is expressed in the form

$$U(\phi, t) = E\left[:\exp\left\{\frac{im}{2\hbar t}\int_0^1 \phi(x)^2 dx + \frac{1+i}{2}\int_0^t\int_0^1 \dot{B}_x(u)^2 dxdu - \frac{i}{2t}\int_0^1 B_x(t)^2 dx\right\}\right.$$
$$\left.\times\exp\left\{-\frac{i}{\hbar}\int_0^t V[q](u)du\right\}:\right].$$

As in [28], after the functional

$$\exp\left\{\frac{1+i}{2}\int_0^t\int_0^1 \dot{B}_x(u)^2dxdu - \frac{i}{2t}\int_0^1 B_x(t)^2dx\right\}$$

is renormalized, the term $(i/2t)\int_0^1 B_x(t)^2dx$ is not important in the evaluation. Thus, the functional $U(\phi, t)$ can be expressed in the form

(5-2)

$$U(\phi, t) = c(t)\exp\left\{\frac{im}{2\mathfrak{h}t}\int_0^1 \phi(x)^2dx\right\} \times \left\langle\!\left\langle :\exp\left\{\frac{1+i}{2}\int_0^t\int_0^1 \dot{B}_x(u)^2dxdu\right\}:,\ \exp\left\{-\frac{i}{\mathfrak{h}}\int_0^t V[q](u)du\right\}\right\rangle\!\right\rangle,$$

where $c(t)$ is a constant depending on t and is independent of the potential V. Since $U(\phi, t)$ does not exist in the case of the polynomial V of degree 2, we are led to define a renormalization $\mathscr{R}U(\phi, t)$ of $U(\phi, t)$ as follows:

We put

$$U_N(\phi, t) = c_N(t)\exp\left\{\frac{im}{2\mathfrak{h}t}\sum_{n=0}^{N-1}\phi_n^2\right\} \times \left\langle\!\left\langle :\exp\left\{\frac{1+i}{2}\sum_{n=0}^{N-1}\int_0^t \dot{B}^{(n)}(u)^2du\right\}:,\ \exp\left\{-\frac{i}{\mathfrak{h}}\int_0^t V[q_N](u)du\right\}\right\rangle\!\right\rangle,$$

where $\phi_n = (\phi, e_n)$ and

$$q_N(u) = \frac{u}{t}\sum_{n=0}^{N-1}\phi_n + \left(\frac{\mathfrak{h}}{m}\right)^{1/2}\sum_{n=0}^{N-1}\left(B^{(n)}(u) - \frac{u}{t}B^{(n)}(t)\right).$$

In case of $V = 0$, we have

$$U_N(\phi, t) = c_N(t)\exp\left\{\frac{im}{2\mathfrak{h}t}\sum_{n=0}^{N-1}\phi_n^2\right\}.$$

As the chain rule

$$\int U_N(\phi^{(2)} - \phi^{(1)}, t_2 - t_1)U_N(\phi^{(3)} - \phi^{(2)}, t_3 - t_2)d\phi^{(2)} = U_N(\phi^{(3)} - \phi^{(1)}, t_3 - t_1)$$

is valid for every particle, we get

$$c_N(t) = \left(\frac{m}{2\pi i\mathfrak{h}t}\right)^{N/2}.$$

A renormalization $\mathscr{R}U(\phi, t)$ of $U(\phi, t)$ is defined by

$$\mathscr{R}U(\phi, t) = \lim_{N\to\infty} \frac{U_N(\phi, t)}{U_{N-1}(0, t)}.$$

This definition is consistent with Kuo's Fourier transform [19]. Thus we can take

$$c(t) = \left(\frac{m}{2\pi i \mathfrak{h} t}\right)^{1/2}.$$

The mean of Feyman's functional I in the present field is now given by

$$E[I] = E\left[:\exp\left\{\frac{im}{2\mathfrak{h}}\int_0^t \left(\frac{\partial q(u, x)}{\partial u}\right)^2 dxdu + \frac{1}{2}\int_0^1\int_0^t \dot{B}_x(u)^2 dudx\right\}\delta(q(t) - \phi):\right]$$

as a generalization of L. Streit-T. Hida formula. We have the following.

PROPOSITION 5-1. *The renormalized mean of Feyman's functional I is expressible as*

$$\mathscr{R}E(I) = c(t)\exp\left\{\frac{im}{2\mathfrak{h}t}\|\phi\|^2\right\}, \qquad \|\cdot\|:\ L^2[0, 1]\text{-norm}.$$

The next Proposition is applicable to the calculation of the functional $U(\phi, t)$ with potential $V = C\int_0^1 q(\cdot, x)dx$, where C is a constant and q is given by (5-1).

