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DETERMINANTAL IDEALS WITHOUT MINIMAL FREE RESOLUTIONS

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Introduction

Let R be a Noetherian commutative ring with unit element, and x_{ij} be variables with $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $S = R[x_{ij}]$ be the polynomial ring over R, and I_t be the ideal in S, generated by the $t \times t$ minors of the generic matrix $(x_{ij}) \in M_{m,n}(S)$. For many years there has been considerable interest in finding a minimal free resolution of S/I_t, over arbitrary base ring R. If we have a minimal free resolution $\boldsymbol{P}.$ over $R = \boldsymbol{Z}$, the ring of integers, then $R' \otimes_{\boldsymbol{Z}} \boldsymbol{P}.$ is a resolution of S/I_t over the base ring R'. When does S/I_t have a minimal free resolution over $\boldsymbol{Z}$, then?

The resolution over $\boldsymbol{Z}$ has been found in the case $t = \min(m, n)$ (Eagon-Northcott complex, [3]) and in the case $t = \min(m, n) - 1$ (Akin-Buchsbaum-Weyman complex, [1]). Of course, in the case $t = 1$, we have the resolution of S/I_t, namely, the Koszul complex. Recently, we proved that S/I_t has a minimal free resolution over $\boldsymbol{Z}$ in the case $m = n = t + 2$ [5]. But our proof consists in showing that the Betti numbers of S/I_t are independent of the characteristic of the ground field, so it does not provide an explicit construction of a resolution.

In this paper, we prove that S/I_t does not have any minimal free resolutions, if R is the ring of integers $\boldsymbol{Z}$, and if $2 \leq t \leq \min(m, n) - 3$, as we announced in [5]. The third Betti number of S/I_t is independent of the characteristic, if $t = 1$ or $t \geq \min(m, n) - 2$ ([5]). To the contrary, it depends on the characteristic if $2 \leq t \leq \min(m, n) - 3$. If the characteristic is 3, then the Betti number gets larger than the characteristic zero case.

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§ 1. Preliminaries

On the characteristic free representation theory of GL, including the notion of partitions, Schur modules (Schur functors) and Schur complexes, tableaux, and Cauchy formulae, we use the notation, the terminology and the results of [2] and [5] freely. But we shall review some facts on the characteristic free representation theory of GL, which will be used later. For the details, see [2] and [5].

Let R be a commutative ring with unit, and α: $0 \to G \xrightarrow{\psi} F \xrightarrow{\varphi} E \to 0$ be a finite free complex of length two. We define the symmetric algebra of α, to be the tensor product: $S\alpha = SE \otimes \wedge F \otimes DG$. $S\alpha$ has a structure of a graded bialgebra over R, with an appropriate anticommutative structure. Moreover, $S\alpha$ has a structure of a chain complex. We define the boundary map $\partial^{S\alpha}$ to be the sum, $\partial^{S\varphi} \otimes 1_{DG} \pm 1_{SE} \otimes \partial^{\wedge\psi}$. The multiplication and the comultiplication of $S\alpha$ are chain maps (see [5, chapter I, § 2]).

Let φ: $F_1 \to F_0$ and ψ: $G_1 \to G_0$ be two morphisms of finite free modules, and k be a nonnegative integer. There is a unique universal natural transformation θ_k, which makes the following diagram commutative;

$$(*)\qquad \begin{tikzcd} \wedge^k\varphi \otimes \wedge^k\psi \arrow[r, "\theta_k"] \arrow[d, "\Delta\otimes\Delta"] & S_k(\varphi\otimes\psi) \arrow[d, "\Delta"] \\ T_k\varphi \otimes T_k\psi \arrow[r, "T"] & T_k(\varphi\otimes\psi) \end{tikzcd}$$

where Δ's in the diagram are appropriate diagonalizations, and the T in the diagram is an appropriate twisting. We define θ: $\wedge\varphi \otimes \wedge\psi \to S(\varphi\otimes\psi)$ given by $\theta = \theta_k$ on $\wedge^k\varphi \otimes \wedge^k\psi$, and $\theta = 0$ on $\wedge^i\varphi \otimes \wedge^j\varphi$ if $i \neq j$. The natural transformation θ is the composite map;

$$\wedge\varphi \otimes \wedge\psi = \wedge F_0 \otimes DF_1 \otimes \wedge G_0 \otimes DG_1$$

$$\xrightarrow{\Delta} \wedge F_0 \otimes \wedge F_0 \otimes DF_1 \otimes DF_1 \otimes \wedge G_0 \otimes \wedge G_0 \otimes DG_1 \otimes DG_1$$
$$\xrightarrow{T} \wedge F_0 \otimes \wedge G_0 \otimes DF_1 \otimes \wedge G_0 \otimes \wedge F_0 \otimes DG_1 \otimes DF_1 \otimes DG_1$$
$$\xrightarrow{\phi^S \otimes \phi^\wedge \otimes \psi^\wedge \otimes \psi^D} S(F_0 \otimes G_0) \otimes \wedge(F_1 \otimes G_0) \otimes \wedge(F_0 \otimes G_1) \otimes D_1(F_1 \otimes G_1)$$
$$\xrightarrow{\simeq} S(F_0 \otimes G_0) \otimes \wedge(F_1 \otimes G_0 \oplus F_0 \otimes G_1) \otimes D(F_1 \otimes G_1) = S(\varphi \otimes \psi)$$

where Δ is the diagonalization, T is an appropriate twisting. ϕ^S, $\phi^\wedge$, $\psi^\wedge$, and ψ^D are the unique universal natural transformations determined as follows. We define $\phi_k^S(F, G)$: $\wedge^k F \otimes \wedge^k G \to S_k(F \otimes G)$ for any nonnegative integer k to be the unique universal natural transformation which makes the following diagram commutative.

$$(**) \qquad \begin{array}{ccc} \wedge^k F \otimes \wedge^k G & \xrightarrow{\phi_k^S} & S_k(F \otimes G) \\ \downarrow{\scriptstyle \Delta \otimes \Delta} & & \downarrow{\scriptstyle \Delta} \\ T_k F \otimes T_k G & \xrightarrow{\simeq} & T_k(F \otimes G) \end{array}$$

We define $\phi^S = \phi_k^S$ on $\wedge^k F \otimes \wedge^k G$ and $\phi^S = 0$ on $\wedge^i F \otimes \wedge^j G$ if $i \neq j$. Thus ϕ^S is a natural transformation which maps $\wedge F \otimes \wedge G$ to $S(F \otimes G)$. The definitions of $\phi^\wedge$, $\psi^\wedge$, and ψ^D are quite similar (see [5, chapter III]). Note that ϕ_k^S is given by

$$\phi_k^S(f_1 \wedge \cdots \wedge f_k \otimes g_1 \wedge \cdots \wedge g_k) = (-1)^{k(k-1)/2} \det(f_i \otimes g_j)_{1 \le i, j \le k}$$

for $f_1, \cdots, f_k \in F$ and $g_1, \cdots, g_k \in G$. Since the diagram $(*)$ commutes, θ_k is a chain map.

