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ON THE CARTAN-NORDEN THEOREM FOR AFFINE KÄHLER IMMERSIONS

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In [N-Pi-Po] the notion of affine Kähler immersion for complex manifolds has been introduced: if M^n is an n-dimensional complex manifold and f: $M^n \to \mathbf{C}^{n+1}$ is a holomorphic immersion together with an antiholomorphic transversal vector field ζ, we can induce a connection ∇ on M^n which is Kähler-like, that is, its curvature tensor R satisfies $R(Z, W) = 0$ as long as Z, W are $(1, 0)$ complex vector fields on M.

This work is aimed at proving a Cartan-Norden-like theorem for affine Kähler immersions, generalizing the classical result in affine differential geometry (see [N-Pi]). In §1 we deal with some preliminaries about affine Kähler immersions in order to make our work self-contained. In §2 we prove our main result: if a non-flat Kähler manifold (M^n, g) can be affine Kähler immersed into $\mathbf{C}^{n+1}$ and the immersion f is non-degenerate, then for every point $x \in M^n$ we can find a parallel pseudokählerian metric in $\mathbf{C}^{n+1}$ such that f is locally isometric around the point x.

§ 1. Preliminaries

Throughout this work we shall refer to [N-Pi-Po] for basic results in the geometry of affine Kähler immersions. We recall here some fundamental equations. Let M^n be an n-dimensional complex manifold with complex structure J and let f: $M^n \to \mathbf{C}^{n+1}$ be a holomorhic immersion. We denote by D the standard flat connection in $\mathbf{C}^{n+1}$, a transversal $(1, 0)$ vector field $\zeta = \xi - iJ\xi$ along f is said to be antiholomorphic if $D_Z\zeta = 0$ for every complex vector field Z of type $(1, 0)$ on M^n.

If X and Y are real vector fields on M^n, we can write

$$D_X(f_*Y) = f_*(\nabla_X Y) + h(X, Y)\xi + k(X, Y)J\xi \tag{1.1}$$

thus defining a torsionfree affine connection ∇ and symmetric tensors h

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and k on M^n. Since f is holomorphic and J is D-parallel, we get that $\nabla J = 0$ and $k(X, Y) = -h(JX, Y) = -h(X, JY)$. We can also write

$$D_X \xi = -f_*(AX) + \mu(X)\xi + \nu(X)J\xi \tag{1.2}$$

defining the shape operator A and two 1-forms μ and ν. An easy calculation shows that the transversal vector field ζ is antiholomorphic if and only if $AJ = -JA$ and $\nu(X) = \mu(JX)$ for every real tangent vector field X. By extending h as a complex bilinear function on complex tangent vectors, we get for $Z = X - iJX$ and $W = Y - iJX$

$$h(Z, W) = 2(h(X, Y) + ik(X, Y)) \tag{1.3}$$

and

$$h(Z, \overline{W}) = 0$$

so that we can write for complex vector fields Z, W

$$D_Z(f_* W) = f_*(\nabla_Z W) + h(Z, W)\zeta\,. \tag{1.4}$$

The covariant symmetric tensor h is called the second fundamental form for f and we shall say that f is non-degenerate if the tensor h is non-degenerate; it is very easy to see that this condition is actually independent of the choice of a transversal vector field (holomorphic, antiholomorphic or whatever).

Moreover by putting $S = A - iJA$ and $\tau = \mu - i\nu$ we can write

$$D_Z \zeta = -S(Z) + \tau(Z)\zeta \tag{1.5}$$

for every (1, 0)-complex vector field Z.

We are now going to write down the fundamental equations of Gauss, Codazzi and Ricci in the real representation; for the complex version we refer to [N-Pi-Po]. Henceforth U, X, Y will indicate real vector fields. We have the equation of Gauss

$$\begin{aligned} R(X, Y)U = h(Y, U)AX - h(X, U)AY + h(JY, U)AJX \\ - h(JX, U)AJY, \end{aligned} \tag{1.6}$$

the two equations of Codazzi

$$\begin{aligned} &(\nabla_X h)(Y, U) + \mu(X)h(Y, U) + \mu(JX)h(JY, U) \\ &\quad = (\nabla_Y h)(X, U) + \mu(Y)h(X, U) + \mu(JY)h(JX, U) \end{aligned} \tag{1.7}$$

$$\begin{aligned} &(\nabla_X A)Y - \mu(X)AY - \mu(JX)JAY \\ &\quad = (\nabla_Y A)X - \mu(Y)AX - \mu(JY)JAX \end{aligned} \tag{1.8}$$

