J. K. Verma
Nagoya Math. J.
Vol. 118 (1990), 155-163

JOINT REDUCTIONS OF COMPLETE IDEALS

J. K. VERMA

§ 1. Introduction

The aim of this paper is to extend and unify several results concerning complete ideals in 2-dimensional regular local rings by using the theory of joint reductions and mixed multiplicities. The theory of complete ideals in a 2-dimensional regular local ring was developed by Zariski in his 1938 paper [Z]. This theory is presented in a simpler and general form in [ZS, Appendix 5] and [H2].

The Zariski's product theorem asserts that the product of complete ideals in a 2-dimensional regular local ring is again complete. Counterexamples to such a statement in 3-dimensional regular local rings have been given by Huneke in [H1, § 3]. A surprising generalization of Zariski's product theorem was obtained by Huneke and Sally in [H-S, Theorem 4.1]; Let (R, m) be a d-dimensional Cohen-Macaulay local ring satisfying (R_2). Let I be a height two complete ideals of analytic spread two. Then I^k is complete for all positive integers k. Inspired by this theorem, Huneke asked the following

QUESTION. Let I and J be height two complete ideals of analytic spread two in a d-dimensional Cohen-Macaulay local ring satisfying (R_2). Suppose that IJ has analytic spread two. Is it true that IJ is complete?

We shall use joint reductions to prove that Huneke's question has an affirmative answer. For this purpose we shall prove our main theorem 2.1 in § 2 that if I and J are m-primary complete ideals in a two dimensional regular local ring (R, m) with infinite residue field then there exists elements $a \in I$ and $b \in J$ such that $aJ + bI = IJ$. By taking $I = J$ we obtain the Lipman-Teisier theorem: $I^2 = (a, b)I$, which they proved in [L-T, Corollary 5.4].

Inspired by the equation $aJ + bI = IJ$ in § 3, we study the Bhattacharya function $B(r, s) = \text{length } (R/I^rJ^s)$ for m-primary ideals

Received February 21, 1989.

I and J in a two dimensional Cohen-Macaulay local ring R. Bhattacharya showed in [B] that there is a polynomial $P(r, s)$ of total degree 2 in r and s such that $B(r, s) = P(r, s)$ for all large values of r and s. Let $e_1(I|J)$ denote the coefficient of rs in $P(r, s)$. If I and J are complete m-primary ideals in a two dimensional regular local ring R then it follows by a result of Lipman [L, Corollary 3.7] that

$$(*) \qquad e_1(I|J) = l(R/IJ) - l(R/I) - l(R/J),$$

where l denotes length. As a consequence of our main theorem 2.1 we deduce this formula. We note that our treatment of (*) is much simpler since Lipman's proof makes use of the theory of point bases of ideals and a formula of Hoskin and Deligne which in turn is proved in [L] by using sheaf cohomology.

For techniques and methods of proof we often use Huneke's lucid exposition of theory of complete ideals in a two dimensional regular local ring [H2] and Huneke and Sally's paper [H-S]. Our sole contribution is a careful blend of joint reductions and mixed multiplicities in their proofs thereby achieving a unification of many beautiful theorems about complete ideals.

§ 2. The main theorem

In this section we prove our main theorem concerning joint reductions of complete ideals in a two dimensional regular local ring. We recall the concept of joint reductions introduced by Rees in [R1]. Let $I_1, \cdots, I_r$ be ideals of a ring R. A set of elements $(x_1, \cdots, x_r)$ with $x_1 \in I_1, \cdots, x_r \in I_r$ is called a joint reduction of the set of ideals $(I_1, \cdots, I_r)$ if there exists positive integers $a_1, \cdots, a_r$ so that

$$I_1^{a_1} I_2^{a_2} \cdots I_r^{a_r} = \sum_{i=1}^{r} x_i I_1^{a_1} \cdots I_i^{a_i - 1} \cdots I_r^{a_r}.$$