For a partition λ with $lg(\lambda) = q$ and $|\lambda| = r$, we define θ_λ: $\wedge_\lambda \varphi \otimes \wedge_\lambda \psi \to S_r(\varphi \otimes \psi)$ to be the composite map;

$$\wedge_\lambda \varphi \otimes \wedge_\lambda \psi = \wedge^{\lambda_1}\varphi \otimes \cdots \otimes \wedge^{\lambda_q}\varphi \otimes \wedge^{\lambda_1}\psi \otimes \cdots \otimes \wedge^{\lambda_q}\psi$$
$$\xrightarrow{T} \wedge^{\lambda_1}\varphi \otimes \wedge^{\lambda_1}\psi \otimes \cdots \otimes \wedge^{\lambda_q}\varphi \otimes \wedge^{\lambda_q}\psi$$
$$\xrightarrow{\theta_{\lambda_1} \otimes \cdots \otimes \theta_{\lambda_q}} S_{\lambda_1}(\varphi \otimes \psi) \otimes \cdots \otimes S_{\lambda_q}(\varphi \otimes \psi) \xrightarrow{m} S_r(\varphi \otimes \psi)$$

where T is an appropriate twisting, and m is the (iterated) multiplication. We also define:

$$M_\lambda(\theta) = \sum_{\substack{|\mu| = r \\ \mu \ge \lambda}} \operatorname{Im} \theta_\mu \quad \text{and} \quad \dot{M}_\lambda(\theta) = \sum_{\substack{|\mu| = r \\ \mu > \lambda}} \operatorname{Im} \theta_\mu$$

For $r \in \boldsymbol{N}_0$, $\{M_\lambda(\theta)\}_{|\lambda| = r}$ gives a filtration of $S_r(\varphi \otimes \psi)$.

The Cauchy formula holds for $S(\varphi \otimes \psi)$ via the pairing θ.

LEMMA 1.1 ([5, Proposition III, 2.6]). *Let $\lambda \in \Omega_k^+$ and $\mu \in S_\square(\lambda)$. The following diagram is commutative.*

$$\begin{array}{ccccc}
\wedge_\mu\varphi \otimes \wedge_\lambda\psi & \xrightarrow{\mathrm{id}\otimes\widetilde{\square}_\mu(\wedge\psi)} & \wedge_\mu\varphi \otimes \wedge_\mu\psi & \xleftarrow{\widetilde{\square}_\mu(\wedge\varphi)\otimes\mathrm{id}} & \wedge_\lambda\varphi \otimes \wedge_\mu\psi \\
\downarrow{\scriptstyle \square_\lambda(\wedge\varphi)\otimes\mathrm{id}} & & \downarrow{\scriptstyle \theta_\mu} & & \downarrow{\scriptstyle \mathrm{id}\otimes\square_\lambda(\wedge\psi)} \\
\wedge_\lambda\varphi \otimes \wedge_\lambda\psi & \xrightarrow{\theta_\lambda} & S_k(\varphi\otimes\psi) & \xleftarrow{\theta_\lambda} & \wedge_\lambda\varphi \otimes \wedge_\lambda\psi
\end{array}$$

THEOREM 1.2 ([5: Theorem III. 2.7]). *Let $k \in N_0$, and $\varphi\colon F_1 \to F_0$ and $\psi\colon G_1 \to G_0$ be morphism of finite free R-modules. If $\lambda \in \Omega_k^-$, then θ_λ induces the* isomorphism *of complexes $\beta_\lambda\colon L_\lambda\varphi \otimes L_\lambda\psi \to M^\lambda(\theta)/\dot{M}^\lambda(\theta)$ which makes the following diagram commutative;*

$$\begin{array}{ccc}
\wedge_\lambda\varphi \otimes \wedge_\lambda\psi & \xrightarrow{\theta_\lambda} & M^\lambda(\theta) \\
\downarrow{\scriptstyle d_\lambda\otimes d_\lambda} & & \downarrow{\scriptstyle \mathrm{proj.}} \\
L_\lambda\varphi \otimes L_\lambda\psi & \xrightarrow{\beta_\lambda} & M^\lambda(\theta)/\dot{M}^\lambda(\theta)
\end{array}$$

where L_λ is the Schur complex with respect to the shape λ. Hence, the associated graded complex of the filtration $\{M^\lambda(\theta)\}_{\lambda\in\Omega_k^-}$ is $\sum_{\lambda\in\Omega_k^-} L_\lambda\varphi \otimes L_\lambda\psi$.

Now we fix positive integers m, n, and t with $t \leq \min(m, n)$, and we consider free R-modules F and G with $\operatorname{rank} F = m$ and $\operatorname{rank} G = n$. We let $S = S(F\otimes G)$ so that S is isomorphic to the polynomial ring with $m\cdot n$ variables over R. We define I_t to be the ideal of S generated by $\operatorname{Im}\phi_t^S$ and call I_t *a determinantal ideal.* For $r \in N_0$, we denote $S_r(F\otimes G)$ by S_r, and $S_r \cap I_r$ by $I_{t,r}$. We denote the complex $I_t \otimes_S S(\mathrm{id}_{F\otimes G})$ (resp. $I_t \otimes_S S(\mathrm{id}_F \otimes \mathrm{id}_G)$) by $\mathscr{I}^t$ (resp. $\tilde{\mathscr{I}}^t$). The complex $\mathscr{I}^t$ (resp. $\tilde{\mathscr{I}}^t$) is a graded S-complex so that $\mathscr{I}^t$ (resp. $\tilde{\mathscr{I}}^t$) is decomposed into the direct sum; $\mathscr{I}^t = \sum_{r\in N_0} \mathscr{I}^{t,r}$ (resp. $\tilde{\mathscr{I}}^t = \sum_{r\in N_0} \tilde{\mathscr{I}}^{t,r}$). Since $S(\mathrm{id}_{F\otimes G}) = S \otimes \wedge(F\otimes G)$ is a graded minimal free resolution of $R = S/I_1$, $H_i(\mathscr{I}^{t,r})$ is the degree r component $[\underline{\mathrm{Tor}}_i^S(I_t, S/I_1)]_r$ of the graded S-module $\underline{\mathrm{Tor}}_i^S(I_t, S/I_1)$ for any $i \geq 0$ and $r \geq 0$. On the other hand, we have an isomorphism $H_i(\mathscr{I}^{t,r}) \simeq H_i(\tilde{\mathscr{I}}^{t,r})$ for any i and r [5, Lemma IV. 1.4]. In case $R = K$ is a field of characteristic p, we denote $\dim_K [\mathrm{Tor}_i^S(S/I_t, S/I_1)]_r$, which is invariant under an extention of the base field K, by $\beta_{i,r}^p$. We have the following lemma.

