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ON THE CLASS NUMBER AND UNIT INDEX OF SIMPLEST QUARTIC FIELDS

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§1. Introduction

The term "simplest" field has been used to describe certain totally real, cyclic number fields of degrees 2, 3, 4, 5, 6, and 8. For each of these degrees, the fields are defined by a one-parameter family of polynomials with constant term ± 1. The regulator of these "simplest" fields is small in an asymptotic sense: in consequence, the class number of these fields tends to be large.

The simplest quartic fields are defined by adjunction to $\mathbf{Q}$ of a root of

$$(*) \qquad P_t = X^4 - tX^3 - 6X^2 + tX + 1\,, \qquad t \in \mathbf{Z}^+$$

where $t^2 + 16$ is not divisible by an *odd* square [5]. Here t may be specified greater than zero since P_t and P_{-t} generate the same extension. This polynomial is reducible precisely when $t^2 + 16$ is a square, which occurs only for the excluded cases $t = 0,\ 3$.

Gras [5] shows that the form $T^2 + 16$ represents infinitely many square-free integers, so this family is infinite. As an example of why the odd-square-free restriction is important, note that $t = 22$, for which $t^2 + 16 = 500$, defines the same field as $t = 2$.

Some of the simplest fields arise from torsion elements in $PSL(2, \mathbf{Q})$ acting as linear fractional transformations on one given root [3]. The matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ has order 4; the cyclic Galois action on the roots of $(*)$ is given by $\varepsilon \mapsto (\varepsilon - 1)/(\varepsilon + 1)$.

Remark. References to other examples of simplest fields may be found in [10].

Notation. Throughout K will be a real cyclic quartic field, simplest unless stated otherwise, and k will be the unique quadratic subfield. Let

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$E_K(E_k)$ and $F(m)$ be the units and conductor of $K(k)$ respectively. Let ε_1 be the largest root of (*) and $\varepsilon_2 = \varepsilon_1^\sigma$; let ε_k be the fundamental unit of k. Let σ generate $\mathrm{Gal}(K/\mathbf{Q})$ and χ generate its character group. These groups are both isomorphic to $\mathbf{Z}/4\mathbf{Z}$. The parameter F/m will occur frequently so we denote it by G. By the Conductor-Discriminant Theorem the discriminant $D = mF^2$.

§ 2. Fields with class number at most two

The author published in [11] an effective, unconditional lower bound on the class number of K in terms of t. The present work continues with a complete list of simplest quartic fields of even conductor and class number at most two. Let $h_{K(t)}$ be the class number of the field defined by (*) and let $h_{k(t)}$ be the class number of the quadratic subfield.

THEOREM 1. *For even F, the class number $h_{K(t)} = 1$ if and only if $t \in \{2, 4, 6, 8, 10, 24\}$. $h_{K(t)} = 2$ if and only if $t \in \{12, 16, 20\}$. The quadratic class number $h_{k(t)}$ is one in all these cases except $t = 16$, where it is two.*

The proof is a consequence of the calculations at the end of this paper.

§ 3. The unit index

The *χ-relative units* (a term owing to Leopoldt [13]) are

$$E_\chi = \{\varepsilon \colon N_k^K \varepsilon = \varepsilon^{1+\sigma^2} = \pm 1\}.$$

The *unit index* $Q = Q_K$ is defined as $[E_K \colon E_\chi E_k]$.

PROPOSITION 3.1. (A) (Hasse [7]) *In any real cyclic quartic field $Q \leq 2$.*

(B) (Gras [5]) *In a simplest quartic field, $E_\chi = \langle \varepsilon_1, \varepsilon_2 = \varepsilon_1^\sigma \rangle$, i.e. the roots of (*) are fundamental relative units, whence $Q = [E_K \colon \langle \pm 1, \varepsilon_1, \varepsilon_2, \varepsilon_k \rangle]$. We determine Q in most cases.*

THEOREM 2. *In a simplest quartic field*

F	Q	F	Q
odd, prime	2	16	2
odd, composite, $N_{\mathbf{Q}}^k(\varepsilon_k) = -1$	1, 2	even $\neq 16$	1
odd, composite, $N_{\mathbf{Q}}^k(\varepsilon_k) = 1$	1		

Both possibilities for the undetermined entry appear in Gras's table [5].

Remark. For m a product of primes $\equiv 1 \bmod 4$ the determination of $N_{\mathbf{Q}}^{k}(\varepsilon_k)$ is an open question.