and the equations of Ricci

$$h(X, AY) - h(Y, AX) = 2d\mu(X, Y) \tag{1.9}$$

$$h(AX, JY) = d\nu(X, Y)\,. \tag{1.10}$$

§ 2. On the Cartan-Norden Theorem

We are now going to prove our main theorem

THEOERM 2.1. *Let $f\colon M^n \to \mathbf{C}^{n+1}$ be a non-degenerate affine Kähler immersion. If the induced connection ∇ is non-flat and coincides with the Levi-Civita connection of a pseudo-kählerian metric g on M^n, then for every $x \in M^n$ there is a neighborhood $U(x)$ and a parallel pseudo-kählerian metric $\langle\ \rangle$ on $\mathbf{C}^{n+1}$ so that f is isometric relative to g and $\langle\ \rangle$ and the transversal vector field ζ for f is perpendicular to $f(U(x))$ at each point of $U(x)$.*

Proof. We denote by h the second fundamental form for f and we define the conjugate connection $\tilde{\nabla}$ of ∇ by means of the following equation

$$Xh(Y, U) = h(\nabla_X Y, U) + h(Y, \tilde{\nabla}_X U - \mu(X)U - \nu(X)JU)\,. \tag{2.1}$$

We recall that $\nu(X) = \mu(JX)$. Equation (2.1) defines $\tilde{\nabla}$ uniquely since h is supposed to be non-degenerate and we have easily that $\tilde{\nabla}$ is a complex connection, that is, $\tilde{\nabla}J = 0$; by using the Codazzi equation $\tilde{\nabla}$ turns out to be torsionfree.

LEMMA 2.1. *If the connection ∇ is a Levi-Civita connection, then the 1-form is closed.*

Proof. Indeed from the Gauss equation we get that $\mathrm{Ric}(Y, Z) = -2h(AY, Z)$ since $\mathrm{tr}\, A = \mathrm{tr}\, JA = 0$. Since ∇ is metric, the Ricci tensor is symmetric and from the Ricci equation we have that $(\nabla_X \mu)(Y)$ is symmetric in X and Y, that is, $d\mu = 0$. q.e.d.

LEMMA 2.2. *If ∇ comes from a pseudo-kählerian metric g, then the conjugate connection $\tilde{\nabla}$ is locally pseudo-kählerian.*

Proof. We define the (1, 1) tensor B by setting $g(X, Y) = h(BX, Y)$; we note that since g is hermitian, we have that

$$h(BX, Y) = h(BJX, JY) = h(JBJX, Y)$$

hence $B = JBJ$. We now define

$$\tilde{g}(X, Y) = vh(B^{-1}X, Y)$$

for a suitable positive function v in order to have that $\tilde{\nabla}\tilde{g} = 0$. We note that

$$Zh(X, B^{-1}Y) - h(X, \nabla_Z B^{-1}Y) - h(\nabla_Z B^{-1}X, Y) = (\nabla_Z g)(B^{-1}X, B^{-1}Y) = 0$$

and that

$$h(X, JB^{-1}Y) = -h(JB^{-1}X, Y).$$

Using these identities we have

$$\begin{aligned} Z\tilde{g}(X, Y) &- \tilde{g}(\tilde{\nabla}_Z X, Y) - \tilde{g}(X, \tilde{\nabla}_Z Y) \\ &= Z(v)h(B^{-1}X, Y) + vZh(B^{-1}X, Y) - v[Zh(X, B^{-1}Y) \\ &\quad - h(X, \nabla_Z B^{-1}Y - \mu(Z)B^{-1}Y - \mu(JZ)JB^{-1}Y)] \\ &\quad - v[Zh(B^{-1}X, Y) - h((\nabla_Z B^{-1}X, Y) \\ &\quad + \mu(Z)h(B^{-1}X, Y) + \mu(JZ)h(Y, JB^{-1}X)] \\ &= [Z(v) - 2v\mu(Z)]h(B^{-1}X, Y). \end{aligned}$$

So $\tilde{g}$ turns out to be $\tilde{\nabla}$-parallel if and only if we can choose a positive function v so that $Z(v) = 2v\mu(Z)$; since μ is closed by Lemma 1, we can find locally a function λ so that $\mu = d\lambda$ and then we can put $v = \exp(2\lambda) > 0$. q.e.d.

