M. Kotani
Nagoya Math. J.
Vol. 118 (1990), 55-64

A DECOMPOSITION THEOREM OF 2-TYPE IMMERSIONS

MOTOKO KOTANI

§ 1. Introduction

One branch of the research of submanifolds was introduced by Chen in terms of *type* in [2]. Type of a submanifold makes clear how the eigenspace decomposition of the Laplacian (of the ambient space) preserve after restricted to the submanifold.

We will review the definition of type of a submanifold M in the unit sphere $S^m(1)$ in the Euclidean space E^{m+1}. Let x be the canonical coordinate in E^{m+1}. We call M k-type if x is decomposed into k maps $x_1, \cdots, x_k$ such that

$$x = x_1 + \cdots + x_k,$$
$$\Delta x_i = \lambda_i x_i \qquad \text{for } i = 1, \cdots, k$$

as a vector valued function, where Δ is the Laplacian of M. As coordinate functions generate the 1st eigenspace of $S^m(1)$, k-type means that the 1st eigenspace of $S^m(1)$ restricted to M is decomposed into k eigenspaces of M. We can generalize the definition to the k-type via l-th eigenspace of other ambient spaces in the same way. But here as we are concerned only with surfaces of 2-type in $S^m(1)$, we will not refer to it anymore. For the precise definitions, see § 5. See [1], [5] etc. for other relevant results for the general case.

The immersion ι: $M \rightarrow S^m$ is called *mass-symmetric* if the center of mass of $\iota(M)$ coincides with the center of S^m.

In terms of the type of immersions, a well known theorem of Takahashi [4] states that an n-dimensional compact submanifold M of E^{m+1} is 1-*type if and only if M is a minimal submanifold of a hypersphere S^m* of E^{m+1}, and any compact minimal submanifold of S^m is known to be mass-symmetric. Our results can be stated as follows.

THEOREM 1. *Any mass-symmetric and proper 2-type immersion of a*

Received September 26, 1988.

topological 2-sphere into a unit hypersphere $S^m(1) \subset E^{m+1}$ *is the direct sum of two minimal into spheres. That is we can write*

$$x = x_p \oplus x_q \in E^{r+1} \oplus E^{m-r} = \ = E^{m+1}$$

such that

$$x_p\colon M \longrightarrow S^r(\cos\theta) \subset E^{r+1} \quad \textit{and}$$
$$x_q\colon M \longrightarrow S^{m-r-1}(\sin\theta) \subset E^{m-r}$$

are minimal immersions with respect to the induced metrics.

COROLLARY. *If the immersion in Theorem* 1 *is full, then m is odd and greater than* 5.

Remark. There is a mass-symmetric and 2-type immersion of a flat torus which does not admit a decomposition in the sense of Theorem 1. And to the remark for Corollary no examples of 2-type surfaces are known in even-dimensional spheres.

By Theorem 1 a mass-symmetric and proper 2-type immersion of a 2-sphere is decomposed into two minimal immersions. Hence we reduce the problem to determine the space of all 2-type immersions of S^2 into the sphere to that to know when (S^2, g) admits more than two distinct minimal immersions into spheres.

(S^2, g) of constant curvature has been the only known example having countably infinite minimal immersions. Moreover we get the following when the dimension m is small.

THEOREM 2. *If a 2-sphere admits a mass-symmetric and proper 2-type immersion into* $S^9(1)$, *then the 2-sphere is of constant curvature.*

Though the hyperbolic space H^m is not compact, we can define the notion of mass-symmetric and 2-type immersions into the hyperbolic space as follows.

Let L^{m+1} be the $(m+1)$-Euclidean space with the inner-product $\langle\ ,\ \rangle$ of signature $(-, +, \cdots, +)$. It is well known that H^m can be realized as

$$H^m = \{x \in L^{m+1}\colon \langle x, x\rangle = -1\}\,.$$

Let $x\colon M \to H^m$ be an isometric immersion. We can easily see that the mean curvature vector H of M in H^m is given by

$$\Delta x = n(H + x),$$

where $n = \dim M$.

