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Nagoya Math. J.
Vol. 118 (1990), 177-193

ON A SYSTEM OF ELLIPTIC MODULAR FORMS ATTACHED TO THE LARGE MATHIEU GROUP

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§ 1. Introduction and statement of results

This paper is a continuation of two previous papers of the author. In the first [4] we discussed a Thompson series associated with the group M_{24} in which each of the modular forms $\eta_g(\tau)$ attached to elements $g \in M_{24}$ are primitive cusp-forms. In the second [5] we showed how, given a rational G-module V for an arbitrary finite group G, it is possible to attach to each pair of commuting elements (g, h) in G a certain q-expansion $f(g, h; \tau) = \sum_{n\geq 1} a_n(g, h)q^{n/D}$ (for $q = \exp(2\pi i\tau)$, τ in the upper half-plane $\mathfrak{h}$, and D an integer depending only on (g, h)) such that the follow ing hold:

(1.1) $$f(g, h; \tau) = f(g^x, h^x; \tau), \quad x \in G$$

(1.2) For each $\gamma \in \Gamma = SL_2(\boldsymbol{Z})$ we have

$$f(g, h; \tau)|_k\gamma = (\text{constant})f((g, h)\gamma; \tau)$$

where $k = \frac{1}{2}\dim C_V(\langle g, h\rangle)$. Here the left-side is the usual slash operator on modular forms of weight k and on the right we have

$$(g, h)\gamma = (g^a h^c, g^b h^d) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(1.3) For each $g \in G$ and $n \in \boldsymbol{N}$ the map

$$h \longmapsto a_n(g, h)$$

is a virtual character of $C_G(g)$.

We call an assignment $(g, h) \mapsto f(g, h; \tau)$ satisfying (1.1)–(1.3) an *elliptic system* for G, and the purpose of this paper is to study in detail the

Received March 20, 1989.
* Research supported by the National Science Foundation and the S.E.R.C. of Great Britain.

elliptic system for M_{24} corresponding to its usual permutation representation on 24 letters. We will see that this system has remarkable multiplicative properties.

The definition of $f(g, h; \tau)$ in [6] is quite complicated and will not be repeated here, but in certain cases it can be written as a "Frame shape." For this purpose we make the following definition:

(1.4) The commuting pair (g, h) is called *rational* if h acts rationally on each of the g-eigenspaces of $V \otimes_Q C$.

If (g, h) is a rational pair and g has order r then on the $\exp(2\pi ji/r)$-eigenspace of g on $V \otimes_Q C$, h has a Frame shape, say

$$\prod_{m|s}^{n} m_j^{e(mj)}$$

where $s =$ order of h. Then we have

$$f(g, h; \tau) = \prod_{j|r} \prod_{d|j} \prod_{m_j|s} \eta(m_j\tau/d)^{e(mj)\mu(r/dj)} \tag{1.5}$$

where μ is the Möbius function.

If $g = 1$ then (1.5) reduces to $f(1, h; \tau) = \Pi\eta(m_j\tau)^{e(mj)}$ and is precisely the form $\eta_h(\tau)$ discussed in [4]. Thus (1.5) represents the generalization of "Frame shape" to rational pairs.

We use the term "primitive" cusp-form as in [3]. The main result of that paper is that the primitive cusp-forms of the type

$$p(\tau) = \prod_{i=1}^{s} \eta(k_i\tau)^{e_i}, \qquad 1 \leq k_1 < k_2 < \cdots, e_i > 0 \tag{1.6}$$

are precisely those for which the corresponding partition $(k_1^{e_1}, \cdots, k_s^{e_s})$ is a "balanced" partition of 24. In other words, we have

(1.7) (i) $\sum k_i e_i = 24$
(ii) $k_1 | k_i, \quad i \geq 1$
(iii) If $N = k_1 k_s$, then $N = k_i k_{s+1-i}$, $i \geq 1$,
(iv) $e_i = e_{s+1-i}, \quad i \geq 1$.

We call the integer N in (iii) the *balancing number* of the partition.

Now each $h \in M_{24}$ has a balanced Frame shape, so that each $\eta_h(\tau)$ is a primitive cusp-form of the preceding type. Moreover, of the 28 cusp-forms in [3] which satisfy (1.6) and (1.7), 22 appear as $\eta_h(\tau)$ for $h \in M_{24}$. One of the main results of the present paper is to extend these observations to the contex of our elliptic system, and to explain how *every* form satisfying (1.6) and (1.7) appears. To state these results we need some notation.

$$N_g = \text{balancing number of } g \in M_{24}\,.$$

For a pair (g, h) of commuting elements we set

$$N_{(g,h)} = N_g N_h\,,$$

and for an abelian subgroup $A \leq M_{24}$ with at most 2 generators we set

$$N_A = \min\{N_{(g,h)} \mid \langle g, h\rangle = A\}\,.$$

Finally, let $m(g, h; \tau) = f(g, h; N_g\tau)$, We will establish the following:

I. To each $A \leq M_{24}$ is attached a primitive cusp-form $p_A(\tau) = p(\tau)$ satisfying (1.6) and (1.7) and the following:

(a) If $\langle g, h\rangle = A$ then $m(g, h, \tau) = p(\tau)$, if and only if, $N_{(g,h)} = N_A$.