The proof uses the techniques of Section 6 and is therefore postponed.

§ 4. A lower bound for the class number

Both the class number bound and the computation of the list of specific fields in Theorem 1 come from Dirichlet's analytic class number formula. For a real quartic this simplifies to

$$h = \frac{\sqrt{D}\ \prod_{\chi\neq 1} L(1,\chi)}{8R} \tag{ACF}$$

where D is the discriminant, R is the regulator, and the product is over the non-trivial primitive characters of the Galois group $\mathrm{Gal}(K/\mathbf{Q})$. We bound the numerator of (ACF) from below and the denominator from above, thereby bounding h itself from below.

The conductors and discriminants of K and k depend upon the 2-adic valuation $\nu_2(t)$.

TABLE 4.1.

CASE	$\nu_2(t)$	F	m	G	D
1	0	t^2+16 *	t^2+16 *	1	$(t^2+16)^3$
2	1	t^2+16	$\frac{t^2+16}{4}$ *	4	$\frac{(t^2+16)^3}{4}$
3	2	$\frac{t^2+16}{2}$	$\frac{t^2+16}{4}$	2	$\frac{(t^2+16)^3}{16}$
4	≥ 3	$\frac{t^2+16}{2}$	$\frac{t^2+16}{16}$ *	8	$\frac{(t^2+16)^3}{64}$

The stars indicate the odd values of F and m. Cases 1 through 4 will have the meaning from this table throughout.

LEMMA 4.2. *For a simplest quartic field K_t with $t > 57$, $\prod_{\chi\neq 1} L(1,\chi) > 0.066 D^{-1/4}$.*

Proof. In [11] this is deduced, with weaker constant, from Theorem 1′ of Stark's famous paper [17]. The constant $c_2 = 2/\log 3$ in [11, 17] may be replaced by $c_2 = 0.186$ because Stark's proof requires only $c_2 \geq 4/\log D$. $\square$

PROPOSITION 4.3 ([9], Chapter 12.13). *For any quadratic field k of conductor m, $\varepsilon_k < (e\sqrt{m})^{\sqrt{m}}$.*

PROPOSITION 4.4 ([3, 11]). *When t is even, ε_k is given by*

$$\varepsilon_k = \begin{cases} \dfrac{(t/2) + \sqrt{(t/2)^2 + 4}}{2} & t \equiv 2 \bmod 4 \\ \dfrac{1 + \sqrt{5}}{2} & t = 8 \\ (t/4) + \sqrt{(t/4)^2 + 1} & \textit{otherwise}. \end{cases}$$

We approximate the actual regulator R with

$$\tilde{R} = -\begin{vmatrix} \log \varepsilon_1 & \log \varepsilon_2 & -\log \varepsilon_1 \\ \log \varepsilon_2 & -\log \varepsilon_1 & -\log \varepsilon_2 \\ \log \varepsilon_k & -\log \varepsilon_k & \log \varepsilon_k \end{vmatrix} \tag{4.1}$$
$$= 2 \log \varepsilon_k (\log^2 \varepsilon_1 + \log^2 \varepsilon_2).$$

We see that $\tilde{R} > 0$ and $\tilde{R}/R = Q_K$. Also define the *relative regulator* $R^- = R/R_k$; let $\tilde{R}^- = \tilde{R}/R_k = Q_K R^-$. Define $A_0 = 0.00826$.

THEOREM L ([11], with weaker constant). *For K_t a simplest quartic field with $t^2 + 16$ odd-square-free, the class number $h_{K(t)}$ is bounded*:

$$h_{K(t)} > \begin{cases} \dfrac{3Q_K A_0 D^{1/12}}{\tilde{R}^- \log D} & t \textit{ odd} \\ \dfrac{Q_K A_0 D^{1/4}}{\tilde{R}} & t \textit{ even} \end{cases}$$

Proof. By (ACF), Lemma 4.2 and, for t odd, Proposition 4.3. $\square$

We can rewrite this result in terms of t instead of D using the quartic formula for ε_1 and ε_2, Table 4.1, Equation (4.1), and Propositions 4.3 and 4.4; cf. [11], Corollary 6.