We call the immersion x mass-symmetric and 2-type when x can be given by

$$x = x_p + x_q \in L^{m+1},$$

where $\Delta x_p = \lambda_p x_p$ and $\Delta x_q = \lambda_q x_q$, $\lambda_p < \lambda_q$. We note that x is the eigenfunction of the Laplacian of the hyperbolic space H^m.

By the same argument as in Theorem 1, we can see that

$$x_p \colon S^2 \longrightarrow L^{m+1}$$

is an immersion, whose induced metric is homothetic to the original one, into the space

$$H^m((\lambda_q - \lambda_p)/(\lambda_q + 2)) = \{x \in L^{m+1};\ \langle x, x\rangle = -(\lambda_q + 2)/(\lambda_q - \lambda_p)\},$$

that is, x_p is a minimal immersion of a 2-sphere into the hyperbolic space, which is impossible. Hence we get the following.

THEOREM 3. *There is no mass-symmetric and 2-type immersion of a topological 2-sphere into the hyperbolic space.*

The author wishes to express her gratitude to Professors B.Y. Chen and K. Ogiue for their valuable suggestions.

§ 2. Preliminaries

We assume that $x\colon M \to S^m(1)$ is a mass-symmetric and 2-type immersion of a Riemannian surface M into the unit hypersphere $S^m(1)$ in E^{m+1} centered at the origin of E^{m+1}. In terms of an isothermal coordinate $z = x + iy$, the induced metric is given by $g = \rho^2|dz|^2$. Denote by ∇ and $\tilde{\nabla}$ the Riemannian connections of M and E^{m+1} respectively, and by $\tilde{H}$, $\tilde{\sigma}$ and $\tilde{D}$ the mean curvature vector, the second fundamental form and the normal connection of M in E^{m+1} and H, σ and D the mean curvature vector, the second fundamental form and the normal connection of M in $S^m(1)$. By an easy calculation we obtain

$$\Delta^D \xi = 4\rho^{-2}(\partial_{\bar z}\partial_z \xi + R^D_{\partial_{\bar z}\partial_z}\xi)\,, \tag{2.1}$$

$$H = 2\rho^{-2}\sigma_{z\bar z}\,, \tag{2.2}$$

where ξ is a normal vector field, R^D is the normal where ξ and $\sigma_{z\bar z} = \sigma(\partial_z, \partial_{\bar z})$.

The Codazzi equation and the Ricci equation are given respectively by

$$\partial_z H = 2\rho^{-2}\partial_{\bar z}\sigma_{zz}\,, \tag{2.3}$$

$$R^D_{\partial_{\bar z}\partial_z}\xi = 2\rho^{-2}(\langle\sigma_{\bar z\bar z}, \xi\rangle\sigma_{zz} - \langle\sigma_{zz}, \xi\rangle\sigma_{\bar z\bar z})\,. \tag{2.4}$$

From the definition of a mass-symmetric and 2-type immersion, x is decomposed as follows:

$$x = x_p + x_q\,, \tag{2.5}$$

$$\Delta x = \lambda_p x_p + \lambda_q x_q\,. \tag{2.6}$$

Then we see

$$\Delta(\Delta x) = (\lambda_p + \lambda_q)\Delta x - \lambda_p\lambda_q x\,. \tag{2.7}$$

On the other hand, the mean curvature vectors $\tilde H$ in E^{m+1} and H in $S^m(1)$ are given by

$$\tilde H = H - x = -\frac{1}{2}\Delta x\,. \tag{2.8}$$

Hence x_p and x_q can be written as

$$x_p = (2\tilde H + \lambda_q x)/(\lambda_p - \lambda_q) = \{2H + (\lambda_q - 2)x\}/(\lambda_q - \lambda_p)\,, \tag{2.9}$$

$$x_q = (2\tilde H + \lambda_p x)/(\lambda_q - \lambda_p) = \{2H + (\lambda_p - 2)x\}/(\lambda_p - \lambda_q)\,. \tag{2.10}$$