In our main theorem 2.1 we show that if I and J and complete m-primary ideals of a two dimensional regular local ring and (a, b) is a joint reduction of the set (I, J) then $IJ = aJ + bI$. For this purpose we need to recall several facts from Zariski's theory of complete ideals. Let (R, m) be a two dimensional regular local ring. An ideal I is called contracted if there exists $x \in m \backslash m^2$ such that $I = IR[m/x] \cap R$. We say that I is contracted from $R[m/x]$. By Lemma 2 of [ZS, Appendix 5] I is contracted from $R[m/x]$ if and only if $I:m = I:x$ and by Lemma 3 of [*loc. cit.*] if I and J are contracted from $R[m/x]$ then so is IJ. Rees [R2] and Lipman [L, Corollary 3.2] showed that if an m-primary ideal I is contracted then $\mu(I) = 1 + \mathrm{ord}(I)$ where $\mu(I) = l(I/mI)$ and $\mathrm{ord}(I) = \max\{n \mid I \subset m^n\}$. Huneke and Sally proved [H-S, Theorem 2.1] that if $\mu(I) = 1 + \mathrm{ord}(I)$ then I is contracted provided R/m is infinite.

Let I be an m-primary ideal contracted from $S = R[m/x]$. If N is any maximal ideal of S containing mS then S_N is a 2-dimensional regular local ring. If $\mathrm{ord}(I) = r$ then $IS = x^r I'S$ for an ideal I' in S. I' is called transform of I in S and $I^{\sim} = I'_N$ is called tranform of I in S_N. By Proposition 5 of [ZS, Appendix 5], if I is complete then I' and hence $I^{\sim}$ are complete. A complete ideal is contracted [H2, Proposition 3.1]. Finally, by Proposition 3.6 of [H2], $e(I) > e(I^{\sim})$ where e denotes multiplicity.

THEOREM 2.1. *Let (R, m) be a 2-dimensional regular local ring. Let I and J be complete m-primary ideals. Then for any joint reduction (a, b) of (I, J); $aJ + bI = IJ$.*

Proof. First we remark that if R/m is infinite then joint reductions do exist by [R1]. We prove this theorem by induction on $t = \max(e(I), e(J))$. If $t = 1$ then $e(I) = e(J) = 1$. We may assume without loss of generality that R/m is infinite. Then there exist elements c, d in I such that (c, d) is a reduction of I, i.e., $(c, d)I^n = I^{n+1}$ for some n by [NR]. It follows that $e((c, d)) = e(I)$. Since R is regular, it is Cohen-Macaulay. Hence $e(I) = e((c, d)) = l(R/(c, d)) = 1 = l(R/m)$. Thus $m = (c, d) = I$. Similarly $J = m$. In this case $m^2 = cm + dm$. Suppose that the theorem has been proved for all positive integers less than t. Let (a, b) be a joint reduction of (I, J). We show that the ideal $K = aJ + bI$ is contracted.

There exists an n so that $aI^{n-1}J^n + bI^nJ^{n-1} = I^nJ^n$. This implies that a (resp. b) is part of a minimal basis of I (resp. J). Indeed, if $a \in mI$ then $I^nJ^n = aI^{n-1}J^n + bI^nJ^{n-1} \subset mI^nJ^n + bI^nJ^{n-1} \subset I^nJ^n$. Thus $I^nJ^n = mI^nJ^n + bI^nJ^{n-1}$. By Nakayama's lemma, $bI^nJ^{n-1} = I^nJ^n$. This implies that (b) is a reduction of I which is a contradiction since I is m-primary. Similarly b is part of a minimal basis of J. Put $\mathrm{ord}(I) = r$ and $\mathrm{ord}(J) = s$. Since I and J are complete, they are contracted. Hence $\mu(I) = r + 1$ and $\mu(J) = s + 1$. choose $a_1, a_2, \cdots, a_r \in I$ and $b_1, b_2, \cdots, b_s \in J$ such that $I = (a_0, a_1, \cdots, a_r)$ and $J = (b_0, b_1, \cdots, b_s)$ where $a = a_0$ and $b = b_0$. Then

$$(*) \qquad K = (a_0b_0, a_0b_1, \cdots, a_0b_s, b_0a_1, b_0a_2, \cdots, b_0a_r).$$