LEMMA 1.3. *There is a minimal free resolution of S/I_t in the case $R = Z$, if and only if $\beta_{i+1,r}^p = \operatorname{rank} H_i(\tilde{\mathscr{I}}^{t,r})$ is independent of the characteristic p of the base field $R = K$ for any $i \geq 0$.*

For the proof of the lemma, see [9, Proposition 2 of chapter 4] or

[5, Proposition II. 3.4].

Now we shall prepare some additional notation. We define $\pi\colon \mathrm{id}_F\otimes\mathrm{id}_G \to \mathrm{id}_{F\otimes G}$ and $\iota\colon \mathrm{id}_{F\otimes G}\to\mathrm{id}_F\otimes\mathrm{id}_G$ to be the morphisms of complexes given by:

$$\begin{tikzcd}
\mathrm{id}_F\otimes\mathrm{id}_G = 0 \arrow[r] \arrow[d, "\pi"'] & F\otimes G \arrow[r, "\binom{-1}{1}"] \arrow[d, "0"] & F\otimes G\oplus F\otimes G \arrow[r, "(1{,}1)"] \arrow[d, "(1{,}1)"] & F\otimes G \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{id}_{F\otimes G} = 0 \arrow[r] & 0 \arrow[r] & F\otimes G \arrow[r, "1"] & F\otimes G \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{id}_{F\otimes G} = 0 \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[r] \arrow[d, "0"] & F\otimes G \arrow[r, "1"] \arrow[d, "(1{,}0)"] & F\otimes G \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{id}_F\otimes\mathrm{id}_G = 0 \arrow[r] & F\otimes G \arrow[r, "\binom{-1}{1}"] & F\otimes G\oplus F\otimes G \arrow[r, "(1{,}1)"] & F\otimes G \arrow[r] & 0
\end{tikzcd}$$

It is easy to see that $\pi\circ\iota = \mathrm{id}_{\mathrm{id}_{F\otimes G}}$. For $r\in \boldsymbol{N}_0$, $S_r\pi$ maps $\tilde{\mathscr{J}}^{t,r}$ onto $\mathscr{J}^{t,r}$, and $S_r\iota$ maps $\mathscr{J}^{t,r}$ into $\tilde{\mathscr{J}}^{t,r}$. Since $H_*(\tilde{\mathscr{J}}^{t,r})\simeq H_*(\mathscr{J}^{t,r})$ and $H_*(S_r\pi)\circ H_*(S_r\iota)=\mathrm{id}$, $H_*(S_r\pi)$ gives an explicit isomorphism between them. We define $\alpha^r\colon \wedge^r\mathrm{id}_F\otimes\wedge^r\mathrm{id}_G\to S_r(\mathrm{id}_{F\otimes G})$ to be the composition $S_r\pi\circ\theta_r$ for $r\in\boldsymbol{N}_0$. It is clear that α^r maps $L_k^{t,(r)}=\sum_{i+j=k}L_{i,j}^{t,(r)}$ to $\mathscr{J}_k^{t,r}$ for $t,k\in\boldsymbol{N}_0$ (for the definition of $L_{i,j}^{t,\lambda}$ (λ a partition), see [5, Definition IV. 1.5]). Note that $L^{t,(r)}$ is nothing but the complex $\{\boldsymbol{U}^t(F,G),\partial^t\}$ defined in [1, Definition 3.7]. The map α^r coincides with the map defined in [1, Remark 3.19]. If R contains $\boldsymbol{Q}$, then $\alpha_k^{t,t+k}\colon L_k^{t,(t+k)}\to Z_{k+1}^{t,t}=\partial_k^{-1}(\mathscr{J}_{k-1}^{t+1,t+k})$ is surjective, but this is not true in general (see section 3).

We fix ordered bases $X=X_0\cup X_1$ of $\mathrm{id}_F\colon F_1\to F_0$ and $Y=Y_0\cup Y_1$ of $\mathrm{id}_G\colon G_1\to G_0$, where $X_0=\{x_1<\cdots<x_m\}$, $X_1=\{x_1'<\cdots<x_m'\}$, $Y_0=\{y_1<\cdots<y_n\}$ and $Y_1=\{y_1'<\cdots<y_n'\}$ are bases of F_0, F_1, G_0 and G_1, respectively. The ordering is given by $X_0<X_1$ and $Y_0<Y_1$. For simplicity of notation, we may denote x_i and y_i by i, and x_i' and y_i' by i', if there is no danger of confusion.

For a tableau $S\in\mathrm{Tab}_{\lambda/\mu}(X)$ and subsets $I\subset X$ and $N\subset\boldsymbol{N}$, we denote $\#\{(i,j)\in\Delta_{\lambda/\mu}\mid i\in N \text{ and } T(i,j)\in I\}$ by $n_N(T,I)$. In this notation, an element $x\in X$ (resp. $i\in\boldsymbol{N}$) may stand for the singleton $\{x\}$ (resp. $\{i\}$). We denote $n_i(S,X_1)$ by $n_i(S)$, and $n_{\boldsymbol{N}}(S,X_1)$ by $n(S)$. We will use a similar convention for a tableau $T\in\mathrm{Tab}_{\lambda/\mu}(Y)$.

Let $\lambda\in\Omega^-$, $S\in\mathrm{Tab}_\lambda X$, and $T\in\mathrm{Tab}_\lambda Y$. We use the bitableau notation as in [2]. We denote $\theta_\lambda(S\otimes T)$ by $(S\mid T)$. More generally, we will denote

$\theta_\lambda(a \otimes b)$ by $(a|b)$ for $a \in \wedge_\lambda \mathrm{id}_F$ and $b \in \wedge_\lambda \mathrm{id}_G$. The set of tableaux, $\{S \in \mathrm{Tab}_\lambda X \mid S$ is row-standard mod $X_1\}$, is denoted by X_λ. The set Y_λ is defined similarly.