(b) $p(\tau)$ is a primitive cusp-form of level N_A and integral weight $k_A = \frac{1}{2}\dim C_V(A)$ for some Dirichlet character $\varepsilon_A \pmod{N_A}$ which is trivial if, and only if, k_A is even.

(c) If $\langle g, h\rangle = A$ then $m(g, h; \tau)$ can be derived from $p(\tau)$ by applying a succession of operators of the form $|_k T_{Q^{-1}}$ and $|_k W_N$ where $T_{Q^{-1}} = \begin{pmatrix} 1 & Q^{-1} \\ 0 & 1 \end{pmatrix}$, $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and Q, N are suitably chosen integers.

(d) If $p(\tau) = \sum_{n=1}^{\infty} a_n q^n$ then there is a root of unity λ such that $m(g, h; \tau) = \sum_{n=1}^{\infty} b_n q^n$ where either $b_n = 0$ or $b_n = \lambda^{n-1} a_n$.

(e) The majority of the forms $m(g, h; t)$ have multiplicative coefficients, in particular this is true of each rational pair (g, h).

II. Because of (1.3) the forms $m(g, h; \tau)$ for fixed g form a Thompson series for $C_{M_{24}}(g)$ which we may write either as $\sum_{n\geq 1}\chi_n^g q^n$ for $\chi_n^g \in RC(g)$, χ_n^g being the coefficient of q^n in $m(g, h; \tau)$, or as a formal Dirichlet series

$$L(C(g), s) = \sum_{n=1}^{\infty} \frac{\chi_n^g}{n^s}\,.$$

(a) If we take $g = 1$ the series $L(M_{24}, s)$ has an Euler product which is exactly that discussed in [4].

(b) Similarly, several other of the L-series $L(C(g), s)$ also have Euler products (e.g., if g is an involution, because of $I(e)$). They exhibit a "ramified" behavior at the primes dividing the order of g. For example, if g is of type 2A (Frame shape $1^8 2^8$) then $C = C(g) \cong 2^{1+8}$. $L_3(2)$ and we have

$$L(C, s) = \prod_{p \text{ odd}} \left(1 - \frac{\chi_p^g}{p^s} + \frac{\psi_p^g}{p^{2s}}\right)^{-1} \left(1 + \frac{S}{2^s}\right)^{-1} \left(1 + \frac{S - T}{2^s}\right).$$

Here, $T = -\chi_2^g$ is the character of C of degree 8 realized on the (-1)-eigenspace of g on V and S is the permutation character of C on the 8 order orbits of g of length 2. Moreover, on the $(+1)$-eigenspace of g on V the action of $C/\langle g\rangle = \bar{C}$ induces an embedding $\bar{C} \leq SO(15, \boldsymbol{R})$ and then ψ_p is determined via $p\psi_p^g = \beta_p^{\text{or}}$ where β_p^{or} is the oriented Bott cannibalistic class of $SO(16, \boldsymbol{R})$ of degree p^8, restricted to $\bar{C}$ and lifted to C. (See [5] for a (general) discussion of this particular virtual character in the present context.)

(c) In general, g acts on the virtual module affording χ_p^g as a scalar. Thus we may think of χ_n^g as affording a *projective* character of $\bar{C} = C/\langle g\rangle$, which we write as $\hat{\chi}_n^g$. Then in *every case* the projectivized Dirichlet series has an Euler product, i.e.,

$$\hat{L}(\bar{C}, s) = \sum_{n \geq 1} \frac{\hat{\chi}_n^g}{n^s} = \prod_p \left(1 - \frac{\hat{\chi}_p^g}{p} + \frac{\hat{\psi}_p^g}{p^{2s}}\right)^{-1}$$

where again $\hat{\psi}_p^g$ is of Bott type arising from the induced embedding $\bar{C} \leq SO(C_V(g))$.

(d) After (c) we may combine the Euler products together to obtain a bundle version. For the $\hat{\chi}_n^g$ and $\hat{\psi}_p^g$ for fixed n, p and g ranging over $G = M_{24}$ define a *virtual projective G-bundle over G*, where by a projective G-bundle over G we mean that for each $g \in G$ we have a projective space P_g and conjugation by x induces a linear isometry $l(x)\colon P_g \to P_{xgx^{-1}}$ satisfying $l(x) = \text{id.}$ on P_x and $l(xy) = l(x) \circ l(y)$. If we write C_n, B_p for the virtual projective bundles corresponding to $\{\hat{\chi}_n^g\}$, $\{\hat{\psi}_p^g\}$ respectively then we have

$$\sum_{n \geq 1} \frac{C_n}{n^s} = \prod_p \left(1 - \frac{C_p}{p^{2s}} + \frac{B_p}{p^{2s}}\right)^{-1},$$

an Euler product with coefficients in the Grothendieck ring $KP_G(G)$ of such bundles. As in [4], this latter equality may be formulated in terms of the existence of a certain formal group with coefficients in $KP_G(G)$.