We show that the generators of K displayed in $(*)$ form a minimal basis of K. Suppose that we have a relation

$$\sum_{i=0}^{s} (a_0b_i)u_i + \sum_{j=1}^{r} (b_0a_j)v_j = 0$$

where $u_0, \cdots, u_s, v_1, v_2, \cdots, v_r \in R$. Since (a, b) is a joint reduction of (I, J), (a, b) is m-primary. Consequently a and b are coprime. Thus there exists an $f \in R$ such that

$$a_0f = \sum_{j=1}^{r} a_jv_j \quad \text{and hence} \quad b_0f = \sum_{i=0}^{s} b_iu_i .$$

This implies that $f, v_1, v_2, \cdots, v_r \in m$ and $u_0 - f, u_1, \cdots, u_s \in m$. Consequently the generators displayed in $(*)$ form a minimal basis of K. Since (a, b) is a joint reduction of (I, J), there exists n such that $(aJ + bI)(IJ)^n = (IJ)^{n+1}$. Hence $\operatorname{ord}(aJ + bI) = \operatorname{ord}(IJ) = r + s$. Thus $\mu(aJ + bI) = 1 + r + s$ which implies that $aJ + bI$ is contracted. We may assume that K and IJ are contracted from $S = R[m/x]$. Write $a = a'x^r$ and $b = b'x^s$ for some $a', b' \in S$ and $I = x^rI'S$ and $J = x^sJ'S$. Then

$$KS = (a'J' + b'I')x^{r+s}S \quad \text{and} \quad IJS = x^{r+s}I'J'S .$$

Thus (a', b') is a joint reduction of (I', J'). Hence it is enough to prove that $a'J' + b'I' = I'J'$ since IJ and K are contracted. To prove that the last equation holds we may localize at any maximal ideal N containing $a'J' + b'I'$. By Huneke's theorem $e(I'S_N) < e(I)$ and $e(J'S_N) < e(J)$. Thus by induction hypothesis $(a'J' + b'I')S_N = I'J'S_N$ for all N containing $a'J' + b'I'$. Hence $a'J' + b'I' = I'J'$ and consequently $IJ = aJ + bI$.

COROLLARY 2.2 (Lipman-Teissier). *Let (R, m) be a 2-dimensional regular local ring. Then for any reduction (a, b) of an m-primary complete ideal I, $(a, b)I = I^2$.*

Proof. If (a, b) is a reduction of I then there is an n such that $(a, b)I^n = I^{n+1}$. Hence $(a, b)I^{2n+1} = I^{2n+2}$ which can be rewritten as $aI^nI^{n+1} + bI^nI^{n+1} = I^{n+1}I^{n+1}$. Thus (a, b) is a joint reduction of the set (I, I). Hence $aI + bI = I^2$ which completes the proof.

§ 3. Applications of the main theorem

In this section we study the Bhattacharya polynomial of two m-primary ideals in a 2-dimensional Cohen-Macaulay local ring. By combining this with our main theorem we give simpler proofs of several results about complete ideals in a 2-dimensional regular local ring.

Let (R, m) be a two dimensional local ring. For m-primary ideals I and J consider the function $B(r, s) = l(R/I^rJ^s)$. By [B], there exists a polynomial $P(r, s)$ with rational coefficients such that for large r and s, $P(r, s) = B(r, s)$. The polynomial $P(r, s)$ is called the Bhattacharya polynomial of I and J. The total degree of $P(r, s)$ is 2 and it can be written in the form

$$P(r, s) = e(I)\binom{r}{s} + e_1(I \mid J)rs + e(J)\binom{s}{2} + fr + gs + h\,,$$

where $e_1(I \mid J)$ is a positive integer called the mixed multiplicity of I and J and f, g, h are integers. By a theorem of Rees in [R1], for any joint reduction (a, b) of (I, J), $e((a, b)) = e_1(I \mid J)$. We begin with a lemma.