From

$$\langle x, x\rangle = 1,\ \langle x, \tilde H\rangle = -1 \quad \text{and} \quad \langle\Delta(\Delta x), x\rangle = \langle\Delta(-2\tilde H), x\rangle = 2|\tilde H|^2$$

we easily get

$$|\tilde H|^2 = |H|^2 + 1 = 1 - \frac{1}{4}(\lambda_p - 2)(\lambda_q - 2)\,, \tag{2.11}$$

$$\langle x_p, x_p\rangle = \{4|H|^2 + (\lambda_q - 2)^2\}/(\lambda_q - \lambda_p)^2 = (\lambda_q - 2)/(\lambda_q - \lambda_p)\,, \tag{2.12}$$

$$\langle x_q, x_q\rangle = \{4|H|^2 + (\lambda_p - 2)^2\}/(\lambda_p - \lambda_q)^2 = (\lambda_p - 2)/(\lambda_p - \lambda_q)\,, \tag{2.13}$$

$$\langle x_p, x_q\rangle = -\{4|H|^2 + (\lambda_p - 2)(\lambda_q - 2)\}/(\lambda_p - \lambda_q)^2 = 0\,. \tag{2.14}$$

These imply that x_p and x_q are maps into spheres. In the same way, Chen gives the following formula in [2].

$$\begin{aligned} -\frac{1}{2}\Delta(\Delta x) = \Delta\tilde H = \Delta^D H &+ \frac{4}{\rho^4}\{\langle H, \sigma_{\bar z\bar z}\rangle\sigma_{zz} - \langle H, \sigma_{zz}\rangle\sigma_{\bar z\bar z}\} \\ &+ \mathrm{tr}(\nabla\langle\sigma_{zz}, H\rangle) + 2|\tilde H|^2\tilde H \end{aligned} \tag{2.15}$$

where Δ^D is the normal Laplacian of M.

§ 3. Some lemmas

In this section we are preparing some lemmas to prove Theorem 1.

LEMMA 1. *If M is mass-symmetric and 2-type, then $\langle H, \sigma_{zz}\rangle$ is a holomorphic function. Moreover, if M is a topological S^2, then M is pseudo-umbilic, i.e., $\langle H, \sigma_{zz}\rangle = 0$.*

Proof. In terms of the isothermal coordinate, $\operatorname{tr}(\nabla\langle\sigma_{zz}, H\rangle)$ is given as

$$\begin{aligned}\operatorname{tr}(\nabla\langle\sigma_{zz}, H\rangle) &= 2\rho^{-2}\{(\langle\sigma_{zz}, \partial_{\bar z}H\rangle + \langle H, \partial_{\bar z}\sigma_{zz}\rangle)\partial_{\bar z} + (\langle H, \partial_z\sigma_{\bar z\bar z}\rangle + \langle\partial_z H, \sigma_{\bar z\bar z}\rangle)\partial_z\} \\ &= 2\rho^{-2}(\partial_{\bar z}\langle\sigma_{zz}, H\rangle\partial_{\bar z} + \partial_z\langle H, \sigma_{\bar z\bar z}\rangle\partial_z)\,.\end{aligned}$$

As M is mass-symmetric and proper 2-type, it follows that $\operatorname{tr}(\nabla\langle\sigma_{zz}, H\rangle) = 0$ by comparing the tangent parts of (2.7) and (2.15). Hence we get $\partial_{\bar z}\langle\sigma_{zz}, H\rangle = 0$. ##

LEMMA 2. *Let x: $S^2 \to S^m(1)$ be mass-symmetric and 2-type. Then*

$$\Delta^D H = (\lambda_p\lambda_q/2)H\,. \tag{3.1}$$

Proof. From the normal parts of (2.7) and (2.15) we obtain

$$(\lambda_p + \lambda_q)\tilde H + (\lambda_p\lambda_q/2)x = \Delta\tilde H = \Delta^D H + 2(|H|^2 + 1)(H - x)\,.$$