III. All but 2 of the 28 forms satisfying (1.6) and (1.7) appear as $p_A(\tau)$ for some A. Moreover the remaining 2 appear in the elliptic system attached to O, or even to its maximal 2-local $2^{12} \cdot M_{24}$.

The paper is arranged as follows: in section 2 we describe all 2-generator abelian subgroups of M_{24} and study their action on the 24 letters.

In section 3 we list the forms $m(g, h; \tau)$ and study their q-expansions, and in particular give the proofs of the preceding assertions.

Thanks are due to A.O.L. Atkin for providing some numerical data and thereby influencing my ideas about the forms $m(g, h; \tau)$, to S. P. Norton for correspondence which convinced me of the usefulness of introducing projective characters (though its utility is admittedly not quite evident in the foregoing), and to P. Landweber for supplying a list of errata in an earlier version.

§ 2. Hypothesis "Even"

Let G be a finite group with ρ an even-dimensional representation of G by real unimodular matrices

$$\rho\colon G \longrightarrow SL(2d, \boldsymbol{R}). \tag{2.1}$$

In the following we shall frequently abuse notation by omitting ρ and thereby identifying $\rho(g)$ with g. We let V be the $\boldsymbol{R}G$-module affording the representation ρ, and for a subgroup $H \leq G$ we set $V_H = \{v \in V \mid h.v = v \text{ for all } h \in H\}$.

Lemma 2.1. *If H is either cyclic or abelian of odd order then V_H has even dimension.*

Proof. As V affords a real representation of G, the non-real irreducible constituents of the action of H on $\overline{V} = V \otimes_{\boldsymbol{R}} \boldsymbol{C}$ occur in conjugate pairs. Thus if $\overline{U}$ is the sum of such constituents and $\overline{W}$ the sum of the real constituents then $\overline{V} = \overline{U} \oplus \overline{W}$ and each of $\overline{U}$, $\overline{W}$ is of even dimension.

If $|H|$ is odd then $\overline{W}$ is a trivial H-module, so $\overline{W} = \overline{V}_H$ and we are done in this case. If H is cyclic then a generator h of H has only the eigenvalues ± 1 on $\overline{W}$ and $\overline{W} = \overline{V} \oplus \overline{V}_{-1}$ where V_{-1} is the -1 eigenspace of h on V. Since $\det h = 1$ we have $\dim V_{-1}$ even, so also $\dim V_H$ is even as required.

Lemma 2.2. *Suppose that* $\operatorname{codim} V_{\langle x \rangle} \equiv 0 \pmod 4$ *for each involution $x \in G$. Then* $\dim V_H$ *is even for each $H \cong Z_2 \times Z_2$.*

Proof. If x_i and the involutions of H, $1 \leq i \leq 3$, we have the fixed-point formula

$$\dim V = \dim V_H + \sum_{i=1}^{3} \dim (V_{\langle x_i \rangle} / V_H).$$

The result follows from this.

The following situation is relevant.

HYPOTHESIS EVEN. *ρ is as in* (2.1) *and we have*
(2.2) dim V_H *is even for each 2-generator abelian subgroup $H \leq G$.*

LEMMA 2.3. *Hypothesis Even is equivalent to the following condition*:
(2.3) $C_G(h) \subseteq SL(V_{\langle h\rangle})$ *for each 2-element h. This means that $C_G(h)$ acts on $V_{\langle h\rangle}$ as a group of unimodular matrices.*

Proof. Suppose that (2.3) holds. If $H = \langle h, k\rangle$ is abelian with h a 2-element then dim $V_{\langle x\rangle}$ is even by Lemma 2.1 and $H \subseteq SL(V_{\langle x\rangle})$ by hypothesis. Now apply Lemma 2.1 to the action k on $V_{\langle x\rangle}$ to see that $(V_{\langle x\rangle})_{\langle k\rangle} = V_H$ has even dimension.

This shows that (2.2) holds at least for abelian 2-groups with at most 2 generators. For an arbitrary such abelian group H we may write $H = T \times K$ where T is a 2-Sylow of H. Then V_T is even-dimensional and affords a real representation of K, whence $V_H = (V_T)_K$ is even dimensional by the argument of Lemma 2.1.

The proof that (2.2) implies (2.3) is left to the reader.

We turn now to the application of these ideas to M_{24}. Specifically we take

$$\rho\colon M_{24} \longrightarrow SL(24, \boldsymbol{R}) \tag{2.4}$$

to be the usual permutation representation of M_{24} on 24 letters.