PROPOSITION 5-2. *The $\mathscr{T}$-transform for*

$$:\exp\left\{\frac{1+i}{2}\sum_{n=0}^{N-1}\int_0^t \dot{B}^{(n)}(u)^2 du\right\}:$$

is expressed by

$$\mathscr{T}\left[:\exp\left\{\frac{1+i}{2}\sum_{n=0}^{N-1}\int_0^t \dot{B}^{(n)}(u)^2 du\right\}:\right](f)$$

$$= \exp\left\{-\frac{1}{2}\int_0^t\int_0^1 f(u, x)^2 dudx + \frac{1-i}{2}\sum_{n=0}^{N-1}\int_0^t \left(\int_0^1 f(u, x)e_n(x)dx\right)^2 du\right\}. \tag{5-3}$$

Proof. The functional

$$:\exp\left\{\frac{1+i}{2}\sum_{n=0}^{N-1}\int_0^t \dot{B}^{(n)}(u)^2 du\right\}:$$

is approximated by

$$\Phi_{\Pi_K}(W) \equiv \exp\left\{\frac{1+i}{2}\sum_{n=0}^{N-1}\sum_{k=1}^{K}\left(\frac{\langle W, 1_{\Delta_k}\otimes e_n\rangle}{|\Delta_k|}\right)^2|\Delta_k|\right\}.$$

It is shown that

$$\begin{aligned}
&\mathscr{T}\left[\frac{\Phi_{\Pi_K}}{E[\Phi_{\Pi_K}]}\right](\xi)\\
&\quad = \exp\left\{-\frac{1}{2}\|\xi\|^2 + \frac{1-i}{2}\sum_{n=0}^{N-1}\sum_{k=1}^{K}\left(\frac{1}{|\Delta_k|}\int_{\Delta_k}\int_0^1 \xi(u, x)e_n(x)dxdu\right)^2|\Delta_k|\right\}\\
&\quad = \exp\left\{-\frac{1}{2}\|\xi\|^2 + \frac{1-i}{2}\sum_{n=0}^{N-1}\sum_{k=1}^{K}\left(\frac{1}{|\Delta_k|}\int_{\Delta_k}\int_0^1 \xi(u, x)e_n(x)dxdu\right)^2|\Delta_k|\right\}.
\end{aligned}$$

Let K tend to ∞, we obtain (5-3). (Q.E.D.)

We can see Lévy's Laplacian arises in the Schrödinger equation for $\Phi(\dot{b}, t) = U^{-1}[\mathscr{R}U(\cdot, t)]$.

THEOREM 5-1. *The functional* $\Phi(\dot{b}, t)$ *with a polynomial potential* V *of degree* 2 *satisfies the equation*:

$$\frac{1}{i}\cdot\frac{\partial\Phi(\dot{b}, t)}{\partial t} = \frac{\mathfrak{h}}{2m}(\Delta_L + \Delta_V)\Phi(\dot{b}, t) + \frac{1}{\mathfrak{h}}U^{-1}V[\dot{b}]\diamond\Phi(\dot{b}, t), \tag{5-4}$$

where $\Phi\diamond\Psi$, $\Phi, \Psi \in (E)^*$, *means the Wick product, defined by*

$$U[\Phi\diamond\Psi](\xi) = U[\Phi](\xi)U[\Psi](\xi) \qquad (\textit{See } [23]).$$

Proof. It is sufficient to prove (5-4) for a case of

$$V[q](u) = \frac{g}{2}\int_0^1 q(u, x)^2dx, \qquad g\text{: a constant}.$$

The functional $U_N(\phi, t)$ is expressed in the form

$$\begin{aligned}
U_N(\phi, t) = c_N(t)\exp&\left\{\left(\frac{im}{2\mathfrak{h}t} - \frac{igt}{6\mathfrak{h}}\right)\sum_{n=0}^{N-1}\phi_n^2\right\}\\
&\times\left\langle\!\!\left\langle :\exp\left\{\frac{1+i}{2}\sum_{n=0}^{N-1}\int_0^t \dot{B}^{(n)}(u)^2du\right\}:,\right.\right.\\
&\quad\exp\left\{-\frac{ig}{2m}\sum_{n=0}^{N-1}\int_0^t\left(B^{(n)}(u) - \frac{u}{t}B^{(n)}(t)\right)^2du\right.\\
&\quad\left.\left.\left. - \frac{ig}{t\sqrt{\mathfrak{h}m}}\sum_{n=0}^{N-1}\phi_n\int_0^t u\left(B^{(n)}(u) - \frac{u}{t}B^{(n)}(t)\right)du\right\}\right\rangle\!\!\right\rangle.
\end{aligned}$$