Noting that $H = \tilde H + x$ is normal to x, we see that

$$(\lambda_p + \lambda_q)H = \Delta^D H - (\lambda_p\lambda_q - 2\lambda_p - 2\lambda_q)/2\,.$$

Thus we obtain $\Delta^D H = (\lambda_p\lambda_q/2)H$. ##

LEMMA 3. *Let x: $S^2 \to S^m(1)$ be mass-symmetric and proper 2-type. Then the following equations hold.*

1) $\langle\partial_z^k H, \partial_z^l H\rangle = 0$,
2) $\langle\partial_z^k H, \partial_z^l\sigma_{zz}\rangle = 0$,
3) $\langle\partial_z^k\sigma_{zz}, \partial_z^l\sigma_{zz}\rangle = 0$.

Proof. We shall prove the result by induction. To this end we define the condition [N] as follows.

[N]–1 $\langle\partial_z^k H, \partial_z^l H\rangle = 0$ for all $k + l \leqq N$.
[N]–2 $\langle\partial_z^k\sigma_{zz}, \partial_z^l H\rangle = 0$ for all $k + 1 \leqq N - 1$,
[N]–3 $\langle\partial_z^k\sigma_{zz}, \partial_z^l\sigma_{zz}\rangle = 0$ for all $k + l \leqq N - 2$,

[N]–4 $\partial_{\bar z}\partial_z(\partial_z^k H)$ is a linear combination of $\partial_z^k H$, $\partial_z^{k-1}H$, $\cdots$, H, σ_{zz}, $\partial_z\sigma_{zz}$, $\cdots$, $\partial_z^{k-2}\sigma_{zz}$ for all $k \leqq N-2$,

[N]–5 $\partial_{\bar z}\partial_z(\partial_z^k\sigma_{zz})$ is a linear combination of $\partial_z^{k+2}H$, σ_{zz}, $\partial_z\sigma_{zz}$, $\cdots$, $\partial_z^k\sigma_{zz}$ for all $k \leqq N-3$.

In what follows we write ∂_z, $\partial_{\bar z}$ and σ_{zz} simply as ∂, $\bar\partial$ and σ, respectively.

Now we know that M is pseudo-umbilic and has constant mean curvature. Moreover it follows from Lemma 2 that its mean curvature vector satisfies the equation $\Delta^D H = \lambda\rho^{-2}H$. Hence using (2.3) we get

$$\begin{aligned}
&\langle\sigma, H\rangle = 0\,,\\
&\langle H, \partial H\rangle = \rho^2\langle H, \bar\partial\sigma\rangle = 0\,,\\
&\langle H, \bar\partial H\rangle = \rho^2\langle H, \partial\bar\sigma\rangle = 0\,,\\
&\bar\partial\partial H = \rho^2\Delta^D H - \rho^{-2}\{\langle\sigma, H\rangle\bar\sigma - \langle\bar\sigma, H\rangle\sigma\} = \rho^2\Delta^D H = \lambda H\,,\\
&\bar\partial\langle\partial H, \partial H\rangle = 2\langle\lambda H, \partial H\rangle = 0\,.
\end{aligned}$$

As a global holomorphic on differential S^2 is identically zero, the last equation implies

$$\langle\partial H, \partial H\rangle = 0\,.$$

Similarly, noting that

$$\bar\partial\langle\partial H, \sigma\rangle = \langle\lambda H, \sigma\rangle + \rho^{-2}\langle\partial H, \partial H\rangle/2 = 0\,.$$

we get

$$\langle\partial H, \sigma\rangle = \partial\langle H, \sigma\rangle - \langle H, \partial\sigma\rangle = -\langle H, \partial\sigma\rangle = 0\,.$$

We also get

$$\begin{aligned}
&\bar\partial\langle\sigma, \sigma\rangle = \rho^2\langle\partial H, \sigma\rangle = 0, \quad \text{i.e.} \quad \langle\sigma, \sigma\rangle = 0\,.\\
&\langle\partial^2 H, \partial H\rangle = 2\partial\langle\partial H, \partial H\rangle = 0\,.
\end{aligned}$$

These imply that the condition [2] holds.