COROLLARY 4.5. *There exists a constant C, independent of the field, such that for K_t as above and $t \gg 0$.*

$$h_{K(t)} > \begin{cases} C\dfrac{t^{1/2}}{\log^3 t} & t \text{ odd} \\ C\dfrac{t^{3/2}}{\log^3 t} & t \text{ even.} \end{cases}$$

In particular, we may take $C = (0.00338, 0.00288, 0.00204, 0.00143)$ *in Cases* 1 *to* 4, *respectively.*

COROLLARY 4.6. *To ensure* $h_K > 1$ *in Theorem L it suffices to take*

$$t \geq (8.5\cdot 10^{13}, 3222, 4356, 5942).$$

The magnitude of t in the first case corresponds to fields whose discriminants have 84 decimal digits. This is less surprising in view of the following.

CONJECTURE. The polynomial $T^2 + 16$ represents a prime infinitely often. (See Hardy and Wright [6], II. 2.8).

CONJECTURE (Gauss). The field $\mathbf{Q}[\sqrt{m}]$ has class number one for infinitely many primes $m \equiv 1 \bmod 4$.

Assuming that these classical conjectures are independent, as seems likely, it is reasonable to hypothesize:

CONJECTURE. The quadratic subfields have class number one for infinitely many simplest quartic fields of Case 1.

This has consequences for the quartic class number computations, because of class field theory.

PROPOSITION 4.7. $h_{k(t)} \mid h_{K(t)}$.

Proof. Since K is cyclic and ramification begins at the top, K/k is ramified at all primes dividing m. Since K/k has no non-trivial unramified abelian subextension, $h_{k(t)} \mid h_{K(t)}$. □

Efficient calculation of the class number and the proof of Theorem 2 both involve writing elements of K in a special basis.

§ 5. Hasse's basis and $\tau(\chi)$

In the following proposition the quartic field is real cyclic but not necessarily simplest. For convenience, however, we restrict to the case that G is a power of 2. Similar results obtain in general. Since G ($G/2$ if and only if $2\mid m$) is the conductor of a primitive quadratic character [7], $\nu_2(G) \in \{0, 1, 2, 3\}$.

PROPOSITION 5.1 ([12]). *For m odd and G a fixed power of 2, there is a one-to-one correspondence between*

1. *Real quartic fields of conductor F.*
2. *Conjugate pairs of even numerical quartic characters of conductor F.*
3. *Representation of $m = a^2 + b^2$ where $a + bi$ is primary and furthermore $b > 0$. Here $m \equiv 1 \bmod 8$ if $G = 1$, $m \equiv 5 \bmod 8$ if $G = 4$, and m is unrestricted if $G = 8$.*
4. *Primary Gaussian integers $\psi = a + bi$ of norm m, up to complex conjugation, with m restricted as in 3.*

The restrictions on m insure that K is real. The case m even is handled similarly, except that $\psi = 2(1 + i)\psi_0$, ψ_0 primary, with each choice of ψ_0 corresponding to a different field of conductor F.

Hasse [7] expressed elements of real cyclic fields in terms of a $\mathbf{Q}$-basis of four elements. The 4-tuple $[x_0, x_1, y_0, y_1]$ represents the number

$$\frac{1}{4}(x_0 \pm x_1\sqrt{m} + (y_0 + iy_1)\tau(\chi) + (y_0 - iy_1)\overline{\tau(\chi)}) \tag{5.1}$$

where $\tau = \tau(\chi)$ is the Gauss sum $\sum_{j=1}^{F-1} \chi(j)\zeta_F^j$. The ambiguous sign is determined below. An element is an integer of K if and only if $x_0, x_1, y_0, y_1 \in \mathbf{Z}$ and

$$m \text{ odd} \qquad x_0 \equiv x_1\,, \quad \frac{x_0 + x_1}{2} \equiv Gy_0\,, \quad \frac{x_0 - x_1}{2} \equiv Gy_1 \bmod 2$$

$$m \text{ even} \qquad x_0 \equiv 0 \bmod 4\,, \qquad x_1 \equiv 0 \bmod 2\,.$$

Galois action is described easily in this basis:

$$\sigma\colon [x_0, x_1, y_0, y_1] \longmapsto [x_0, -x_1, -y_1, y_0]$$

Define μ_2 to be $\mu(m/2^{\nu_2(m)})$, where μ is the Möbius function. (The subscript is selected to emphasize use of the 2-free-part.) Assign δ to the ambiguous sign in (5.1). Then from [7], § 7 Equation (12) we deduce that $\delta = \mu_2$ except when $G = 8$, where $\delta = (-1)^{(m-1)/4}\mu_2$.