LEMMA 3.1. *Let (R, m) be a local ring of dimension ≥ 2. Let I and J be ideals of R. If $a \in I$ and $b \in J$ are such that (a, b) is an R-sequence, then the R-module homomorphism*

$$f\colon\ R/I \oplus R/J \longrightarrow (a, b)/(aJ + bI)$$

defined as $f(\bar{x}, \bar{y}) = (xb + ya) + (aJ + bI)$ is an isomorphism.

Proof. It is clear that f is a surjective R-module homomorphism. Suppose that $xb + ya \in aJ + bI$. Then $xb \in (a, bI)$. Choose $c \in R$ and $d \in I$ so that $xb = ca + bd$. Hence $b(x - d) = ca$ which gives $x - d \in (a\colon b) = (a)$. Thus $x \in I$. Similarly $y \in J$. Hence f is injective.

Let I and J be complete m-primary ideals in a two dimensional regular local ring R. By a result of Lipman [L. Corollary 3.7]

$$e_1(I \mid J) = l(R/IJ) - l(R/I) - l(R/J)\,.$$

In our next result we generalize that above "*mixed multiplicity formula of Lipman*" to 2-dimensional Cohen-Macaulay local rings.

THEOREM 3.2. *Let (R, m) be a two dimensional Cohen-Macaulay local ring with R/m infinite. Let I and J be m-primary ideals. Then the following are equivalent:*

(a) *There exist* $a \in I$ *and* $b \in J$ *with* $aJ + bI = IJ$.

(b) *For all integers* $r, s \geq 1$; $a^r J^s + b^s I^r = I^r J^s$.

(c) $l(R/I^r J^r) = l(R/I^r) + rse_1(I \mid J) + l(R/J^s)$ *for all* r, s.

(d) $e_1(I \mid J) = l(R/IJ) - l(R/I) - l(R/J)$.

Proof. (a) $\Rightarrow$ (b). By symmetry, it is enough to show that $a^r J + bI^r = I^r J$ for all r. Use induction on r. Assume that this equation holds for r. Then

$$\begin{aligned} I^{r+1}J &= I(I^r J) \\ &= I(a^r J + bI^r) \\ &= a^r IJ + bI^{r+1} \\ &= a^r(aJ + bI) + bI^{r+1} \\ &= a^{r+1}J + bI^{r+1}. \end{aligned}$$

(b) $\Rightarrow$ (c). By the lemma 3.1 we get

$$R/I^r \oplus R/J^s \simeq (a^r, b^s)/a^r J^s + b^s I^r$$

since the equation $a^r J^s + b^s I^r = I^r J^s$ forces (a^r, b^s) to be an R-sequence in view of the Cohen-Macaulayness of R. Hence for all $r, s \geq 1$

$$l(R/I^r) + l(R/J^s) = l(R/I^r J^s) - l(R/(a^r, b^s)).$$

Since R is Cohen-Macaulay, $l(R/(a^r, b^s)) = rsl(R/(a, b)) = rse((a, b)) = rse_1(I \mid J)$ by [R1, Theorem 2.4]. Thus (c) follows.

(c) $\Rightarrow$ (d). Clear.

(d) $\Rightarrow$ (a). Since R/m is infinite there exists a joint reduction (a, b) of (I, J) by [R1, Corollary (i)]. By Rees' theorem 2.4 in [R1], $e_1(I \mid J) = e((a, b)) = l(R/(a, b))$. By Lemma 3.1

$$l(R/I) + l(R/J) = l(R/aJ + bI) - l(R/(a, b)).$$

Hence $e_1(I \mid J) = l(R/aJ + bI) - l(R/I) - l(R/J)$. Comparing this expression with the formula for $e_1(I \mid J)$ in (d) we conclude that $aJ + bI = IJ$.