Next we will show that [N] holds if [N–1] holds. From the Ricci equation, we get

$$\bar\partial\partial(\partial^k H) = \partial(\bar\partial\partial)(\partial^{k-1}H) + \rho^{-2}\{\langle\bar\sigma, \partial^k H\rangle\sigma - \langle\sigma, \partial^k H\rangle\bar\sigma\}\,.$$

As $k \leqq N-2$, we obtain $\langle\sigma, \partial^k H\rangle = 0$ by [N–1]–2. Then combining this with [N–1] we get [N]–4. Similarly, from the Ricci equation we get

$$\bar\partial\partial(\partial^k\sigma) = \partial(\bar\partial\partial)(\partial^{k-1}\sigma) + \rho^{-2}\{\langle\partial^k\sigma, \bar\sigma\rangle\sigma - \langle\partial^k\sigma, \sigma\rangle\bar\sigma\}\,.$$

By using [N–1]–3 and [N–1]–5, we get [N]–5. Finally we prove [N]–1$\sim$3. We remark that

$$\begin{aligned}\langle\partial^k\sigma, \partial^l\sigma\rangle &= \partial\langle\partial^{k-1}\sigma, \partial^l\sigma\rangle - \langle\partial^{k-1}\sigma, \partial^{l+1}\sigma\rangle\\ &= -\langle\partial^{k-1}\sigma, \partial^{l+1}\sigma\rangle = (-1)^k\langle\sigma, \partial^{l+k}\sigma\rangle\,.\end{aligned}$$

$$\langle\partial^k H, \partial^l\sigma\rangle = (-1)^l\langle\partial^{k+1}H, \sigma\rangle\,,$$

$$\langle\partial^k H, \partial^l H\rangle = (-1)^{l-1}\langle\partial^{k+l-1}H, \partial H\rangle\,.$$

$$\begin{aligned}\bar\partial\langle\sigma, \partial^k H\rangle &= \rho^2\langle\partial H, \partial_k H\rangle/2 + \langle\sigma, \bar\partial\partial(\partial^{k-1}H)\rangle\\ &= \rho^2\langle\partial H, \partial^k H\rangle/2 \text{ linear combination of}\\ &\qquad\langle\sigma, \partial^{k-1}H\rangle, \cdots, \langle\sigma, H\rangle, \langle\sigma, \sigma\rangle, \cdots, \langle\sigma, \partial^{k-3}\sigma\rangle\\ &= \rho^2\langle\partial H, \partial^k H\rangle/2\,,\end{aligned}$$

$$\begin{aligned}\bar\partial\langle\sigma, \partial^k\sigma\rangle &= \rho^2\langle\partial H, \partial^k\sigma\rangle/2 - \langle\sigma, \bar\partial\partial(\partial^{k-1}\sigma)\rangle = \rho^2\langle\partial H, \partial^k\sigma\rangle/2\\ &\quad + \text{linear combination of } \langle\sigma, \partial^{k+1}H\rangle, \langle\sigma, \partial^{k-1}\sigma\rangle, \cdots, \langle\sigma, \sigma\rangle\\ &= \rho^2\langle\partial H, \partial^k\sigma\rangle/2\,.\end{aligned}$$

In these equations we use the assumption [N–1] and the Codazzi equation (2.3). Noting that holomorphic form on S^2 is identically zero, we may prove $\bar\partial\langle\partial H, \partial^k H\rangle = 0$ for all $k \leqq N-1$ to get [N]–1 $\sim$ 3.