Write $\psi = a + bi$, as defined and normalized in Proposition 5.1. Multiplication of elements can be accomplished by the following table[1].

[1]This table appears in Hasse [7]; the similar table in Gras [5] confuses Hasse's sign factor σ [our δ] with Galois action, which Hasse called S. As a result, Gras's tuples for the relative fundamental units incorrectly lack sign factors although her formula for the minimal polynomial of a generic element is correct.

$$(5.2) \qquad \tau\bar{\tau} = F\,, \quad \tau^2 = G\psi\delta\sqrt{m}\,, \quad \bar{\tau}^2 = G\bar{\psi}\delta\sqrt{m}\,, \quad \sqrt{m}\,\tau = \delta\psi\bar{\tau}\,.$$

PROPOSITION 5.2. *The following elements and parameter assignments satisfy* (*), *and must therefore be one of the fundamental relative units.*

Case 1. $G = 1,\ b = 4$: $[|a|, \operatorname{sgn} a, 1, 0]$

Case 2. $G = 4,\ b = 2$: $[2|a|, 2\operatorname{sgn} a, 1, 0]$

Case 3. $G = 2,\ a = \pm 2$: $[2b, -2, 1, 1]$

Case 4. $G = 8,\ a = \pm 1$: $[4b, -4, 1, 1]$.

Proof. This is shown in [12] using the formula for the minimal polynomial over $\mathbf{Q}$ of a generic element of $K\backslash k$. For these elements the minimal polynomial is (*). □

Remark. The first component equals $\mathrm{Tr}_{\mathbf{Q}}^{K}\varepsilon = t$.

This proposition shows how to pick out the character of simplest field from other characters of the same conductor: either a or b is predetermined.

§ 6. Proof of Theorem 2

Until Proposition 6.6 K is a real cyclic quartic field, not necessarily simplest.

LEMMA 6.1 (Hasse [7]). *The following are equivalent*:

1. $Q = 2$.
2. *There exists* $\hat{\varepsilon}$ *such that* $E_K = \langle -1, \hat{\varepsilon}, \hat{\varepsilon}^{\sigma}, \hat{\varepsilon}^{\sigma^2}\rangle$.
3. *There exists* $\hat{\varepsilon}$ *such that* $E_k = \langle -1, \hat{\varepsilon}^{1+\sigma^2}\rangle$.
4. *There exists* $\hat{\varepsilon}$ *such that* $E_\chi = \langle -1, \hat{\varepsilon}^{1+\sigma}, \hat{\varepsilon}^{\sigma(1+\sigma)}\rangle$.

Each $\hat{\varepsilon}$ appearing above is the same, up to sign and conjugation by $\pm\sigma$. Such an element $\hat{\varepsilon}$ is called a *Minkowski unit.*

LEMMA 6.2 (Hasse [7]). *If there exist a Minkowski unit* $\hat{\varepsilon}$, *then*

(A) $\hat{\varepsilon}^2 = \pm\,\varepsilon_k\varepsilon_\chi^{1-\sigma}$.

(B) $N_k^K(\varepsilon_\chi) = N_{\mathbf{Q}}^k(\varepsilon_k) = N_{\mathbf{Q}}^K(\hat{\varepsilon})$.

Proof. (A) From Lemma 6.1, we have $\hat{\varepsilon}^{1+\sigma} = \pm\,\varepsilon_k$ and $\hat{\varepsilon}^{1+\sigma^2} = \pm\,\varepsilon_\chi$. It follows that $\hat{\varepsilon}^2\varepsilon_k^{-1}\varepsilon_\chi^{\sigma-1} = \pm 1$.

(B) Immediate from Lemma 6.1.

LEMMA 6.3.

1. *K has a Minkowski unit.*

2. $Q = 2$.
3. *Every totally positive unit is a square.*
4. *There exists a unit* ε *with* $N_{\mathbf{Q}}^{K}(\varepsilon) = -1$.

1. $\Leftrightarrow$ 2., 3. $\Leftrightarrow$ 4., *and* 4. $\Rightarrow$ 2. *If either* $N_{k}^{K}(\varepsilon_{\chi}) = -1$ *or* $N_{\mathbf{Q}}^{k}(\varepsilon_{k}) = -1$, *then also* 2. $\Rightarrow$ 4.