Let $R = K$ be an infinite field, and M be a polynomial representation of $GL(F)$ (i.e., M be a $K[\mathrm{End}(F)]$-module with $\dim_K M < \infty$, and the representation map ρ: $\mathrm{End}(F) \to \mathrm{End}(M)$ be a regular morphism). We identify $\mathrm{End}(F)$ with $M_n(K)$ via the basis $X = \{x_1, \cdots, x_m\}$. For a sequence $\alpha = (\alpha_1, \cdots, \alpha_m) \in N_0^m$, we define the subspace M_α of M by

$$M_\alpha = \{a \in M \mid {}^\forall(t_1, \cdots, t_m) \in K^m \ \ \rho(t_1 \oplus \cdots \oplus t_m) \cdot a = t_1^{\alpha_1} \cdots t_m^{\alpha_m} \cdot a\}$$

where $t_1 \oplus \cdots \oplus t_m$ is a diagonal matrix whose (i, i) content is t_i. We call M_α the *α-weight submodule of M*, and α its *weight*. The representation M is decomposed into the direct sum of M_α. Any morphism of polynomial representations of $GL(F)$ preserves weight. So any chain complex of polynomial representations of $GL(F)$, say $\boldsymbol{P}$, is decomposed into the direct sum; $\boldsymbol{P} = \sum_\alpha \boldsymbol{P}_\alpha$.

We will consider complexes of polynomial representations of $GL(F) \times GL(G)$ in section 3. Such a complex, say $\boldsymbol{C}$, is decomposed into the direct sum of biweight subcomplexes $\boldsymbol{C}_\alpha$ corresponding to the biweight $\alpha = (\alpha(F); \alpha(G))$. For example, the biweight $(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n)$ submodule of $S_k(\mathrm{id}_F \otimes \mathrm{id}_G)$ is generated by:

$$\begin{aligned}\{(S \mid T) \mid {}^\exists\lambda \in \Omega_k^-,\ S \in X_\lambda,\ T \in Y_\lambda,\ {}^\forall i\ (1 \leq i \leq m) \\ n_N(S, \{x_i, x_i'\}) = \alpha_i,\ {}^\forall j\ (1 \leq j \leq n)\ n_N(T, \{y_j, y_j'\}) = \beta_j\}\end{aligned}$$

Any universally free functor L on F and G that we will consider will always be a polynomial functor. So $L(F, G)$ is a polynomial representation of $GL(F) \times GL(G)$.

§ 2. The filtration of $\mathscr{I}^{t,r}$

We have calculated β_3^p in the case $t \geq \min(m, n) - 2$, in [5], using the natural filtration $\{M^{t,\lambda}\}_{\lambda \in \Omega_r^-}$ of $\mathscr{I}^{t,r}$. We can associate with this filtration the usual spectral sequence whose E^1-term is $E_*^{1,t,\lambda} = H_*(M^{t,\lambda}/\dot{M}^{t,\lambda})$. We use the following facts on the homology of the associated graded complex of this filtration.

PROPOSITION 2.1. *Let m, n, r and t be positive integers with* $\min(m, n) \geq t$, *and $\lambda \in \Omega_r^-$. Then we have:*

(1) $E_1^{1,t,\lambda} = 0$, *except for the case $\lambda = (t+1)$. In particular,* $H_1(\mathscr{I}^{t,r})$

$= 0$ *except for the case* $r = t + 1$.

(2) $E_2^{1,t,\lambda} = 0$, *except for the following three cases.*

(i) $\lambda = (t + 2)$

(ii) $r = 2t + 1$, $\lambda = (t + 1, t)$, $1/(t + 1) \notin R$, *and* $\min(m, n) \geq r$

(iii) $t < r \leq 2t$, $\lambda = (t, r - t)$, $1/(r - t) \notin R$, *and* $\min(m, n) \geq r$

(3) *If the following two conditions hold, then* $E_3^{1,t,\lambda} = 0$.

(i) $\lambda_1 = t$ *or* $\lambda_2 < t$

(ii) $lg(\lambda) \geq 3$, *or equivalently*, $\lambda_3 \neq 0$

Proof. (3) is [5, Proposition IV. 3.1]. (2) is a little stronger than [5, Proposition IV. 2.3]. We have to show that $E_2^{1,t,\lambda} = 0$ if $\lambda \neq (t + 2)$ and if $r - t$ is invertible in R.

We use the same spectral sequence argument used in the proof of [5, Proposition IV. 2.3]. By Lemma IV. 2.4 and Lemma IV. 2.7 of [5], we have only to show that $E_{1,1}^2 = H_1^G(H_1^F(M_{*,*}^{t,\lambda}/\dot{M}_{*,*}^{t,\lambda})) = 0$.

First we consider the case $t < r \leq 2t$, and $\lambda = (t, r - t)$. In this case the same argument as in the proof of [5, Lemma IV. 2.8] works. In fact, any element of $E_{1,1}^2$ is represented by $A = \sum_{S,T} c_{S,T}(S \mid T)$, where S is standard mod X_1, T is standard mod Y_1, and $n_1(S) = n_1(T) = 0$. So we can write $\sum_S c_{S,T}\partial_F^\lambda S = \sum_{\mu \in S_{\square}(\lambda)} \square_\lambda^\mu(a_\mu^T)$, where $a_\mu^T \in \wedge_\mu F$. But since $\square_\lambda^{(r)}(a_{(r)}^T) = 1/(r - t)\square_\lambda^{(r-1,1)}(\square_{(r-1,1)}^{(r)}(a_{(r)}^T))$, we may assume that $a_{(r)}^T = 0$, after replacing $a_{(r-1,1)}^T$ by $a_{(r-1,1)}^T + 1/(r - t)\square_{(r-1,1)}^{(r)}(a_{(r)}^T)$. So this case is clear.

We consider the case $\lambda = (t + 1, t)$. Any element of $E_{1,1}^2$ is represented by $A = \sum_{S,T} c_{S,T}(S \mid T)$, where $S \in X_\lambda$, $T \in Y_\lambda$, S is standard mod X_1, T is standard mod Y_1, and $n(S) = n(T) = 1$. We claim that for each pair (S, T), which appears in the sum with $n_1(S) = n_1(T) = 1$, it holds

$$(S \mid T) \in \theta_\lambda(L_{1,1}^{t,\lambda,1}) + \dot{M}_{1,1}^{t,\lambda} + \partial_F(M_{2,1}^{t,\lambda}) + \partial_G(M_{1,2}^{t,\lambda}).$$

If the claim is true, we may assume that $A \in \theta_\lambda(L_{1,1}^{t,\lambda,1}) + \partial_G(M_{1,2}^{t,\lambda})$. So we can write $A = A' + \partial_G B$ with $A' \in \theta_\lambda(L_{1,1}^{t,\lambda,1})$ and $B \in M_{1,2}^{t,\lambda}$. It is easy to see that there exists some $B' \in \theta_\lambda(L_{1,2}^{t,\lambda,1})$ such that $\partial_F(B - B') \in \dot{M}_{0,2}^{t,\lambda}$ (see the proof of [5, Lemma IV. 2.4]). Replacing $A = A' + \partial_G B$ by $A' + \partial_G B'$, we may assume $A \in \theta_\lambda(L_{1,1}^{t,\lambda,1})$. So the proof of [5, Lemma IV. 2.8] is still valid by [5, Lemma IV. 2.6], and it suffices to prove the claim.