PROPOSITION 2.4. *If ρ is as in* (2.4) *then Hypothesis Even is satisfied.*

Proof. We will need a few properties of M_{24} which can be found in [1] or [2], for example. First, the involutions are of shape $1^8 2^8$ or 2^{12}. They therefore satisfy the hypothesis of Lemma 2.2, so that result tells us that dim V_H is even for $H \cong Z_2 \times Z_2$.

Now these involutions have centralizers of shape $2^{1+6}\cdot L_3(2)$ and $2^6\cdot\Sigma_5$, respectively, so in each case if x is an involution with centralizer C then C is generated by its involutions. Also, by the first paragraph we see that involutions of C lie in $SL(V_{\langle x\rangle})$, so in fact $C \subseteq SO(V_{\langle x\rangle})$.

Let now h be any 2-element with centralizer C. If $x \in C$ is an involution then $h \in C(x)$, so $\langle x, h\rangle \subseteq SL(V_{\langle x\rangle})$ by the last paragraph, so $V_{\langle x,h\rangle}$ has even dimension by Lemma 2.1, so $x \in SL(V_{\langle x\rangle})$. Now as in the last paragraph we get $C_1 \subseteq SL(V_{\langle x\rangle})$ where C_1 is generated by $\langle h\rangle$ together with the involutions of C.

If h has order 8 then $C(h) \cong Z_2 \times Z_8$ so that $C_1 = C \subseteq SL(V_{\langle x\rangle})$. If h has order 4 then h is conjugate to one of $4A \sim 2^4 4^4$, $4B \sim 1^4 \cdot 2^2 \cdot 4^4$ or $4C \sim 4^6$. The first and third of these satisfy $C(h) \cong (Z_4 * D_8 * D_8) \cdot \Sigma_3$ resp. $Z_4 \times \Sigma_4$ and hence $C_1 = C$ in these cases.

From these reductions together with Lemma 2.3 we see that if the proposition is false, with dim V_H odd for a suitable H, then in fact $H \cong Z_4 \times Z_4$ and H contains only elements of order 4 which are of type $4B$. But here we compute directly that

$$\dim V_H = 1/16(24 + 3.8 + 12.4) = 6\ .$$

(Here we used $\dim V_H = \langle \chi | H, 1_H \rangle_H$ where χ is the character afforded by ρ and satisfying $\chi(g) = \sharp$ of letter s fixed by g.) The proposition is proved.

We wish now to give all 2-generator abelain subgroups of M_{24}—not up to conjugacy necessarily, but by listing the number of elements of each cycle shape that they contain. Table 1 names the elements (cycle shapes) following [2]; table 2 names the non-cyclic 2-generator abelian subgroups together with the elements they contain.

Table 1

Elt.	Shape	Elt.	Shape
1A	1^{24}	7A	$1^3 \cdot 7^3$
2A	$1^8 \cdot 2^8$	8A	$1^2 \cdot 2 \cdot 4 \cdot 8^2$
2B	2^{12}	10A	$2^2 \cdot 10^2$
3A	$1^6 \cdot 3^6$	11A	$1^2 \cdot 11^2$
3B	3^8	12A	$2 \cdot 4 \cdot 6 \cdot 12$
4A	$2^4 \cdot 4^4$	12B	12^2
4B	$1^4 \cdot 2^2 \cdot 4^4$	14A	$1 \cdot 2 \cdot 7 \cdot 14$
4C	4^6	15A	$1 \cdot 3 \cdot 5 \cdot 15$
5A	$1^4 \cdot 5^4$	21A	$3 \cdot 21$
6A	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	23A	$1 \cdot 23$
6B	6^4		

Table 2 I. $Z_2 \times Z_2$

Name	# Elts	
	2A	2B
A	3	0
B	0	3
C	2	1
D	1	2

II. $Z_2 \times Z_4$

	2A	2B	4A	4B	4C
A	3	0	4	0	0
B	2	1	2	2	0
C	1	2	4	0	0
D	1	2	0	0	4
E	3	0	0	4	0
F	1	2	0	4	0

III. $Z_4 \times Z_4$

	2A	2B	4A	4B	4C
A	1	2	4	0	8
B	3	0	8	4	0
C	3	0	0	12	0

IV. $Z_2 \times Z_8$

	2A	2B	4A	4B	4C	8A
A	1	2	0	4	0	8

V. $Z_2 \times Z_6$

	2A	2B	3A	3B	6A	6B
A	3	0	2	0	6	0
B	0	3	0	2	0	6

VI. $Z_2 \times Z_{10}$

	2A	2B	5A	10A
A	0	3	4	12

VII. $Z_3 \times Z_3$

	3A	3B
A	8	0
B	2	6

As to the correctness of the tables, IV–VIII are readily deduced from the relevant information in [2], so that only I–III need be considered further. Let us therefore take $H \leq M_{24}$ with $H \cong Z_{2a} \times Z_{2b}$, $1 \leq a \leq b \leq 2$, and first show that H is necessarily one of the types in I–III. The condition imposed by Proposition 2.4 is sufficient to show that only one possibility not listed might occur, namely $a = b = 2$ with H containing 2 2A, 1 2B, 2 4A, 6 4B and 4 4C.