We now compute the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$: we have

$$\begin{aligned} UZh(X, Y) = {} & h(\nabla_U \nabla_Z X, Y) + h(\nabla_Z X, \tilde{\nabla}_U Y - \mu(U)Y - \mu(JU)JY) \\ & + h(\nabla_U X, \tilde{\nabla}_Z Y) + h(X, \tilde{\nabla}_U \tilde{\nabla}_Z Y - \mu(U)\tilde{\nabla}_Z Y - \mu(JU)J\tilde{\nabla}_Z Y) \\ & - U\mu(Z)h(X, Y) - \mu(Z)U(h(X, Y) - U\mu(JZ)h(X, JY) \\ & - \mu(JZ)Uh(X, JY). \end{aligned}$$

Interchanging U and Z and subtracting $[U, Z]h(X, Y)$, we get

$$h(R(U, Z)X, Y) + h(X, \tilde{R}(U, Z)Y) - 2d\nu(U, Z)h(X, JY) = 0.$$

Using now the structure equations (1.6), (1.10) and the fact that h is non-degenerate, we have

$$\begin{aligned} \text{(2.2)} \qquad \tilde{R}(U, Z)Y = {} & 2h(AU, JZ)JY - h(AU, Y)Z + h(AZ, Y)U \\ & - h(Y, AJU)JZ + h(Y, AJZ)JU. \end{aligned}$$

Taking trace we have that $\widetilde{\mathrm{Ric}}(X, Y) = 2(n+1)h(ZX, Y)$ and by equation (2.2), it follows that the space $(M^n, \tilde{g})$ is H-projectively flat (see e.g. [Y], Chapter XII, (3.16)); so the space $(M^n, \tilde{g})$ has constant holomorphic sectional curvature and in particular it is Einstein, hence

$$h(AX, Y) = \lambda \tilde{g}(X, Y) = \lambda v h(B^{-1}X, Y) \tag{2.3}$$

for some function λ, which is constant if $n \geq 2$ (see [K-N], p. 168). By (2.3) we have $A = \lambda v B^{-1}$ and

$$g(AX, Y) = \lambda v g(B^{-1}X, Y) = \lambda v h(X, Y)\,. \tag{2.4}$$

We now state the following

LEMMA 2.3. *There is a nowhere vanishing C^∞ function ϕ such that*

$$g(AX, Y) = \phi h(X, Y) \tag{2.5}$$

for all real vector fields X, Y and

$$d\phi = 2\phi\mu\,. \tag{2.6}$$

Proof. We have already established the first assertion (2.5); the function ϕ can be taken to be λv, where v is the function found in Lemma 2.2 and λ is a constant if $n \geq 2$; so (2.6) follows from the proof of Lemma 2.2 if $n \geq 2$. In the general case we differentiate (2.5)

$$Zg(AX, Y) = (Z\phi)h(X, Y) + \phi Zh(X, Y)$$

hence

$$g((\nabla_Z A)X, Y) + g(A(\nabla_Z X), Y) + g(AX, \nabla_Z Y) = (Z\phi)h(X, Y) + \phi Zh(X, Y)$$

that is

$$g((\nabla_Z A)X, Y) - \phi(\nabla_Z h)(X, Y) = (Z\phi)h(X, Y) \tag{2.7}$$

and

$$g((\nabla_X A)Z, Y) - \phi(\nabla_X h)(Z, Y) = (X\phi)h(Z, Y)\,. \tag{2.8}$$

If we now subtract (2.8) from (2.7) and use the equations of Codazzi we obtain

$$\begin{aligned}(Z\phi)h(X, Y) - (X\phi)h(Z, Y) &\\ = g(\mu(Z)AX + \mu(JZ)JAX - \mu(X)AZ - \mu(JX)JAZ, Y)&\\ - \phi h(\mu(X)Z + \mu(JX)JZ - \mu(Z)X - \mu(JZ)JX, Y)&\\ = \phi h(2\mu(Z)X - 2\mu(X)Z, Y)&\end{aligned}$$

hence

$$(Z\phi)X - (X\phi)Z = 2\phi[\mu(Z)X - \mu(X)Z]$$

that is

$$Z\phi = 2\phi\mu(Z) .$$

Since the function v satisfies the same differential equation $dv = 2v\mu$ and does not vanish anywhere, it follows that λ is a constant. If λ were 0, we would have from equation (2.4) that A vanishes identically, hence that ∇ is flat. q.e.d.