We shall prove the claim.

We put;

$$S = \begin{matrix} a_1 \cdots a_t a'_{t+1} \\ b_1 \cdots b_t \end{matrix} \quad \text{and} \quad T = \begin{matrix} \alpha_1 \cdots \alpha_t \alpha'_{t+1} \\ \beta_1 \cdots \beta_t \end{matrix}$$

where a_i and b_j are elements of X_0, and α_i and β_j are elements of Y_0. We may assume that α_i ans β_j are all distinct (if not, then the claim is (essentially) proved in [5, Lemma IV. 2.5]). If we set;

$$S' = \begin{matrix} a_1 \cdots a_t a_{t+1} \\ b_1 \cdots b'_t \end{matrix}$$

then we have

$$\begin{aligned}(S\,|\,T) - (S'\,|\,T) &= \frac{1}{t+1}\cdot\partial_G\left(S\;\middle|\;\sum_{j=1}^{t}(-1)^{t-j}\begin{matrix}\alpha_1 \overset{\overset{j}{\wedge}}{\cdots} \alpha_t\alpha'_j\alpha'_{t+1} \\ \beta_1 \cdots \beta_t\end{matrix}\right) \\ &+ \frac{1}{t+1}\cdot\partial_G\left(S'\;\middle|\;t\cdot\begin{matrix}\alpha_1 \cdots \alpha_t\alpha'_{t+1} \\ \beta_1 \cdots \beta'_t\end{matrix}\;\sum_{j=1}^{t}(-1)^{t-j}\begin{matrix}\alpha_1 \overset{\overset{j}{\wedge}}{\cdots} \alpha_t \;\; \alpha_{t+1}\alpha'_j \\ \beta_1 \cdots \beta_{t-1}\beta'_t\end{matrix}\right) \\ &+ \frac{1}{t+1}\cdot\left(S - S'\;\middle|\;\sum_{j=1}^{t+1}(-1)^{t-j+1}\begin{matrix}\alpha_1 \overset{\overset{j}{\wedge}}{\cdots} \alpha_{t+1}\alpha'_j \\ \beta_1 \cdots \beta_t\end{matrix}\right),\end{aligned}$$

where each symbol $\overset{j}{\wedge}$ indicates the deletion of the j-th member in the sequence. Hence, it suffices to show that the element

$$C = \frac{1}{t+1}\cdot\left(S - S'\;\middle|\;\sum_{j=1}^{t+1}(-1)^{t-j+1}\begin{matrix}\alpha_1 \overset{\overset{j}{\wedge}}{\cdots} \alpha_{t+1}\alpha'_j \\ \beta_1 \cdots \beta_t\end{matrix}\right)$$

is contained in $\partial_F(M^{t;\lambda}_{2,1})$. We shall calculate C. If we put

$$U = \sum_{j=1}^{t+1}(-1)^{t-j+1}\begin{matrix}\alpha_1 \overset{\overset{j}{\wedge}}{\cdots} \alpha_{t+1} \\ \beta_1 \cdots \beta_t \\ \alpha'_j\end{matrix} \in [\wedge_{(t,t,1)}\mathrm{id}_G]_1\,,$$

then using Lemma 1.1, we have

$$\begin{aligned}C &= \frac{1}{t+1}\cdot\left(\begin{matrix}a_1 \cdots a_t \\ b_1 \cdots b_t \\ a'_{t+1}\end{matrix} + \sum_{j=1}^{t}(-1)^{t+1-j}\begin{matrix}a_1 \overset{\overset{j}{\wedge}}{\cdots} a_t a'_{t+1} \\ b_1 \cdots\cdots b_t \\ a_j\end{matrix}\;\middle|\;U\right) \\ &- \frac{1}{t+1}\cdot\left(\sum_{j=1}^{t+1}(-1)^{t-j+1}\begin{matrix}a_1 \cdots\overset{\overset{j}{\wedge}}{\cdots}\cdots a_{t+1} \\ b_1 \cdots b_{t-1}b'_t \\ a_j\end{matrix}\;\middle|\;U\right) \\ &= \frac{1}{t+1}\cdot\partial\left(\sum_{j=1}^{t}(-1)^{t-j}\begin{matrix}a_1 \overset{\overset{j}{\wedge}}{\cdots} a_t a'_{t+1} \\ b_1 \cdots\cdots b_t \\ a'_j\end{matrix} - \sum_{j=1}^{t+1}(-1)^{t-j}\begin{matrix}a_1 \cdots\overset{\overset{j}{\wedge}}{\cdots}\cdots a_{t+1} \\ b_1 \cdots b_{t-1}b'_t \\ a'_j\end{matrix}\;\middle|\;U\right) + D,\end{aligned}$$

where $D \in M_{2,0}^{t,(t,t,1)}$ is of the form $D = (V\,|\,\partial_G U)$. Since

$$\partial_G U = \sum_{j=1}^{t+1}(-1)^{t-j+1}\begin{matrix} \alpha_1 \cdots \overset{j}{\hat{}} \cdots \alpha_{t+1} \\ \beta_1 \cdots \beta_t \\ \alpha_j \end{matrix}$$

and

$$\sum_{j=1}^{t+1}(-1)^{t-j+1}\alpha_1 \wedge \cdots \overset{j}{\hat{}} \cdots \wedge \alpha_{t+1} \otimes \alpha_j = \Delta(\alpha_1 \wedge \cdots \wedge \alpha_{t+1})$$

it holds that $D \in M_{2,0}^{t,\lambda}$. Hence,

$$\frac{1}{t+1}\cdot\partial\left(\sum_{j=1}^{t}(-1)^{t-j}\begin{matrix} a_1 \cdots \overset{j}{\hat{}} \cdots a_t a'_{t+1} \\ b_1 \cdots\cdots b_t \\ a'_j \end{matrix} - \sum_{j=1}^{t+1}(-1)^{t-j}\begin{matrix} a_1 \cdots \overset{j}{\hat{}} \cdots a_{t+1} \\ b_1 \cdots b_{t-1} b'_t \\ a'_j \end{matrix}\middle| U\right) + D$$

is a cycle of $M^{t,(t,t,1)}/M^{t,\lambda}$. By (3) of this proposition (see also Proposition 2.3 below), $C - D$ is a boundary of $M^{t,\lambda}$ so that $C \in \partial_F(M_{2,1}^{t,\lambda})$. This proves our claim, so we have completed the proof of (2).