To eliminate this, take $x \in M_{24}$ of type 4C with $F = C(x)$. Then $F \cong Z_4 \times \Sigma_4$, so that certainly there is only one type of $Z_4 \times Z_4$ containing x. We assert that F is transitive on the 24 letters. If not then F has two orbits, each of length 12, and if X is one of them then a point-stabilizer in F is $D \cong D_8$. Let $D_0 = D \cap O_2(F) \cong Z_2 \times Z_2$. Clearly each involution of D_0 is of type 2A, and if they are the only such involutions in $O_2(F)$ then $D_0 \trianglelefteq F$ and D_0 fixes each letter in X. This being impossible, $O_2(F)$ must contain 6 involutions of type 2A and 1 of type 2B. As all elements of order 4 in $O_2(F)$ have square equal to x^2 they are of type 4^6. Now we see that $O_2(F)$ has $1/16(24 + 6.8) = 4\frac{1}{2}$ orbits, an absurdity. So indeed F is transitive.

Let F_0 be a point stabilizer in F, a group of order 4. We must show that $F_0 \cong Z_2 \times Z_2$. Indeed if $Z_3 \cong R \leq F$ then $N = N(R) \cong \Sigma_3 \times L_2(7)$ and $x \in O^\infty(N)$. Then an involution $t \in O_\infty(N)$ lies in $F \backslash O_2(F)$ and is of type 2A as it centralizes an element of order 7 in N. Thus we may take $t \in F_0 \backslash F$, whence $F_0 \cong Z_2 \times Z_2$ as required.

As explained above, it is now sufficient to show that each of the types listed in I–III above actually occur in M_{24}. First, type $Z_4 \times Z_4 A$

exists by the foregoing argument. Also, the stabilizer of 3 points in M_{24} is $M_{21} \cong L_3(4)$ and contains a $Z_4 \times Z_4$ necessarily of type C.

Consider next the centralizer $B = C(f)$ of an element of type 4A. We have $B \cong (Z_4 * D_8 * D_8) \cdot \Sigma_3$, and the 8 fixed letters of f^2 and their complement are the 2 orbits of B. So a point-stabilizer of the longer orbit (in B) is isomorphic to Σ_4 and hence contains an element g of type 4B. So $\langle f, g \rangle$ must be of type $Z_4 \times Z_4 B$.

As for $Z_2 \times Z_4$ subgroups, type C and D can be found in a $Z_4 \times Z_4 A$, type E in $Z_4 \times Z_4 C$, and type A in $Z_4 \times Z_4 B$. A $Z_2 \times Z_4 F$ lies in $Z_2 \times Z_8 A$, so only $Z_2 \times Z_4 B$ remains to be accounted for. But from the structure of $B = C(f)$ in the last paragraph we see that if y is an involution in $B \setminus O_2(B)$ then $Z_2 \times Z_4 \cong \langle f, y \rangle$ and is *not* contained in *a* $Z_4 \times Z_4$ or $Z_2 \times Z_8$ subgroup. Thus from the preceding $\langle f, y \rangle$ must be of type $Z_2 \times Z_4 B$ as required. We leave verification of table 2I to the reader.

Finally we remark that because of Proposition 2.4, each of the forms $f(g, h; \tau)$ (or $m(g, h; \tau)$) attached to M_{24} has integral weight 1/2 dim $C_V(\langle g, h \rangle)$.

§ 3. The associated forms

We begin by listing the forms $m(g, h; \tau) = f(g, h; N_g \tau)$ as discussed in section 1. To make the computations one uses tables 1 and 2 of section 2 in order to compute the characteristic polynomial of h on each g-eigenspace. If (g, h) is a rational pair then (1.5) yields $f(g, h; \tau)$, and in any case one can use the original definition [6, equation (3.7)]. One can also make use of Lemmas 3.1 and 3.2 below. We remark that in [6, equation (3.7)] the form $f(g, h; \tau)$ is seen to have the shape $q^d \sum_{n \geq 0} a_n q^n$ for a certain rational number d [6, equation (3.3)], but one readily verifies that $d = 1/N_g$ in the present situation, so that $m(g, h; \tau) = q + \cdots$.

One caveat to the foregoing is that only for those pairs (g, h) which are rational do we explicitly record $m(g, h; \tau)$, as a Frame shape. Moreover we do not repeat $m(1, h; \tau)$, which is given in Table 1 of section 2; and of the pairs (g, h), (h, g) we often list only one (cf. Lemma 3.1).