We are now going to define the parallel pseudo-kählerian metric $\langle\ \rangle$ in $\mathbf{C}^{n+1}$ by means of the following

$$\langle f_*X, f_*Y\rangle = g(X, Y) , \qquad \langle f_*X, \xi\rangle = \langle f_*X, J\xi\rangle = 0 ,$$
$$\langle \xi, J\xi\rangle = 0 , \qquad \langle \xi, \xi\rangle = \langle J\xi, J\xi\rangle = \phi ,$$

where ϕ is the function given by Lemma 2.3. We have to verify that $\langle\ \rangle$ is D-parallel, that is

$$Z\langle U, V\rangle = \langle D_Z U, V\rangle + \langle U, D_Z V\rangle \tag{2.9}$$

for all vector fields U and V along f and a vector field Z on M^n. If $U = f_*X$ and $V = f_*Y$, then (2.9) reduces to $\nabla_Z g = 0$. If $U = f_*X$ and $V = \xi$, then (2.9) gives condition (2.5) and if $U = V = \xi$, then (2.9) reduces to (2.6). The other possibilities are easily seen to be automatically satisfied. q.e.d.

COROLLARY 2.1. *Let (M^n, g) be a non-flat kählerian manifold and let $f\colon M^n \to \mathbf{C}^{n+1}$ be a non-degenerate affine Kähler immersion. Then the Ricci tensor of (M^n, g) is positive- or negative-definite. Moreover the pseudo-kählerian metric $\langle\ \rangle$ in $\mathbf{C}^{n+1}$- given by Theorem 2.1 is positive-definite if and only if the Ricci tensor of (M^n, g) is negative-definite.*

Proof. Using the Gauss equation, we have the following expression for the Ricci tensor

$$\mathrm{Ric}(X, Y) = -2h(AX, Y)$$

for all real vectors X and Y. Using Lemma 2.3 we have (locally)

$$\mathrm{Ric}(X, X) = -\frac{2}{\phi}g(A^2X, X) = -\frac{2}{\phi}g(AX, AY) .$$

Since h is non-degenerate, we see from (2.5) that the (1, 1) tensor A is one-to-one, hence the Ricci tensor is definite. Moreover Ric is negative-definite if and only if the function ϕ is everywhere positive. q.e.d.

EXAMPLE. In order to show that the Ricci tensor can be positive-definite, we give the following example. Let $\Omega = \{z \in \mathbf{C};\ \mathrm{Re}\, z < 0\}$; we define $f\colon \Omega \to \mathbf{C}^2$ by $f(z) = (z, \exp(z))$ and take $\zeta = (\exp(\bar{z}), 1)$ as an anti-holomorphic transversal vector field. Actually ζ is perpendicular to $f(\Omega)$ at each point of Ω with respect to the Lorentzian metric of $\mathbf{C}^2$ of signature $(1, 1)$. The induced Kähler metric g on Ω is given by

$$g(\partial/\partial z, \partial/\partial \bar{z}) = 1 - \exp(2\,\mathrm{Re}\, z) > 0\,, \qquad z \in \Omega\,,$$

and it is easy to see that the second fundamental form h is

$$h(\partial/\partial z, \partial/\partial z) = \frac{\exp(z)}{1 - \exp(2\,\mathrm{Re}\, z)}$$

so that f is non-degenerate. Moreover the Ricci tensor of (Ω, g) is (see [K-N], p. 158)

$$R_{1\bar{1}} = -\frac{\partial^2 \log(1 - \exp(2\,\mathrm{Re}\, z))}{\partial/\partial z \partial/\partial \bar{z}} = \exp(2\,\mathrm{Re}\, z)\frac{1 + \exp(2\,\mathrm{Re}\, z)}{1 - \exp(2\,\mathrm{Re}\, z)} > 0\,.$$

This shows that (Ω, g) can not be obtained as a complex hypersurface of $\mathbf{C}^2$ endowed with the euclidean metric (see [K-N], p. 177, Prop. 9.4).