(1) can be proved quite similarly to (2), and so we omit the proof.

Remark 2.2. From (1) of (2.1), we can conclude that $\beta_{2,r} = 0$, unless $r = t + 1$. Furthermore, we can see that $X_2^t = H_1(\mathscr{I}^{t,t+1}) = E_1^{1,t,(t+1)}$ is a homomorphic image of $H_1(L^{t,(t+1)})$ by the morphism $H_1(\alpha^{t,t+1})$. Using this fact, it is not difficult to see that X_2^t is generated by the elements of the following form;

$$\partial(i_2 \cdots i_t i'_1 i'_{t+1} \mid j_1 \cdots j_{t+1}) \quad \text{with } \begin{matrix} i_1 \cdots i_t \\ i_{t+1} \end{matrix} \text{ and } j_1 \cdots j_{t+1} \text{ are both standard}$$

and

$$\partial(i_2 \cdots i_{t+1} i'_1 \mid j_1 \cdots j_t j'_{t+1}) \quad \text{with } i_1 \cdots i_{t+1} \text{ and } \begin{matrix} j_1 \cdots j_t \\ j_{t+1} \end{matrix} \text{ are both standard}$$

where ∂ is the boundary map of $S(\mathrm{id}_F \otimes \mathrm{id}_G)$. Since

$$\operatorname{rank} X_2^t = \operatorname{rank}\,[L_{(t,1)}F \otimes \wedge^{t+1}G \oplus \wedge^{t+1}F \otimes L_{(t,1)}G]\,,$$

these elements are a free basis of X_2^t.

These facts were first proved essentially by Kurano [6].

PROPOSITION 2.3. *We let $\lambda_0 = (3, 2)$ if $t = 2$, and $\lambda_0 = (t, 3)$ if $t \geq 3$. Then $E_2^{1,t,\lambda_0} \simeq E_2^{\infty,t,\lambda_0}$. In particular, if we have $E_2^{1,t,\lambda_0} \neq 0$, then $\beta_{3,t+3} \neq 0$.*

Proof. If μ is a partition of weight $t + 3$ with $\mu < \lambda_0$ in the lexicographic order, then μ satisfies the conditions (i) and (ii) of (3) in Proposition 2.1, so that $E_3^{1,t,\mu}=0$. We have $E_1^{1,t,\nu}=0$ for any partition ν of weight $t+3$ by (1) of the proposition. With these facts and the standard spectral sequence argument, it is easy to see that $E_2^{1,t,\lambda_0}\simeq E_2^{\infty,t,\lambda_0}$. The second assertion is now clear and the proof is complete.

By Lascoux's resolution [7], we know that $\beta_{3,t+3}^0=0$. Furthermore, we can see that $\beta_{3,t+3}^p=0$ if p is a prime number with $p\neq 3$ by (2) of Proposition 2.1. We shall show that $\beta_{3,t+3}^3\neq 0$, if $2\leq t\leq \min(m,n)-3$.

§ 3. The main result

This section is devoted to prove the next theorem.

THEOREM 3.1. *Let m, n and t be positive integers with $2\leq t\leq \min(m,n)-3$. Then the third Betti number β_3 of S/I_t depends on the characteristic. In this case, S/I_t does not have any minimal free resolutions over $\mathbf{Z}$.*

Proof. By the argument in section 2 and Lemma 1.3, we see that it sufficies to show that $E_2^{1,t,\lambda_0}\neq 0$ when R is an infinite field K of characteristic three, where λ_0 is the partition defined in Proposition 2.3. Each $M^{t,\lambda}$ is decomposed into the direct sum of the summands indexed by the *bicontents* (see section 1). So it is sufficient to show that the biweight

$$\alpha=(\underbrace{1,1,\cdots,1}_{t+3},0,\cdots,0;\ \underbrace{1,1,\cdots,1}_{t+3},0,\cdots,0)$$

submodule of E_2^{1,t,λ_0} is not zero. We shall show that $E=E_{2,\alpha}^{1,t,\lambda_0}=[E_2^{1,1,\lambda_0}]_\alpha$ is not zero. To this end, we construct a non-zero linear form $h\colon E\to K$.

(i) *case* 1. $t=2$.

First, we construct a linear form $g\colon L_{1,1,\alpha}^{t,\lambda_0}\to K$. Note that $L_{1,1,\alpha}^{t,\lambda_0}=L_{1,1,\alpha}^{t,\lambda_0,1}\oplus L_{1,1,\alpha}^{t,\lambda_0,2}$. It holds that

$$L_{1,1,\alpha}^{t,\lambda_0,2}=[\wedge^2F\otimes D_1F\otimes\wedge^2F]_{\alpha(F)}\otimes[\wedge^2G\otimes D_1G\otimes\wedge^2G]_{\alpha(G)}$$

where $[\ \ ]_{\alpha(F)}$ and $[\ \ ]_{\alpha(G)}$ indicates the weight $(1,1,1,1,1,0,0,\cdots)$-submodule. Hence, the basis element of $L_{1,1,\alpha}^{t,\lambda_0,2}$ is of the form

$$S\otimes T=\begin{matrix}\sigma1\sigma2(\sigma3)'\\ \sigma4\sigma5\end{matrix}\otimes\begin{matrix}\tau1\tau2(\tau3)'\\ \tau4\tau5\end{matrix}$$

with $\sigma,\tau\in\mathfrak{S}_5$, and S and T both row-standard (mod X_1 or mod Y_1). For such a basis element, define $g(S\otimes T)=(-1)^{\sigma\tau}$. We define g to be zero on $L_{1,1,\alpha}^{t,\lambda_0,1}$. This gives the definition of g. We shall see that g induces a linear form $\bar{g}$; $M_{1,1,\alpha}^{t,\lambda_0}/\dot{M}_{1,1,\alpha}^{t,\lambda_0} \to K$. To see this, it suffices to prove that g vanishes on

$$\begin{aligned}
&(\operatorname{Ker} \bar{\theta}) \\
&= [\square_{\lambda_0}^{(5)}(\wedge^4 F \otimes D_1 F) + \square_{\lambda_0}^{(4,1)}(\wedge^3 F \otimes D_1 F \otimes \wedge^1 F)] \otimes [\wedge^2 G \otimes D_1 G \otimes \wedge^2 G]_{\alpha(G)} \\
&\quad + [\wedge^2 F \otimes D_1 F \otimes \wedge^2 F]_{\alpha(F)} \otimes [\square_{\lambda_0}^{(5)}(\wedge^4 G \otimes D_1 G) + \square_{\lambda_0}^{(4,1)}(\wedge^3 G \otimes D_1 G \otimes \wedge^1 G)] \\
&\quad + (\operatorname{Ker} \bar{\theta}) \cap L_{1,1,\alpha}^{t,\lambda_0,1}
\end{aligned}$$

where $\bar{\theta}$ is the composite map:

$$L_{1,1,\alpha}^{t,\lambda_0} \xrightarrow{\theta_{\lambda_0}} M_{1,1,\alpha}^{t,\lambda_0} \xrightarrow{\text{proj.}} M_{1,1,\alpha}^{t,\lambda_0}/\dot{M}_{1,1,\alpha}^{t,\lambda_0}.$$

