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ALGEBRAIC SINGULARITIES HAVE MAXIMAL REDUCTIVE AUTOMORPHISM GROUPS

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§ 1. Introduction

Let $X = \mathcal{O}_n/\mathfrak{i}$ be an analytic singularity where $\mathfrak{i}$ is an ideal of the C-algebra $\mathcal{O}_n$ of germs of analytic functions on $(C^n, 0)$. Let $\mathfrak{m}$ denote the maximal ideal of X and $A = \operatorname{Aut} X$ its group of automorphisms. An abstract subgroup $G \leqslant A$ equipped with the structure of an algebraic group is called *algebraic subgroup* of A if the natural representations of G on all "higher cotangent spaces" $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ are rational. Let π be the representation of A on the first cotangent space $\mathfrak{m}/\mathfrak{m}^2$ and $A_1 = \pi(A)$.

Cartan's Uniqueness Theorem [8] asserts that every reductive algebraic subgroup of A is faithfully represented by π. This was strengthened by the second author in [9]: Any two reductive algebraic subgroups G, H of A are conjugate if and only if $\pi(G)$ and $\pi(H)$ are conjugate in A_1.

Since A_1 is an algebraic subgroup of $\mathrm{GL}(\mathfrak{m}/\mathfrak{m}^2)$ it has by [7, Chapter VIII, Theorem 4.3] a Levi subgroup, i.e. a reductive subgroup containing every reductive subgroup of A_1 up to conjugacy. (Hence a Levi subgroup is a maximal reductive subgroup, unique up to conjugacy.) A reductive algebraic subgroup G of A will be called a *Levi subgroup* of A if $\pi(G)$ is a Levi subgroup of A_1. It follows from the result cited above that a Levi subgroup of A (if it exists) contains every reductive algebraic subgroup of A up to conjugacy. Let us mention an interesting consequence hereof. A rational action of a reductive algebraic group on a singularity $X = \mathcal{O}_n/\mathfrak{i}$ can be lifted to an action on $\mathcal{O}_n$, linear in suitable coordinates. In the presence of a Levi subgroup of $\operatorname{Aut} X$ this linearization can be done simultaneously for (up to conjugacy) all reductive group actions on X.

In [9] it was shown that weighted homogeneous singularities with positive weights and complete intersections with isolated singularity admit

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a Levi subgroup in their group of automorphisms. In the present paper we shall extend this by proving

THEOREM 1. *Any algebraic singularity has a Levi subgroup in its group of automorphisms.*

Here a singularity $X=\mathcal{O}_n/\mathfrak{i}$ is called *algebraic* if $\mathfrak{i}$ can be generated by power series algebraic over the polynomials. Special cases are arbitrary isolated singularities (cf. [1, Theorem 3.8]) and plane curves (possibly non-reduced, cf. [5, 1.11]). The main step in the proof of Theorem 1 is

THEOREM 2. *If a reductive algebraic group acts rationally on the completion of an algebraic singularity then it also acts on the singularity itself (with the same representation on the cotangent space).*

Theorem 2 also yields an extension of Saito's characterization of weighted homogeneous isolated hypersurface singularities: If $f\in\mathcal{O}_n$ is algebraic over the polynomials and belongs to $\mathfrak{m}\cdot j(f)$ then f is weighted homogeneous in suitable coordinates. (Here $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_n$ and $j(f)$ the Jacobian ideal of f.)

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§ 2. Proofs

Let $\mathrm{GL}(C^n)$ act contragrediently on $\mathcal{O}_n$ and its completion $\hat{\mathcal{O}}_n$. We shall prove the following more precise version of Theorem 2:

THEOREM 2′. *Let $G\leqslant \mathrm{GL}(C^n)$ be reductive. Suppose that the ideal $\mathfrak{i}\leqslant\mathcal{O}_n$ is generated by power series algebraic over the polynomials. Then $\mathfrak{i}$ is equivalent to a G-stable ideal $\mathfrak{j}\leqslant\mathcal{O}_n$ if and only if $\mathfrak{i}\cdot\hat{\mathcal{O}}_n$ is formally equivalent to a G-stable ideal $\mathfrak{j}'\leqslant\hat{\mathcal{O}}_n$.*

Theorem 1 is a corollary of Theorem 2′ by

LEMMA. *Let $X=\mathcal{O}_n/\mathfrak{i}$ be an arbitrary analytic singularity. Then* Aut X *has a Levi subgroup if and only if the assertion of Theorem 2′ holds for every reductive subgroup $G\leqslant\mathrm{GL}(C^n)$.*

Proof. "if". Take a Levi subgroup G of A_1. By [9, Satz 4] there is a faithful rational action $G\to\mathrm{Aut}(\hat{\mathcal{O}}_n/\mathfrak{i}\cdot\hat{\mathcal{O}}_n)$. Hence by the formal version of [9, Satz 6] there is a faithful rational representation $G\to\mathrm{GL}(C^n)$ such that $\mathfrak{i}\cdot\hat{\mathcal{O}}_n$ is formally equivalent to a G-stable ideal of $\hat{\mathcal{O}}_n$. By the assertion of Theorem 2′ we obtain a faithful rational action $\alpha\colon G\to\mathrm{Aut}(\mathcal{O}_n/\mathfrak{i})$.