Table 3

$\langle g, h\rangle$	(g, h)	$m(g, h; \tau)$	$N = N_g N_h$	multiplicative
$Z_2 \times Z_2 A$	$(2A, 2A)$	2^{12}	4	yes
$Z_2 \times Z_2 B$	$(2B, 2B)$	4^6	16	yes
$Z_2 \times Z_2 C$	$(2A, 2A)$	$1^4 \cdot 2^2 \cdot 4^4$	4	yes
	$(2A, 2B)$	$2^{14}/1^4$	8	yes
	$(2B, 2A)$	$4^{14}/8^4$	8	yes
$Z_2 \times Z_2 D$	$(2B, 2A)$	$2^4 4^4$	8	yes
	$(2B, 2B)$	$4^{16}/2^4 8^4$	16	yes
$Z_2 \times Z_4 A$	$(4A, 2A)$	4^6	16	yes
	$(4A, 4A)$	$8^{18}/4^6 \cdot 16^6$	64	yes
$Z_2 \times Z_2 B$	$(2A, 4B)$	$1^2 \cdot 2 \cdot 4 \cdot 8^2$	8	yes
	$(2A, 4A)$	$2^7 8^2/1^2 \cdot 4$	16	yes
	$(2B, 2B)$	$2^2 8^7/4 \cdot 16^2$	16	yes
	$(2B, 4A)$	$4^5 8^5/2^2 \cdot 16^2$	32	yes
	$(4B, 4B)$	irrational	32	no
$Z_2 \times Z_4 C$	$(4A, 2B)$	$4^2 \cdot 8^2$	32	yes
	$(4A, 4A)$	$8^8/4^2 \cdot 16^2$	64	yes
$Z_2 \times Z_4 D$	$(4C, 2A)$	$4^2 \cdot 8^2$	32	yes
	$(4C, 2B)$	$8^8/4^2 16^2$	64	yes
	$(4C, 4C)$	irrational	256	yes
$Z_2 \times Z_4 E$	$(4B, 2A)$	$2^4 \cdot 4^4$	8	yes
	$(4B, 4B)$	$4^{16}/2^4 \cdot 8^4$	16	yes
$Z_2 \times Z_4 F$	$(4B, 2B)$	4^6	16	yes
	$(4B, 4B)$	4^6	16	yes
$Z_4 \times Z_4 A$	$(4C, 4A)$	$8 \cdot 16$	128	yes
	$(4C, 4C)$	$16^4/8 \cdot 32$	256	yes

$\langle g, h\rangle$	(g, h)	$m(g, h; \tau)$	$N = N_g N_h$	multiplicative
$Z_4 \times Z_4 B$	$(4A, 4B)$	$4^2 \cdot 8^2$	32	yes
	$(4A, 4A)$	$8^8/4^2 \cdot 16^2$	64	yes
$Z_4 \times Z_4 C$	$(4B, 4B)$	4^6	16	yes
$Z_2 \times Z_8 A$	$(8A, 2B)$	$4^2 \cdot 8^2$	32	yes
	$(8A, 4B)$	$4^2 \cdot 8^2$	32	yes
	$(8A, 8A)$	$8^2/4^2 \cdot 16^2$	64	yes
$Z_2 \times Z_6 A$	$(6A, 2A)$	$2^3 \cdot 6^3$	12	yes
	$(6A, 6A)$	irrational	36	no
$Z_2 \times Z_6 B$	$(6B, 2B)$	12^2	144	yes
	$(6B, 6B)$	irrational	1269	yes
$Z_2 \times Z_{10} A$	$(10A, 2B)$	$4 \cdot 20$	80	yes
	$(10A, 20A)$	irrational	400	no
$Z_3 \times Z_3 A$	$(3A, 3A)$	3^8	9	yes
$Z_3 \times Z_3 B$	$(3B, 3A)$	$3^2 \cdot 9^2$	27	yes
	$(3B, 3B)$	irrational	81	yes
$Z_2 A$	$(2A, 2A)$	$2^{32}/1^8 \cdot 4^8$	4	yes
$Z_2 B$	$(2B, 2B)$	$4^{36}/2^{12} \cdot 8^{12}$	16	yes
$Z_3 A$	$(3A, 3A)$	irrational	9	no
$Z_2 B$	$(3B, 3B)$	irrational	81	yes
$Z_4 A$	$(4A, 2A)$	$4^{16}/2^4 \cdot 8^4$	16	yes
	$(4A, 4A)$	irrational	64	no
$Z_4 B$	$(2A, 4B)$	$4^{14}/8^4$	8	yes
	$(4B, 2A)$	$2^{14}/1^4$	8	yes
	$(4B, 4B)$	irrational	16	no
$Z_4 C$	$(4C, 2B)$	$8^{18}/4^6 \cdot 16^6$	64	yes
	$(4C, 4C)$	irrational	256	yes