The equation is a consequence of Theorem 1.2. We consider the linear form: g_F: $[\wedge_{\lambda_0} \mathrm{id}_F]_{1,\alpha(F)} \to K$ defined by:

$$\begin{aligned}
&g_F \text{ is zero on } [\wedge^3 F \otimes \wedge^1 F \otimes D_1 F]_{\alpha(F)} \\
&g_F\begin{pmatrix} \sigma 1 & \sigma 2 & (\sigma 3)' \\ \sigma 4 & \sigma 5 & \end{pmatrix} = (-1)^\sigma \qquad \text{for } \sigma \in \mathfrak{S}_5\,.
\end{aligned}$$

The linear form g_G: $[\wedge_{\lambda_0} \mathrm{id}_G]_{1,\alpha(G)} \to K$ is defined similarly. It holds that $g = g_F \otimes g_G$ on $L_{1,1,\alpha}^{t,\lambda_0}$. We see that;

$$g_F \circ \square_{\lambda_0}^{(5)}(\sigma 1\ \ \sigma 2\ \ \sigma 3\ \ \sigma 4\ \ (\sigma 5)') = (-1)^\sigma \cdot \binom{4}{2} = 0$$

$$g_F \circ \square_{\lambda_0}^{(4,1)}\begin{pmatrix} \sigma 1 & \sigma 2 & \sigma 3 & (\sigma 4)' \\ \sigma 5 & & & \end{pmatrix} = (-1)^\sigma \cdot 3 = 0$$

by a straightforward computation. Hence, g_F vanishes on

$$[\square_{\lambda_0}^{(5)}(\wedge^4 F \otimes D_1 F) + \square_{\lambda_0}^{(4,1)}(\wedge_3 F \otimes D_1 F \otimes \wedge^1 F)] \otimes [\wedge^2 G \otimes D_1 G \otimes \wedge^2 G]_{\alpha(G)}\,.$$

Similar calculation will show that g_G vanishes on

$$[\wedge^2 F \otimes D_1 F \otimes \wedge^2 F]_{\alpha(F)} \otimes [\square_{\lambda_0}^{(5)}(\wedge^4 G \otimes D_1 G) + \square_{\lambda_0}^{(4,1)}(\wedge^3 G \otimes D_1 G \otimes \wedge^1 G)]\,.$$

It is clear that g vanishes on $L_{1,1,\alpha}^{t,\lambda_0,1}$. We conclude that g induces $\bar{g}$. We extend the definition of $\bar{g}$. We define $\bar{g}$ is zero on $M_{2,0,\alpha}^{t,\lambda_0}/\dot{M}_{2,0,\alpha}^{t,\lambda_0} \oplus M_{0,2,\alpha}^{t,\lambda_0}/\dot{M}_{0,2,\alpha}^{t,\lambda_0}$ so that $\bar{g}$ is defined over $M_{2,\alpha}^{t,\lambda_0}/\dot{M}_{2,\alpha}^{t,\lambda_0}$.

Now we shall show that $\bar{g}$ induces h: $E \to K$. To see this, it is sufficient to show that $\bar{g}$ is zero on $[\dot{M}_{2,\alpha}^{t,\lambda_0} + B_2(M_\alpha^{t,\lambda_0})]/\dot{M}_{2,\alpha}^{t,\lambda_0}$. To see this, it is sufficient to show that $\bar{g}$ vanishes on

$$\bar{\theta}(\partial_F(L_{2,1,\alpha}^{t,\lambda_0,2})) + \bar{\theta}(\partial_G(L_{1,2,\alpha}^{t,\lambda_0,2}))$$

since $\bar{g}$ vanishes on

$$[\dot{M}^{t,\lambda_0}_{2,\alpha} + M^{t,\lambda_0}_{2,0,\alpha} + M^{t,\lambda_0}_{0,2,\alpha} + \theta_{\lambda_0}(L^{t,\lambda_0,1}_{1,1,\alpha})]/\dot{M}^{t,\lambda_0}_{2,\alpha}\,.$$

But this is clear from the facts that

$$g_F \circ \partial_F\begin{pmatrix}\sigma 1 & (\sigma 2)' & (\sigma 3)' \\ \sigma 4 & \sigma 5 & \end{pmatrix} = (-1)^\sigma - (-1)^\sigma = 0 \quad \text{and}$$

$$g_G \circ \partial_G\begin{pmatrix}\tau 1 & (\tau 2)' & (\tau 3)' \\ \tau 4 & \tau 5 & \end{pmatrix} = 0$$

for $\sigma,\ \tau \in \mathfrak{S}_5$.

We shall show that h is a nonzero linear form. We let;

$$A = \left(\begin{matrix}1 & 2 & 3' \\ 4 & 5 & \end{matrix} - \begin{matrix}1 & 2 & 3 \\ 4 & 5' & \end{matrix}\;\middle|\; \begin{matrix}1 & 2 & 3' \\ 4 & 5 & \end{matrix} - \begin{matrix}1 & 2 & 3 \\ 4 & 5' & \end{matrix}\right).$$

Then $\partial A = 0$ and $\bar{g}(A) = 1$. This shows that h is non-zero.

(ii) *case* 2. $t \geq 3$.