Remark. In order to clarify the geometrical meaning of the conjugate connection used in the proof of Theorem 2.1, we recall something about the Gauss map for complex hypersurfaces as introduced in [N-S]. Let (M, g) be a kählerian manifold and $f\colon M \to \mathbf{C}^{n+1}$ a non-degenerate complex isometric immersion. We choose a unit real vector field ξ normal to $f(M)$; we recall (see [S], p. 230) that if X is any real vector field on M

$$\nabla_X \xi = -AX + s(X)J\xi$$

where s is a 1-form and with our notation the normal connection form τ is simply given by $\tau(Z) = is(X - iJX)$, where $Z = X - iJX$. From $\langle \xi, Y \rangle = 0$ for every vector Y we get by differentiation

$$g(AX, Y) = h(X, Y)\,. \tag{2.10}$$

Finally the Codazzi equation is now the following (see [S], p. 253)

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) - s(X)JAX + s(Y)JAX = 0\,.$$

According to [N-S], p. 516, we define the Gauss map Φ

$$\Phi\colon M \to \mathbf{CP}^n$$

by putting $\Phi(x) = \pi(\xi)$, where $\pi\colon S^{2n+1} \to \mathbf{CP}^n$ is the canonical projection. It is shown that $\Phi_{*x}(X) = -\pi_{*\xi}(AX)$ for every real tangent vector X at $x \in M$, so that since f is non-degenerate, the rank of A is $2n$ by (2.10) and therefore Φ is an immersion. If now $\tilde{g}$ denotes the Fubini-Study kählerian metric on $\mathbf{CP}^n$, a direct inspection of the results stated in [N-S], § 5, shows that the pull back $\Phi^*\tilde{g}$ is given by

$$\Phi^*\tilde{g}(X, Y) = g(AX, AY) = h(AX, Y) = -\frac{1}{2}\operatorname{Ric}(X, Y).$$

We claim that the conjugate connection $\tilde{\nabla}$ as defined by formula (2.1) is the Levi-Civita connection of the metric $\Phi^*\tilde{g}$. Indeed equation (2.1) reduces to

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \tilde{\nabla}_X Z - s(X)JZ) \tag{2.11}$$

where X, Y, Z are real vector fields on M. We first note that by equation (2.10) we have that

$$(\nabla_X h)(Y, Z) = g((\nabla_X A)(Y), Z). \tag{2.12}$$

We write equation (2.11) in the equivalent form

$$(\nabla_X h)(Y, Z) + h(Y, \nabla_X Z) = h(Y, \tilde{\nabla}_X Z - s(X)JZ) \tag{2.13}$$

and if we interchange X and Z and subtract it from (2.13), we obtain

$$\begin{aligned} g((\nabla_X A)(Z), Y) - g((\nabla_Z A)(X), Y) + h(Y, [X, Z]) \\ = h(Y, \tilde{\nabla}_X Z - \tilde{\nabla}_Z X - s(X)JZ + s(Z)JX). \end{aligned}$$

Using now the Codazzi equation, formula (2.12) and the fact that h is non-degenerate, we get that $\tilde{\nabla}_X Z - \tilde{\nabla}_Z X = [X, Z]$, that is, $\tilde{\nabla}$ is torsionfree.

We now prove that $\tilde{\nabla}\Phi^*\tilde{g} = 0$: indeed

$$\begin{aligned} \Phi^*\tilde{g}(\tilde{\nabla}_X Y, Z) + \Phi^*\tilde{g}(Y, \tilde{\nabla}_X Z) &= h(\tilde{\nabla}_X Y, AZ) + h(AY, \tilde{\nabla}_X Z) \\ &= Xh(Y, AZ) - h(Y, \nabla_X AZ) + s(X)h(Y, JAZ) \\ &\quad + Xh(Z, AY) - h(Z, \nabla_X AY) + s(X)h(Z, JAY) \\ &= Xh(Y, AZ) + Xh(Z, AY) - h(Y, \nabla_X AZ) - h(Z, \nabla_X AY) \end{aligned} \tag{2.14}$$

since $h(Z, JAY) = -h(Z, AJY) = -h(AZ, JY) = -h(JAZ, Y)$. We now note that

$$\begin{aligned} Xh(Z, AX) &= (\nabla_X h)(Z, AY) + h(\nabla_X Z, AY) + h(Z, \nabla_X AY) \\ &= g((\nabla_X A)(Z), AY) + h(\nabla_X Z, AY) + h(Z, \nabla_X AY) \\ &= h((\nabla_X A)(Z), Y) + h(A\nabla_X Z, Y) + h(Z, \nabla_X AY) \\ &= h(\nabla_X AZ, Y) + h(Z, \nabla_X AY) . \end{aligned}$$

If we insert this into (2.14), we obtain

$$\Phi^*\tilde{g}(\tilde{\nabla}_X Y, Z) + \Phi^*\tilde{g}(Y, \tilde{\nabla}_X Z) = Xh(Y, AZ) = X\Phi^*\tilde{g}(Y, Z)$$

and we are done.

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