COROLLARY 3.3 (Lipman). *Let* (R, m) *be a 2-dimensional regular local ring. Then for any m-primary complete ideals I and J*

$$e_1(I \mid J) = l(R/IJ) - l(R/I) - l(R/J).$$

Proof. By passing to $R[x]_{mR[x]}$ we may assume that R/m is infinite. Now use Theorem 3.2 and Theorem 3.1.

COROLLARY 3.4 (Lipman). *Let I be a complete m-primary ideal of a 2-dimensional regular local ring. Then for all n*

$$l(R/I^n) = e(I)\binom{n+1}{2} - [e(I) - l(R/I)]n .$$

Proof. By Corollary 3.3 and Theorem 3.2 we have

$$l(R/I^n) = l(R/I^{n-1}I) = l(R/I^{n-1}) + (n-1)e_1(I \mid I) + l(R/I) .$$

The assertion of the corollary is clearly true for $n = 1$. Suppose it is true for $n - 1$. Then by using $e_1(I \mid I) = e(I)$,

$$\begin{aligned} l(R/I^n) &= e(I)\binom{n}{2} - [(e(I) - l(R/I)](n-1) + (n-1)e(I) + l(R/I) \\ &= e(I)\binom{n}{2} + ne(I) - [e(I) - l(R/I)]n \\ &= e(I)\binom{n+1}{2} - [e(I) - l(R/I)]n . \end{aligned}$$

COROLLARY 3.5. *Let (R, m) be a 2-dimensional regular local ring. Let I and J be m-primary complete ideals. Then for all r, s, ≥ 1,*

$$l(R/I^rJ^s) = e(I)\binom{r}{2} + rse_1(I \mid J) + e(J)\binom{s}{2} + rl(R/I) + sl(R/J) ,$$

where

$$e_1(I \mid J) = l(R/IJ) - l(R/I) - l(R/J) .$$

Proof. By Theorem 3.2, Theorem 2.1 and Corollary 3.4 we get

$$\begin{aligned} l(R/I^rJ^s) &= l(R/I^r) + rse_1(I \mid J) + l(R/J^s) \qquad \text{for all } r, s \\ &= e(I)\binom{r+1}{2} - [e(I) - l(R/I)]r + rse_1(I \mid J) \\ &\quad + e(J)\binom{s+1}{2} - [e(J) - l(R/J)]s \\ &= e(I)\binom{r}{2} + rse_1(I \mid J) + e(J)\binom{s}{2} + rl(R/I) + sl(R/J) . \end{aligned}$$

§ 4. On a question of Huneke

In this section we answer a question of Huneke mentioned in the introduction. Recall that the analytic spread of an ideal I in a local ring (R, m) is, by definition, the dimension of the graded ring $R/m \oplus I/Im \oplus I^2/I^2m \oplus \cdots$. Analytic spread of I is denoted by $a(I)$. It is well known that $htI \leq a(I) \leq \mu(I)$. If R/m is infinite then there is an ideal J contained in I so that $\mu(J) = a(I)$ and $JI^n = I^{n+1}$ for some n. We refer the reader to [NR] for details. An ideal is called equimultiple if $htI = a(I)$. Note that all m-primary ideals are equimultiple. We remark that the proof of the next theorem differs from that of Theorem 4.1 of [H-S] only in the use of joint reductions instead of reductions.