But in fact we can prove that

$$\begin{aligned}\bar\partial\langle\partial H, \partial^k H\rangle &= \langle\lambda H, \partial^k H\rangle + \langle\partial H, \bar\partial\partial(\partial^{k-1}H)\rangle\\ &= \text{linear combination of } \langle\partial H, \partial H\rangle, \cdots, \langle\partial H, \partial^{k-1}H\rangle\\ &= 0\,.\end{aligned}$$

##

Now we can prove Corollary of Theorem 1 independently. Let

$$E = \operatorname{span}\{\partial_z^k H, \partial_z^l\sigma_{zz}\}\,.$$

By Lemma 3, $E\oplus\bar{E}\oplus\{H\}$ then gives an orthogonal decomposition. In the 2-dimensional case, the normal space is spanned by all the derivatives of σ and H with respect to z and $\bar{z}$. But (2.3) combined with [N]–4 and [N]–5 in Lemma 3 show that all these derivatives belong to $E\oplus\bar{E}$. Therefore $E\oplus\bar{E}\oplus\{H\}$ gives a decomposition of the normal space, so that

$$\dim S^m = \dim S^2 + 2\dim E + 1\,.$$

Thus m is odd.

Moreover noting that

$$\langle\sigma_{zz}, \partial_z H\rangle = \langle\sigma_{zz}, \partial_{\bar{z}} H\rangle = 0\,,$$

we can easily see that m is greater than 5 unless H is parallel.

On the other hand, if $S^2 \to S^m$ has parallel mean curvature, then the immersion is minimal in a small hypersphere, which contradicts the mass-symmetry. ##

§ 4. Proof of Theorem 1

Let $x: S^2 \to S^m(1) \subset E^{m+1}$ be a mass-symmetric and 2-type immersion, i.e.,

$$x = x_p + x_q : S^2 \longrightarrow E^{m+1} \tag{4.1}$$

where $\Delta x_p = \lambda_p x_p$ and $\Delta x_q = \lambda_q x_q$.

We already know that x has constant mean curvature

$$|H|^2 = -\frac{1}{4}(\lambda_p - 2)(\lambda_q - 2)$$

and x is pseudo-umbilic i.e. $\langle H, \sigma \rangle = 0$. Moreover x_p and x_q can be written in terms of x and H as in (2.9) and (2.10).

First we will show that the maps x_p, x_q: $(S^2, \rho^2|dz|^2) \to E^{m+1}$ are homothetic immersions into some spheres, so that, on account of Takahashi's theorem, they are minimal in the spheres. We already see in § 2 that x_p and x_q are immersions into spheres whose induced metric is homothetic to the original metric $\rho^2|dz|^2$. Since the differential $(x_p)_*$ of x_p satisfies

$$\begin{aligned}(x_p)_*\partial_z &= \{2\tilde{\nabla}_z H + (\lambda_q - 2)\partial_z\}/(\lambda_q - \lambda_p) \\ &= \{2\partial H + (\lambda_q - 2 - 2|H|^2)\partial_z\}/(\lambda_q - \lambda_p),\end{aligned} \tag{4.2}$$

the induced metric is given by

$$\begin{aligned}\langle (x_p)_*\partial_z, (x_p)_*\partial_z \rangle &= \langle (x_p)_*\partial_{\bar z}, (x_p)_*\partial_{\bar z} \rangle = 0, \\ \langle (x_p)_*\partial_z, (x_p)_*\partial_{\bar z} \rangle &= \{4|\partial_z H|^2 + (\lambda_q - 2 - 2|H|^2)^2\rho^2/2\}/(\lambda_p - \lambda_q)^2 \\ &= \lambda_p(\lambda_q - 2)\rho^2/4(\lambda_q - \lambda_p).\end{aligned}$$

This implies that x_p is a 1-type immersion homothetic to the original metric so that x_p is minimal. The same argument can be applied to x_q.

It remains to prove that $x_p + x_q$ is the direct sum, i.e. $(x_p, x_q) \in S^k \times S^{m-k-1} \subset E^{k+1} \times E^{m-k}$. To show this we prove that all derivertives of x_p with respect to z and $\bar z$ are orthogonal to those of x_q, which implies that all coefficients of the Taylor expansion of x_p around a fixed point are orthogonal to those of x_q. But, by the same argument as in Lemma

3 and Proof of Corollary, we can easily see by induction that it is enough to show

$$\langle \partial_z^l x_p, x_q \rangle = 0 .$$

As $\partial_z^l x_p$ is a linear combination of x_p, ∂_z, $\partial_z^i \sigma_{zz}$ and $\partial_z^j H$, we can show the above equation by Lemma 3 and $\langle x_p, x_q \rangle = 0$. This completes the proof of Theorem 1.