The above pairing $\langle\!\langle\cdot, \cdot\rangle\!\rangle$ is the $\mathscr{T}$-transform

$$\mathscr{T}\left[:\exp\left\{-\frac{1}{2}\langle W, K_N W\rangle\right\}:\exp\left\{-\frac{1}{2}\langle W, L_N W\rangle\right\}\right](f)$$
$$= \det(1 + L_N(1+K_N)^{-1})^{-1/2}\exp\left\{-\frac{1}{2}\langle f, (1+K_N+L_N)^{-1}f\rangle\right\},$$

where

$$K_N W(u,x) = -(1+i)1_{[0,t]}(u)\sum_{n=0}^{N-1} e_n(x)\int_0^1 e_n(y)W(u,y)dy\,,$$
$$L_N W(u,x) = i\omega^2\int_0^t\int_0^1\left(\frac{t}{3}+\frac{r^2}{2t}+\frac{u^2}{2t}-r\vee u\right)\sum_{n=0}^{N-1} e_n(x)e_n(y)W(r,y)dydr\,,$$
$$\omega = \left(\frac{g}{m}\right)^{1/2}, \quad \text{and} \quad f(r,x) = -\frac{g}{t\sqrt{\hbar m}}\left(\frac{t^2}{6}-\frac{r^2}{2}\right)\sum_{n=0}^{N-1}\phi_n e_n(x)\,.$$

The eigenvalue problem for the kernel $1 + L_N(1+K_N)^{-1}$ reads

$$\int_0^t\left(\frac{t}{3}+\frac{r^2}{2t}+\frac{u^2}{2t}-r\vee u\right)\sum_{n=0}^{N-1}\int_0^1 e_n(x)\eta(u,x)dxdu$$
$$= \frac{1-\lambda}{\omega^2}\sum_{n=0}^{N-1}\int_0^1 e_n(y)\eta(r,y)dy\,.$$

This equation can be solved in a straightforward way. The resulting eigenvalues and eigenfunctions are:

$$\lambda_k = 1 - \frac{\omega^2 t^2}{(k\pi)^2}\,, \qquad k = 1, 2, \cdots,$$

and

$$\sum_{n=0}^{N-1}\int_0^1 e_n(x)\eta_k(r,x)dx = \left(\frac{2}{t}\right)^{1/2}\cos\left(\frac{\pi k}{t}r\right), \qquad k = 1, 2, \cdots,$$

respectively. Therefore, we have

$$\det(1 + L_N(1+K_N)^{-1})^{-1/2} = \left(\prod_{k=1}^{\infty}\lambda_k\right)^{-N/2}$$
$$= \left(\frac{\sin(\omega t)}{\omega t}\right)^{-N/2}$$

and

$$\exp\left\{-\frac{1}{2}\langle f, (1+K_N+L_N)^{-1})f\rangle\right\}$$
$$= \exp\left\{\frac{im}{2\hbar}\left(\frac{\omega}{\tan(\omega t)}-\frac{1}{t}+\frac{\omega^2 t}{3}\right)\sum_{n=0}^{N-1}\phi_n^2\right\}.$$

Combining the above results, we conclude that

$$U_N(\phi, t) = \left(\frac{m\omega}{2\pi i\mathfrak{h}\cdot\sin(\omega t)}\right)^{N/2} \exp\left\{\frac{im\omega}{2\mathfrak{h}\cdot\tan(\omega t)} \sum_{n=0}^{N-1} \phi_n^2\right\}.$$

Consequently,

$$\Phi(\dot{b}, t) = \left(\frac{m\omega}{2\pi i\mathfrak{h}\cdot\sin(\omega t)}\right)^{1/2} U^{-1}\left[\exp\left\{\frac{im\omega}{2\mathfrak{h}\cdot\tan(\omega t)} \int_0^1 \phi(x)^2 dx\right\}\right].$$

So we have

$$\frac{\partial\Phi(\dot{b}, t)}{\partial t} = -\frac{\omega}{2\tan(\omega t)}\Phi(\dot{b}, t) - \frac{im\omega^2}{2\mathfrak{h}\sin^2(\omega t)} :\int_0^1 \dot{b}(x)^2 dx: \diamond \Phi(\dot{b}, t),$$

$$\Delta_L\Phi(\dot{b}, t) = \frac{im\omega}{\mathfrak{h}\tan(\omega t)}\Phi(\dot{b}, t),$$

and

$$\Delta_V\Phi(\dot{b}, t) = -\frac{m^2\omega^2}{2\mathfrak{h}\tan^2(\omega t)} :\int_0^1 \dot{b}(x)^2 dx: \diamond \Phi(\dot{b}, t).$$

From these calculations, it is easily checked that $\Phi(\dot{b}, t)$ satisfies the equation (5-4). (Q.E.D.)

The following examples can be immediately obtained by using Theorem 5-1.