Proof. 1. $\Leftrightarrow$ 2. is Lemma 6.1; 3. $\Leftrightarrow$ 4. is Garbanati [4] Theorem 1. $N_{\mathbf{Q}}^{K}(E_{\chi}E_{k}) = \{1\}$ so 4. $\Rightarrow$ 2. By the previous lemmas if 4. is false and 2. is true all three norms must be $+1$. $\square$

In a simplest field $N_{k}^{K}(\varepsilon_{1}) = -1$ and all four conditions are equivalent.

§ 6.1. Sufficient conditions for $Q = 2$

PROPOSITION 6.4. *If the conductor of* K *is a prime power then* $Q = 2$.

Proof. This is [4], Corollary 1, or Hasse [8], page 29. For a module-theoretic proof see Bouvier and Payan [2], Theorem II. 2. $\square$

The converse is false. The only possible non-prime prime power is 16.

§ 6.2. Sufficient conditions for $Q = 1$

PROPOSITION 6.5. *Suppose* m *is odd, and the quadratic character* ξ_{G} *of conductor* G *is odd, i.e.* $\xi(-1) = -1$. *Then not every totally positive unit is a square.*

Proof. If m is odd then $\gcd(m, G) = 1$. The result therefore follows from [4], Lemmas 4 and 12. $\square$

PROPOSITION 6.6. *In Case* 2 *simplest fields,* $Q = 1$.

Proof. Since ξ_{4} is odd, use Lemma 6.3 and the previous Proposition. $\square$

PROPOSITION 6.7. *In a simplest field, if* $N_{\mathbf{Q}}^{k}(\varepsilon_{k}) = 1$ *then* $Q = 1$.

Proof. Immediate from Lemmas 6.2 and 6.3. $\square$

The converse is false.

PROPOSITION 6.8. *In a simplest field of even conductor,* $Q = 1$, *with the unique exception* $F = 16$, *in which* $Q = 2$.

Proof. We already know this is true for Case 2 and the field of conductor 16, so we need only consider Cases 3 and 4. For a Minkowski unit $\hat{\varepsilon}$, $(\mathrm{Tr}_{\mathbf{Q}}^{K}(\hat{\varepsilon}))^2$ is a square in $\mathbf{Z}$. Gras [5] computes a form for this trace using Lemma 6.2. Write

$$\varepsilon_\chi = [x_0, x_1, y_0, y_1]\,, \quad \varepsilon_k = \frac{u + v\sqrt{m}}{2}\,, \quad s = N_k^K(\varepsilon_\chi)\,,$$
$$r = 2s + \frac{x_0^2 - mx_1^2}{4}\,.$$

$Q = 2$ if and only if the following expression is a nonzero perfect square for some choice of sign.

$$\pm\, 2x_0 + \mathrm{sgn}\,(\varepsilon_k \varepsilon_\chi^{1-\sigma})\frac{2(sr + 2)u - sGmv[b(y_0^2 - y_1^2) + 2ay_0y_1]}{4}\,. \tag{6.1}$$

In any simplest field $s = -1$ and $r = -6$. In Case 3, $G = 2$, $a = \pm 2$, $\varepsilon_\chi = [2b, -2, 1, 1]$, $u = b$ and $v = 1$, using Propositions 4.4 and 5.2. Evaluation of (6.1) gives the possibilities

$$\pm\, 2(b \pm 2)^2\,, \qquad \pm\, 2(b^2 + 4)\,.$$

Recalling that b is positive and even, the second expression is a perfect square only when $b = 2$ (the field of conductor 16) and the first is never a nonzero square.

For Case 4 the parameters are $G = 8$, $a = \pm 1$, $\varepsilon_\chi = [4b, -4, 1, 1]$, $u = 2b$, and $v = 2$, except for the special case $t = 8$, $m = 5$, for which $Q = 1$ in Gras's table [5]. Evaluation of (6.1) yields

$$\pm\, 8(b^2 + 1)\,, \qquad \pm\, 8(b \pm 1)^2$$

which can never be perfect squares for postive even b. □

Remark. The conditions of this proposition can not be readily weakened. In Gras's table Q may be either 1 or 2 for non-simplest fields with

$$N_k^K(\varepsilon_\chi) = N_{\mathbf{Q}}^k(\varepsilon_k) = -1\,, \qquad \nu_2(G) \in \{1, 3\}\,.$$

This concludes the proof of Theorem 2.

§7. Proof of Theorem 1

RESTRICTION. Because of the Gauss Conjecture and the intractability of ε_k when t is odd, in this section we will assume that K is a simplest quartic of *even* conductor not 16. Hence k is a simplest quadratic field as defined by Shanks [16].