$\langle g, h\rangle$	(g, h)	$m(g, h; \tau)$	$N = N_g N_h$	multiplicative
Z_5A	$(5A, 5A)$	irrational	25	no
Z_6A	$(3A, 2A)$	$1^2\cdot 2^2\cdot 3^2\cdot 6^2$	12	yes
	$(6A, 2A)$	$2^8\cdot 6^8/1^2\cdot 3^2\cdot 4^2\cdot 12^2$	12	yes
	$(6A, 3A)$	irrational	18	no
	$(6A, 6A)$	irrational	36	no
Z_6B	$(3B, 2B)$	6^4	34	yes
	$(6B, 2B)$	$12^2/6^4\cdot 24^4$	144	yes
	$(6B, 3B)$	irrational	324	yes
	$(6B, 6B)$	irrational	1296	yes
Z_7A	$(7A, 7A)$	irrational	49	no
Z_8A	$(8A, 2A)$	$2^2 8^2/1^2\cdot 4$	16	yes
	$(2A, 8A)$	$2^2 8^2/4\cdot 16^2$	16	yes
	$(8A, 4A)$	irrational	64	no
	$(8A, 8A)$	irrational	64	no
$Z_{10}A$	$(5A, 2B)$	$2^2\cdot 10^2$	20	yes
	$(10A, 2B)$	$4^6\cdot 20^6/2^2\cdot 8^2\cdot 10^2\cdot 40^2$	40	yes
	$(10A, 5A)$	irrational	100	no
	$(10A, 10A)$	irrational	400	no
$Z_{11}A$	$(10A, 10A)$	irrational	121	no
$Z_{12}A$	$(4A, 3A)$	$2\cdot 4\cdot 6\cdot 12$	24	yes
	$(4A, 6A)$	$4^4\cdot 12^4/2\cdot 6\cdot 8\cdot 24$	48	yes
	$(12A, 2A)$	$4^4 12^4/2\cdot 6\cdot 8\cdot 24$	48	yes
	$(12A, 4A)$	irrational	96	no
	$(12A, 3A)$	irrational	72	no
	$(12A, 6A)$	irrational	144	no
	$(12A, 12A)$	irrational	576	no

$\langle g, h\rangle$	(g, h)	$m(g, h; \tau)$	$N = N_g N_h$	multiplicative
$Z_{12}B$	$(4C, 3B)$	12^2	144	yes
	$(4C, 6B)$	$24^6/12^2\cdot 48^2$	576	yes
	$(12B, 2B)$	$24^6/12^2\cdot 48^2$	576	yes
	$(12B, 4C)$	irrational	2304	yes
	$(12B, 3B)$	irrational	1296	yes
	$(12B, 6B)$	irrational	5184	yes
	$(12B, 12B)$	irrational	20736	yes
$Z_{14}A$	$(7A, 2A)$	$1\cdot 2\cdot 7\cdot 1\cdot 4$	14	yes
	$(14A, 2A)$	$2^4\cdot 14^4/1\cdot 4\cdot 7\cdot 28$	28	yes
	$(14A, 7A)$	irrational	98	no
	$(14A, 14A)$	irrational	196	no
$Z_{15}A$	$(5A, 3A)$	$1\cdot 3\cdot 5\cdot 15$	15	yes
	$(15A, 3A)$	irrational	45	no
	$(15A, 5A)$	irrational	75	no
	$(15A, 15A)$	irrational	225	no
$Z_{21}A$	$(7A, 3B)$	$3\cdot 21$	63	yes
	$(21A, 3B)$	irrational	567	yes
	$(21A, 7A)$	irrational	441	no
	$(21A, 21A)$	irrational	3969	no
$Z_{23}A$	$(23A, 23A)$	irrational	529	no

We interpolate some easy lemmas.

LEMMA 3.1. *Let (g, h) be a commuting pair with $N = N_g N_h$ and $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Then*

$$m(g, h; \tau)|_k W_N \sim m(h^{-1}, g; \tau)\,.$$

Proof. We remark that the notation $\sim$ means that the ratio of the two functions in question is constant. As for the proof, if $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then

$$\begin{aligned} m(g, h; \tau)|_k W_N &\sim \tau^{-k} m(g, h; -1/N\tau) \\ &\sim (N_h\tau)^{-k} f(g, h; -1/N_h\tau) \\ &= f(g, h; N_h\tau)|_k S \\ &\sim f(g, h)S^{-1}; N_h\tau) \qquad \text{(by eqn. (1.2))} \\ &= m(h^{-1}, g; \tau) \qquad \text{as required}. \end{aligned}$$

A similar argument yields

LEMMA 3.2. *Let Q be a divisor of N_g. Then*

$$m(g, h; \tau)|_k \begin{pmatrix} 1 & Q^{-1} \\ 0 & 1 \end{pmatrix} \sim m(g, g^{N_g/Q}\cdot h: \tau).$$

Concerning the level of these forms, one easily proves using Lemma 3.2 the following:

LEMMA 3.3. *Let Q be a divisor of D, set $D' = \text{l.c.m.}(Q^2, D)$, and assume that $m(g, h; \tau)$ is on $\Gamma_0(D)$. Then*