Without loss of generality $\pi(\alpha(G)) \leqslant G$. Counting dimensions and numbers of components we conclude $\pi(\alpha(G)) = G$.

"only if" is an immediate consequence of the analytic version of [9, Satz 6].

The proof of Theorem 2′ relies on an approximation theorem for polynomial equations with formal solutions. It was conjectured by Artin [2, Conjecture 1.3] and recently proven by Popescu [10, Theorem 1.3] and Rotthaus [11, Theorem 4.2] that excellent Henselian local rings have the approximation property. This implies (cf. [3, Remark 1.5]) the following approximation theorem with nested subring condition. For a coordinate system $x = (x_1, \cdots, x_n)$ denote by $C\{x\}$ the algebra of convergent power series and by $C\langle x\rangle$ the algebra of algebraic power series, i.e. those $f \in C\{x\}$ which are algebraic over $C[x]$.

THEOREM 3. *If a system of polynomial equations over* $C\langle u, x\rangle$ *admits formal solutions* $\bar{y}(u)$, $\bar{z}(u, x)$,

$$F(u, x, \bar{y}(u), \bar{z}(u, x)) = 0 ,$$

then it has convergent (in fact, algebraic) solutions $y(u)$, $z(u, x)$,

$$F(u, x, y(u), z(u, x)) = 0 ,$$

approximating $\bar{y}(u)$, $\bar{z}(u, x)$ *up to any given order.*

Remark. An example of Gabriélov [6] shows that in general the corresponding statement with $C\langle u, x\rangle$ replaced by $C\{u, x\}$ is false.

Proof of Theorem 2′. One implication being obvious let us assume that $\mathfrak{i}\cdot\hat{\mathcal{O}}_n$ is formally equivalent to a G-stable ideal $\mathfrak{j}' \leqslant \hat{\mathcal{O}}_n$. Let $x_1, \cdots, x_n$ be the natural coordinates on $(C^n, 0)$.

By [9, Hilfssatz 2] there are a rational representation of G on C^m and generators $\bar{g}_1(x), \cdots, \bar{g}_m(x) \in \hat{\mathcal{O}}_n$ of $\mathfrak{j}'$ such that the vector $\bar{g}(x)$ with components $\bar{g}_i(x)$ is G-equivariant. Since G is reductive the C-algebra $C[x]^G$ of invariant polynomials and the $C[x]^G$-module of equivariant polynomial mappings $C^n \to C^m$ are finitely generated, cf. [13, Corollary 2.4.10 and Proposition 2.4.14]. Let $u(x) = (u_1(x), \cdots, u_r(x))$ and $p(x) = (p_1(x), \cdots, p_s(x))$ be corresponding generator systems. We get

$$\bar{g}(x) = \bar{y}(u(x))\cdot p(x) = \bar{y}_1(u(x))\cdot p_1(x) + \cdots + \bar{y}_s(u(x))\cdot p_s(x)$$

with suitable $\bar{y}(u) \in C[[u]]^s$.

Let $f_1(x), \cdots, f_m(x) \in C\langle x\rangle$ generate $\mathfrak{i}$. By assumption there are a formal coordinate system $\bar{z}(x) \in C[[x]]^n$ and a matrix $\bar{M}(x) \in \mathrm{GL}(m, C[[x]])$ such that

$$f(x) = \bar{g}(\bar{z}(x)) \cdot \bar{M}(x)\,,$$

hence

$$f(x) - \bar{y}(u(\bar{z}(x))) \cdot p(\bar{z}(x)) \cdot \bar{M}(x) = 0\,.$$

By Taylor expansion there is an $r \times m$ – matrix $\bar{N}(u, x)$ with entries in $C[[u, x]]$ such that

$$f(x) - \bar{y}(u) \cdot p(\bar{z}(x)) \cdot \bar{M}(x) = (u - u(\bar{z}(x))) \cdot \bar{N}(u, x)\,.$$

This is a system of polynomial equations over $C\langle u, x\rangle$ in unknowns y, z, M, N. By Theorem 3 the formal solutions $\bar{y}(u)$, $\bar{z}(x)$, $\bar{M}(x)$, $\bar{N}(u, x)$ can be approximated up to order 2 by algebraic solutions $y(u)$, $z(u, x)$, $M(u, x)$, $N(u, x)$,

$$f(x) - y(u) \cdot p(z(u, x)) \cdot M(u, x) = (u - u(z(u, x))) \cdot N(u, x)\,.$$