Since $Q = 1$, the only parameter in (ACF) remaining to be calculated is $L(1, \chi)$. The odd part of χ is determined easily by factoring m in $\mathbf{Z}[i]$ and computing quartic residue symbols. The even part is determined by G and the condition that $\chi(-1) = 1$. The obvious methods of evaluating $L(1, \chi)$ are far from optimal, so let us review a preferable technique.

The familiar expressions for $L(1, \chi)$ are

$$\begin{aligned} L(1, \chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n} && \text{(Additive form)} \\ &= \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p}\right)^{-1} && \text{(Euler product)}. \end{aligned}$$

Recall the following standard functions [1]:

$$E_1(z) = \int_z^{\infty} \frac{e^{-x}}{x} dx, \qquad \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-x^2} dx$$

Let χ have conductor F. A formula of Lerch is

$$\begin{aligned} L(1, \chi) &= \frac{\tau(\chi)}{F} \sum_{n=1}^{\infty} \chi^{-1}(n) E_1(n^2\pi/F) \\ &\quad + \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \operatorname{erfc}(n\sqrt{\pi/F}) \qquad \text{(method of } \Theta\text{-functions)}. \end{aligned} \tag{7.1}$$

The Euler product works fairly well in practice but has large theoretical error as a function of the largest prime actually used in finite approximation to the infinite product, even on the Generalized Riemann Hypothesis [3]. On the other hand, the error in replacing the infinite sums in (7.1) with the partial sum to N is at most $F^2\pi^{-2}N^{-3} \exp(-N^2\pi F)$ [15] from which it follows that an excellent approximation to h can be obtained with $N \sim F^{-1/2+\varepsilon}$. As with the Euler product, observed convergence is better than theory. In practice, even though τ is known from (5.2) only up to a root of unity, only one possibility gives h sufficiently close to an integer.

The number of quartic fields to be checked for a fixed class number is greatly reduced by Proposition 4.7. To find all fields with $h_K \leq 2$ we need to consider

$$h_k = 1,\ h^- = 1\,; \quad h_k = 1,\ h^- = 2\,; \quad h_k = 2,\ h^- = 1\,.$$

Mollin and Williams [14] compiled an on-GRH list of quadratic fields with class number one whose discriminant was of the Richaud-Degert form $t^2 + r$, $r \in \{\pm 1, \pm 4\}$. This includes all the fields k which arise in Cases 2, 3, and 4.

PROPOSITION 7.1 (Mollin-Williams [14]). *On GRH, if k has class number one then*

$$m \in \{5, 8, 17, 29, 37, 53, 101, 173, 197, 293, 677\}.$$

If GRH is false, there may be one more such field, but a computer search shows that it would have conductor $m > 10^{13}$. The simplest quartic field lying over this hypothetical counterexample to GRH would have class number greater than one because its conductor would exceed the unconditional bound of Theorem L. All fields K for which these eleven k are the quadratic subfield appear in the tables of Gras [5] so the $h_K \leq 2$, $h_k = 1$ problems are settled without any further calculation.

From Theorem L we have that $t > 12000$ is a sufficiently large value to guarantee $h_K > 2$.

LEMMA 7.2. *For $t \leq 12000$ the class number h_k is two if and only if*

$$m \in \{40, 65, 85, 104, 365, 485, 488, 533, 629, 965, \\ 1157, 1448, 1685, 1853, 2117, 2813, 3365\}.$$

Proof. For $h_k = 2$, m can have at most two factors. The class numbers of all such k were computed with the results above. □

The three largest m do not appear in [5], so we calculated $h^- = h_K/h_k$ by (7.1), with the results

t	m	F	h^-
106	2813	11 252	50
184	2117	16 936	90
232	3365	20 920	82

Because no new values with $h^- = 1$ were found, the verification of Theorem 1 is complete.

The following table displays the amount of real time and the accuracy of three different computations of $\prod_{\chi \neq 1} L(1, \chi)$ for $t = 136$. The infinite sums and product were approximated with upper limit N. All calculations were performed on a Macintosh Plus.

TABLE 7.3.

Method	N	$\lvert\%\ \text{error}\rvert$	seconds
Additive	10 000	0.4917%	20.68
Euler $\prod$	15 000	0.1525%	15.83
Θ-functions	200	$< 10^{-4}\%$	17.42

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