(i) *$m(g, g^{N_g/Q}\cdot h; \tau)$ is on $\Gamma_1(D')$.*

(ii) *If $Q|24$ then $m(g, g^{N_g/Q}\cdot h, \tau)$ is on $\Gamma_0(D')$.*

One can use Lemmas 3.1 and 3.2 to establish assertion $I(c)$ of section 1. We illustrate this with a diagram corresponding to the group $Z_2 \times Z_4B$ (cf. Tables 2 and 3):

$$\begin{tikzcd} (2A, 4B) \arrow[r, leftrightarrow, "W_8"] \arrow[d, leftrightarrow, "T_{1/2}"'] & (4B, 2A) \\ (2A, 4A) \arrow[r, leftrightarrow, "W_{16}"] & (4A, 2A) \arrow[d, leftrightarrow, "T_{1/4}"] \\ (2B, 4A) \arrow[r, leftrightarrow, "W_{32}"] & (4A, 2B) \end{tikzcd}$$

As for $I(d)$, (e) we use the following:

LEMMA 3.4. *Suppose that $m(g, h; \tau) = q\Sigma a_n q^{n-1}$, that there is an integer D such that $a_n = 0$ unless $n \equiv 1 \pmod D$ and that $Q|N_g$. Then the following hold:*

(i) *$m(g, g^{N_g/Q}\cdot h; \tau) = q\Sigma b_n q^{n-1}$ where $b_n = \exp(2\pi i(n-1)/Q)$.*

(ii) *If $\{a_n\}$ is multiplicative then $\{b_n\}$ is also multiplicative if $D|Q$, say $Q = mD$, and either*

(a) *$m|D$, or*

(b) $m = 2$, D *odd*.

Part (i) follows from Lemma 3.2, and (ii) is left to the reader.

One starts with the primitive form $p_A(\tau) = p(\tau)$, which has multiplicative coefficients, and then applies Lemma 3.4 with D being the minimal integer which occurs with non-zero exponent in the Frame shape corresponding to $p(\tau)$. Again, successive applications of this principle together with the action of W_N yields what we need, including the third column of Table 3.

One can also easily write down the Euler p-factors of $q\Sigma b_n q^{n-1}$ from those of $q\Sigma a_n q^{n-1}$. Specifically, if the p-factor of the latter is

$$\left(1 - \frac{a_p}{p^s} + \frac{c_p}{p^{2s}}\right)^{-1}$$

then that of the former in case (ii) (a) of Lemma 3.4 is

$$\left(1 - \frac{\sigma a_p}{p^2} + \frac{\sigma^2 c_p}{p^{2s}}\right)^{-1} \qquad (\sigma = \exp 2\pi i(p-1)/Q)\ ;$$

in case (ii) (b) the odd p-factors remain the same while the 2-factor becomes

$$\left(1 - \frac{2^s}{a_2}\right)^{-1}\left(1 - \frac{2a_2}{2^s}\right)$$

(in this case we always have $c_2 = 0$). Again we illustrate with the group $Z_2 \times Z_4 B$:

$$(2\mathrm{A}, 4\mathrm{B}):\qquad \prod_p \left(1 - \frac{a_p}{p^s} + \frac{c_p}{p^s}\right)^{-1}$$

$$(2\mathrm{A}, 4\mathrm{A}):\qquad \prod_p \left(1 - \frac{a_p}{p^s} + \frac{c_p}{p^{2s}}\right)^{-1}\left(1 - \frac{2a_2}{2^s}\right)$$

$$(2\mathrm{B}, 4\mathrm{B}):\qquad \prod_{p \text{ odd}} \left(1 - \frac{a_p}{p^s} + \frac{c_p}{p^{2s}}\right)^{-1}$$

$$(2\mathrm{B}, 4\mathrm{A}):\qquad \prod_{p\equiv 1(4)} \left(1 - \frac{a_p}{p^s} + \frac{c_p}{p^{2s}}\right) \prod_{p\equiv 3(4)} \left(1 + \frac{a_p}{p^3} + \frac{c_p}{p^{2s}}\right)^{-1}$$

$$(4\mathrm{B}, 4\mathrm{A}):\qquad \sum_{n\geq 1} \frac{\exp(2\pi i(n-1)/4)}{n^s}\,.$$

All of the assertions in of section 1 can be deduced in a like manner from these assertions. Concerning III, the two "missing" primitive forms not listed in Table 3 but satisfying (1.16) and (1.7) correspond to the Frame shapes $2\cdot 22$ and $6\cdot 18$. Now in the maximal 2-local $2^{12}\cdot M_{24}$ of O there is an element with Frame shape $2\cdot 22$ in its action on the Leech lattice. Also we find commuting elements with Frame shape $2^3\cdot 6^3$ and 3^8, and a quick calculation yields that the corresponding form $m(g, h; \tau) = \eta(6\tau)\eta(18\tau)$.

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