§ 5. Proof of Theorem 2

From Theorem 1 this immersion is decomposed into two minimal immersions;

$$(S^2, c_1 g) \longrightarrow S^2(1) \quad \text{and} \quad (S^2, c_2 g) \longrightarrow S^6(1) ,$$

or

$$(S^2, c_1 g) \longrightarrow S^4(1) \quad \text{and} \quad (S^2, c_2 g) \longrightarrow S^4(1) .$$

Theorem 2 is clear in the first case. In the second case we will show that $c_1 = c_2$ by using the result in [2]. If (S^2, g) admits two minimal immersions with $k_2 = 0$ in (71) in [2], from these equations we find that curvature is constant $c_1/3 = c_2/3$. But if $c_1 = c_2$, then the immersion is 1-type.

§ 6. General case

In this section we define k-type via 1th-eigenspace in a general compact manifold. Let M be a compact Riemannian manifold and Δ the Laplacian of M acting on the space $C^\infty(M)$ of all C^∞ functions on M. Then Δ is a self-adjoint elliptic operator and has an infinite, discrete sequence of eigenvalues,

$$0 = \lambda_0 < \lambda_1 < \cdots \qquad \uparrow \infty .$$

Let $V_k = \{f \in C^\infty(M);\ \Delta f = \lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k, which is finite dimensional. Each function $f \in C^\infty(M)$ has the following spectral decomposition:

$$f = \sum_{k=0}^{\infty} f_k \qquad (\text{in } L^2\text{-sense}) ,$$

where $f_k \in V_k$. In particular, there are positive integers $1 \leqq p \leqq q \leqq \infty$ such that $f_p \neq 0$ and $f_q \neq 0$ and

$$f - f_0 = \sum_{k=p}^{q} f_k ,$$

where $f_0 \in V_0$ is a constant.

Let l: $M \to \tilde{M}$ be an isometric immersion of a compact Riemannian manifold into a compact Riemannian manifold. We set

$$C^\infty(M) = \sum_{i=0}^{\infty} V_i(M) \quad \text{and} \quad C^\infty(\tilde{M}) = \sum_{i=0}^{\infty} V_i(\tilde{M}) ,$$

as the eigenspace decompositions. We may consider the following general problem.

PROBLEM 1. What can we know about ι if ι satisfies

$$\iota^*(V_1(\tilde{M})) \subset V_0(M) + V_{i_1}(M) + \cdots + V_{i_k}(M) \qquad \text{for some } l?$$

We call such M *k-type via l-th eigenspace.*

Let $x = (x_1, x_2, \cdots, x_{m+1})$ be the standard coordinates of E^{m+1} and let S^m be a hypersphere of E^{m+1}. Then $V_1(S^m)$ is spanned by $x_1, x_2, \cdots, x_{m+1}$. Our primary concern is the following restricted problem.

PROBLEM 2. Investigate the immersions x: $M \to S^N$ such that

$$\iota^* x_A \in V_0(M) + V_{i_1}(M) + \cdots + V_{i_k}(M) \qquad \text{for all } A.$$

We simply say k-type in these cases.

REFERENCES

[1] M. Barros and B. Y. Chen, Spherical submanifolds which are of 2-type via the second standard immersion of the sphere, Nagoya Math. J., **108** (1987), 77–91.

[2] B. Y. Chen, Total mean curvature and submanifolds of finite type, World Scientific, 1984.

[3] S. S. Chern, On the minimal immersions of the two-sphere in a space of constant curvature, Problems in analysis, A symposium in hornor of Salomon Bochner, Princeton University Press (1970).

[4] T. Takahashi, Minimal immersions of riemannian manifolds, J. Math. Soc. Japan, **18** (1966), 380–385.

[5] A. Ros, On spectral geometry of Kaehler submanifolds, J. Math. Soc. Japan, **36** (1984), 433–447.

Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya, Tokyo 158
Japan