THEOREM 4.1. *Let (R, m) be a d-dimensional Cohen-Macaulay local ring satisfying (R_2). Let I and J be height two equimultiple complete ideal so that IJ is also equimultiple. Then for all r, $s \geq 1$, I^rJ^s is a complete equimultiple ideal.*

Proof. We may assume without loss of generality that R/m is infinite. By [O] there exist $a \in I$ and $b \in J$ so that $aJ + bI$ is a reduction of IJ. It follows that (a, b) is an R-sequence and hence by Lemma 3.1 we get

$$\mathrm{Ass}\,(a, b)/aJ + bI = \mathrm{Ass}\, R/I \cup \mathrm{Ass}\, R/J\,.$$

By considering the exact sequence

$$0 \longrightarrow (a, b)/aJ + bI \longrightarrow R/aJ + bI \longrightarrow R/(a, b) \longrightarrow 0$$

we get $\mathrm{Ass}\,(R/aJ + bI) \subset \mathrm{Ass}\, R/(a, b) \cup \mathrm{Ass}\, R/I \cup \mathrm{Ass}\, R/J$. By Proposition 4.1 of [Mc], I and J are unmixed ideals. Hence $aJ + bI$ is an unmixed ideal. Suppose that $aJ + bI < IJ$. Then there exists a height 2 prime $P \in \mathrm{Ass}\,(R/aJ + bI)$ so that $(aJ + bI)R_P < IJR_P$. It follows that $P \subset I + J$. Since R_P is regular, by Theorem 2.1, $IJR_P = (aJ + bI)R_P$. Hence $aJ + bI = IJ$ and consequently IJ is unmixed. Thus $IJ = \cap\, (IJ)R_P$ where the intersection ranges over all associated primes of IJ. By Zariski's product theorem IJR_P is complete. Therefore IJ is complete. That I^rJ^s is equimultiple follows from Theorem 1 of [O] and Corollary (ii) of [R1]. The completeness of I^rJ^s follows by double induction on r and s.

ACKNOWLEDGEMENTS. I would like to thank Professors W. Heinzer and C. Huneke from whom I learned this subject. Thanks are also due to Professor D. Katz for valuable discussions.

REFERENCES

[B] P. B. Bhattacharya, The Hilbert function of two ideals, Proc. Cambridge Philos. Soc., **53** (1957), 568–575.

[H1] C. Huneke, The primary components of and integral closures of ideals in 3-dimensional regular local rings, Math. Ann., **275** (1986), 617–635.

[H2] C. Huneke, Complete ideals in two-dimensional regular local rings, proceedings of "Microprogram on Commutative Algebra" held at MSRI, Berkeley, CA (1987) Springer-Verlag, New York (1989).

[H-S] C. Huneke and J. D. Sally, Birational extensions in dimension two and integrally closed ideals, J. Algebra, **115** (1988), 481–500.

[L] J. Lipman, On complete ideals in regular local rings, Commutative Algebra and Algebraic Geometry: in honor of M. Nagata.

[L-T] J. Lipman and B. Teissier, Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals, Michigan Math. J., **28** (1981), 97–116.

[Mc] S. McAdam, "Asymptotic prime divisors," Lecture notes in Mathematics, Vol. **1023**, Springer-Verlag, Berlin, 1983.

[NR] D. G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Cambridge Philos. Soc., **50** (1954), 145–158.

[O] L. O'Carroll, On two theorems concerning reductions in local rings, J. Math. Kyoto Univ., **27** (1987), 61–67.

[R1] D. Rees, Generalizations of reductions and mixed multiplicities, J. London Math. Soc., (2) **29** (1984), 397–414.

[R2] D. Rees, Hilbert functions and pseudo-rational local rings of dimension 2, J. London Math. Soc., (2) **24** (1981), 467–479.

[Z] O. Zariski, Polynomial ideals defined by infinitely near points, Amer. J. Math., **60** (1938), 151–204.

[ZS] O. Zariski and P. Samuel, "Commutative Algebra," Vol. 2, van Nostrand, Princeton, 1960.

Department of Mathematics
Indian Institute of Technology
Powai, Bombay-400076
India