EXAMPLE 1 (Free particle). For $V = 0$, we have

$$\Phi(\dot{b}, t) = \left(\frac{m}{2\pi i\mathfrak{h}t}\right)^{1/2} U^{-1}\left[\exp\left\{\frac{im}{2\mathfrak{h}t} \int_0^1 \phi(x)^2 dx\right\}\right].$$

EXAMPLE 2 (Constant external field). For $V = -F\int_0^1 q(\cdot, x)dx$ with a constant F, we have

$$\Phi(\dot{b}, t) = \left(\frac{m}{2\pi i\mathfrak{h}t}\right)^{1/2} U^{-1}\left[\exp\left\{\frac{im}{2\mathfrak{h}t} \int_0^1 \phi(x)^2 dx + \frac{iFt}{2\mathfrak{h}} \int_0^1 \phi(x) dx\right\}\right].$$

EXAMPLE 3 (Harmonic oscillator). For $V = (1/2)g\int_0^1 q(\cdot, x)^2 dx$, $g = m\omega^2$, we have

$$\Phi(\dot{b}, t) = \left(\frac{m\omega}{2\pi i\mathfrak{h}\cdot\sin(\omega t)}\right)^{1/2} U^{-1}\left[\exp\left\{\frac{im\omega}{2\mathfrak{h}\cdot\tan(\omega t)} \int_0^1 \phi(x)^2 dx\right\}\right].$$

§6. Itô's formula for generalized functionals

In [24], the author discussed Itô's formula for generalized functionals by using a shift operator $1/dx$. In this section, without the operator $1/dx$, we shall investigate how Lévy's Laplacian appears in Itô's formula for generalized functionals depending on t. It seems that this formula is useful in constructing a stochastic process whose generator contains Lévy's Laplacian. For instance, consider a generalized functional

$$\Phi(B_\cdot(t)) = \int_0^1 f(x)[\![B_x(t)^2]\!]dx\,, \qquad f \in C(\mathbf{R})\,,$$

in $(E)^*$, $E = \mathscr{S}(\mathbf{R}^2)$, where $[\![B_x(t)^2]\!] = :B_x(t)^2: + E[B(t)^2]$. Its U-functional is given by

$$[U\Phi(B_\cdot(t))](\xi) = \int_0^1 f(x)\Big(\Big(\int_0^t \xi(u, x)du\Big)^2 + t\Big)dx\,, \qquad \xi \in E\,.$$

We can calculate its derivative to have

$$\frac{d[U\Phi(B_\cdot(t))](\xi)}{dt} = 2\int_0^1 f(x)\int_0^t \xi(u, x)du\xi(t, x)dx + \int_0^1 f(x)dx$$

$$\frac{\delta[U\Phi(B_\cdot(t))](\xi)}{\delta\xi(t-, x)} = 2f(x)\int_0^t \xi(u, x)du\,,$$

$$[U\Phi(B_\cdot(t))]_s''(\xi; t-, x) = 2f(x)\,,$$

and

$$[U\Phi(B_\cdot(t))]_r''(\xi; t-, x, t-, x) = 0\,.$$

Therefore, we get

$$d\Phi(B_\cdot(t)) = \int_0^1 \partial_{t-,x}\Phi(B_\cdot(t)) \diamond dB_x(t)dx + \frac{1}{2}\Delta_L\Phi(B_\cdot(t))dt\,, \tag{6-1}$$

where $\partial_{t-,x}$ is the $W(t-, x)$-differentiation. These calculations can be extended to functionals

$$\Phi(B_\cdot(t)) = \int_0^1 \cdots \int_0^1 f(x_1, \cdots, x_n)[\![B_{x_1}(t)^2]\!] \diamond \cdots \diamond [\![B_{x_n}(t)^2]\!]dx_1 \cdots dx_n\,, \qquad f \in C(\mathbf{R}^n)\,. \tag{6-2}$$

Thus, we obtain the following

PROPOSITION 6-1. *The equation* (6-1) *holds for every* $\Phi(B_\cdot(t))$ *given by* (6-2).

Concluding Remark. Proposition 6-1 is expected to hold for much more general functionals which are useful in applications. At least, the equation (6-1) holds for every (finite) linear combination of functionals

$$\int_0^1 \cdots \int_0^1 f(x_1, \cdots, x_n)[\![B_x(t)^{p_1}]\!] \diamond \cdots \diamond [\![B_x(t)^{p_n}]\!] dx_1 \cdots dx ,$$
$$f \in C(\mathbf{R}^n) , \qquad p_1, \cdots, p_n \in \mathbf{N} \cup \{0\} ,$$

where

$$[\![B_x(t)^p]\!] = \sum_{k=0}^{[p/2]} \frac{p!\, t^k}{(2k)!!(p-2k)!} :B_x(t)^{p-2k}: \qquad \text{for } p \in \mathbf{N} \cup \{0\} .$$

Details of the problem arising from Itô's formula will be discussed in a forthcoming paper by the author.

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