We define a linear form g: $L^{t,\lambda_0,1}_{1,1,\alpha} \to K$ as in (i). We define:

$$g\begin{pmatrix}\sigma 1 & \sigma 2 & \cdots\cdots & \sigma t \\ \sigma(t+1) & \sigma(t+2) & \sigma(t+3)' & \end{pmatrix} \otimes \begin{pmatrix}\tau 1 & \tau 2 & \cdots\cdots & \tau t \\ \tau(t+1) & \tau(t+2) & \tau(t+3)' & \end{pmatrix}$$
$$= \begin{cases}(-1)^{\sigma\tau} & (\text{if } \{1, \cdots, t-2\} \subset \{\sigma 1, \cdots, \sigma t\} \cap \{\tau 1, \cdots, \tau t\} \\ 0 & (\text{otherwise})\end{cases}$$

for row-standard bitableaux of shape $\lambda_0 = (t, 3)$ in $L^{t,\lambda_0,1}_{1,1,\alpha}$. Note that g admits an expression $g = g_F \otimes g_G$ in an obvious manner as in case (i). It holds that

$$g_F \circ \square^{(t+2,1)}_{\lambda_0}\begin{pmatrix}\sigma 1 & \sigma 2 \cdots \sigma(t+2) \\ (\sigma(t+3))' & \end{pmatrix} = 0 \quad \text{and}$$

$$g_F \circ \square^{(t+1,2)}_{\lambda_0}\begin{pmatrix}\sigma 1 & \sigma 2 & \cdots \sigma(t+1) \\ \sigma(t+2) & (\sigma(t+3))' & \end{pmatrix} = 0$$

(which can be shown by straightforward computation). Using [5, Lemma I.3.9], it is easy to see that

$$\begin{aligned}&\operatorname{Im} \square_{\lambda_0} \cap \wedge^t F \otimes \wedge^2 F \otimes D_1 F \\ &\quad = \square^{(t+1,2)}_{\lambda_0}(\wedge^{t+1} F \otimes \wedge^1 F \otimes D_1 F) + \square^{(t+2,1)}_{\lambda_0}(\wedge^{t+2} F \otimes D_1 F)\,.\end{aligned}$$

Hence, we have g_F is zero on $[\operatorname{Im} \square_{\lambda_0} \cap \wedge^t F \otimes \wedge^2 F \otimes D_1 F]_{\alpha(F)}$, where $\alpha(F)$ is the weight $(1, 1, \cdots, 1, 0, 0, \cdots)$. Similarly, we have g_G is zero on $[\operatorname{Im} \square_{\lambda_0} \cap \wedge^t G \otimes \wedge^2 G \otimes D_1 G]_{\alpha(G)}$, where $\alpha(G)$ is also the weight $(1, 1, \cdots, 1, 0, 0, \cdots)$. Since $\theta_\lambda(L^{t,\lambda,1}_{1,1,\alpha}) + \dot{M}^{t,\lambda}_{1,1,\alpha} = M^{t,\lambda}_{1,1,\alpha}$ by [5, Lemma IV. 2.2], g induces a linear form $\bar{g}$: $M^{t,\lambda_0}_{1,1,\alpha}/\dot{M}^{t,\lambda_0}_{1,1,\alpha} \to K$, and we extend the definition of $\bar{g}$ as in case 1. By an argument similar to the proof in case (i), it is easy to see that $\bar{g}$ induces $h\colon E \to K$.

We shall show that h is nonzero. If we put

$$A = \sum_{\sigma,\tau\in\mathfrak{S}_{t,3}} (-1)^{\sigma\tau}\left(\begin{matrix}\sigma 1 & \sigma 2 & \cdots\cdots & \sigma t \\ \sigma(t+1) & \sigma(t+2) & \sigma(t+3)' & \end{matrix}\middle|\begin{matrix}\tau 1 & \tau 2 & \cdots\cdots & \tau t \\ \tau(t+1) & \tau(t+2) & \tau(t+3)' & \end{matrix}\right)$$

(remember that $\mathfrak{S}_{i,j} = \{\sigma \in \mathfrak{S}_{i+j} | \sigma 1 < \cdots < \sigma i,\ \sigma(i+1) < \cdots < \sigma(i+j)\}$) then $\partial A \in \dot{M}^{t,\lambda_0}$, and $\bar{g}(A) = \binom{5}{3}^2 = 100 \neq 0$. Hence, we have $h \neq 0$.

By case 1 and case 2 above, we have completed the proof of Theorem 3.1.

COROLLARY 3.2. *The rank of the module X_4^t does not depend on the characteristic, if and only if $t = 1$ or $t \geq \min(m, n) - 2$.*

Proof. The 'if' part is [5, Corollary IV. 2.12]. Since $\mathscr{J}^{t,t+3}$ is a universally free complex, and $H_i(\mathscr{J}^{t,t+3}) = 0$, if $i \neq 2, 3$, the rank of $X_4^t = H_3(\mathscr{J}^{t,t+3})$ depends on the characteristic if $\operatorname{rank} H_2(\mathscr{J}^{t,t+3}) = \beta_{3,t+3}$ depends on the characteristic. So the 'only if' part follows from the theorem.

Remark 3.3. An argument quite similar to the proof of the theorem shows that $E_2^{\infty,1,(2,1)} \neq 0$, and $E_2^{\infty,t,(t,2)} \neq 0$ for $2 \leq t \leq \min(m, n) - 2$, if $R = \boldsymbol{F}_2$. It follows that the natural map $H_2(L^{t,(t+2)}) \to X_3^t$ is not surjective, if $t \leq \min(m, n) - 2$ (even if $t = 1$!) and if $R = \boldsymbol{F}_2$. In fact, if we put

$$A = \sum\nolimits_{\sigma,\tau\in\mathfrak{S}_{t,2}}\left(\begin{matrix}\sigma 1 & \sigma 2 & \cdots & \sigma t \\ \sigma(t+1) & \sigma(t+2)' & & \end{matrix}\middle|\begin{matrix}\tau 1 & \tau 2 & \cdots & \tau t \\ \tau(t+1) & \tau(t+2)' & & \end{matrix}\right),$$

then $\partial A \in \tilde{\mathscr{J}}^{t+1,t+2}$, so $S\pi(A) \in \boldsymbol{Z}_3^t$ ($= Z_3^{t,t}$, in the notation of [1]). But $S\pi(A)$ is not contained in the image of $\alpha^{t,t+2}\colon L^{t,(t+2)} \to \boldsymbol{Z}_3^t$. Since $\partial S\pi(A) \in \boldsymbol{X}_2^{t+1}$, there exists $B \in \operatorname{Im}\alpha^{t,t+2}$ such that $\partial S\pi(A) = \partial B$, by Kurano's first syzygy theorem. Hence, $S\pi(A) - B \in \boldsymbol{X}_3^t$, but $S\pi(A) - B \notin \operatorname{Im} H_2(\alpha^{t,t+2})$.

Therefore, $\boldsymbol{X}_3^t$ does not have a standard basis as $\boldsymbol{X}_2^t$ has, although $\boldsymbol{X}_3^t$ is universally free.

Remark 3.4. We have seen that $\boldsymbol{X}_{\cdot}^t$ is not a universally free $GL(F) \times GL(G)$ complex in the case $2 \leq t \leq \min(m, n) - 3$. Recently, the author [4] proved that the Betti numbers of I_t are independent of the characteristic in the case $t = 1$ or $t \geq \min(m, n) - 2$. So $\boldsymbol{X}_{\cdot}^t$ is universally free in this case, and is the linear part of the resolution.

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