Since the matrix $(\partial_x z(u, x))(0)$ is invertible and $(\partial_u z(u, x))(0) = 0$ there is $w(u, x) \in C\{u, x\}^n$ such that $z(u, w(u, x)) = x$, $(\partial_x w(u, x))(0)$ is invertible, and $(\partial_u w(u, x))(0) = 0$. We conclude

$$f(w(u, x)) - y(u) \cdot p(x) \cdot M(u, w(u, x)) = (u - u(x)) \cdot N(u, w(u, x))\,.$$

Setting $\tilde{w}(x) = w(u(x), x)$ and $\tilde{M}(x) = M(u(x), \tilde{w}(x))$ this implies

$$f(\tilde{w}(x)) = y(u(x)) \cdot p(x) \cdot \tilde{M}(x)\,.$$

Since $\tilde{w}(x)$ is a coordinate system and $\tilde{M}(x) \in \mathrm{GL}(m, C\{x\})$ we have proven that $\mathfrak{i}$ is equivalent to the G-stable ideal of $\mathcal{O}_n$ generated by the components of $y(u(x)) \cdot p(x)$.

Remark. The assertion of Theorem 2′ holds for finite groups G and arbitrary singularities $X = \mathcal{O}_n/\mathfrak{i}$. This is a corollary of the following observation:

Let $G \leqslant \mathrm{GL}(C^n)$ be finite. If a system of analytic equations,

$$F(x, y, z) = 0\,,$$

has formal solutions $\bar{y}(x)$, $\bar{z}(x)$ without constant terms and such that $\bar{y}(x) = (\bar{y}_1(x), \cdots, \bar{y}_m(x))$ is G-equivariant with respect to a representation of G on C^m, then it has convergent solutions $y(x)$, $z(x)$, approximating

$\bar{y}(x)$, $\bar{z}(x)$ up to any given order, and such that $y(x)$ is again G-equivariant. (Note that this is false, in general, for infinite G. Take $G = \boldsymbol{C}^*$ acting on $\boldsymbol{C}^n$ by

$$t \cdot (x_1, \cdots, x_n) = (x_1, \cdots, x_r, t \cdot x_{r+1}, \cdots, t \cdot x_n) .$$

Then $\boldsymbol{C}[x]^G = \boldsymbol{C}[x_1, \cdots, x_r]$ and we can use Gabriélov's example.)

For the proof of the observation write $z = (z_1, \cdots, z_k) = {}_e z$, where e denotes the unit element of G, and introduce dummy-variables ${}_\gamma z = ({}_\gamma z_1, \cdots, {}_\gamma z_k)$ for $e \neq \gamma \in G$. Put ${}_\gamma \bar{z}(x) = \bar{z}(\gamma x)$ for $\gamma \in G$. Then $(\bar{y}(x), {}_\gamma \bar{z}(x), \gamma \in G)$ is equivariant with respect to a suitable representation of G on $\boldsymbol{C}^{m+k \cdot |G|}$. A theorem of Bierstone and Milman [4, Theorem A] yields the desired $y(x)$, $z(x)$.

§ 3. Saito's problem

Let $x_1, \cdots, x_n$ be coordinates on $(\boldsymbol{C}^n, 0)$ and $\lambda, \lambda_1, \cdots, \lambda_n \in \boldsymbol{Z}$. A power series $f \in \hat{\mathcal{O}}_n$ is called weighted homogeneous with weights $\lambda_1, \cdots, \lambda_n$ and degree λ (with respect to the coordinates x) if $\lambda = \lambda_1 \cdot \alpha_1 + \cdots + \lambda_n \cdot \alpha_n$ for all monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of f. This is equivalent to: f is equivariant with respect to the representations of $\boldsymbol{C}^*$ on $\boldsymbol{C}^n$ and $\boldsymbol{C}$ defined by

$$\begin{pmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_n} \end{pmatrix} \quad \text{and} \quad t^\lambda .$$

THEOREM 4. *For an algebraic hypersurface singularity* $X = \mathcal{O}_n/(f)$ *the following conditions are equivalent*:

i) $f \in \mathfrak{m} \cdot j(f)$, ($\mathfrak{m} \leqslant \mathcal{O}_n$ *the maximal ideal*, $j(f) = (\partial_1 f, \cdots, \partial_n f)$).

ii) *There is an analytic coordinate change* $z(x)$ *such that* $g(x) = f(z(x))$ *is weighted homogeneous of non-zero degree.*

Proof. One implication being obvious let us assume that $f \in \mathfrak{m} \cdot j(f)$. By [12, Korollar 3.3 and Lemma 1.4] there is a formal coordinate change $\bar{z}(x)$ such that $\bar{g}(x) = f(\bar{z}(x))$ is weighted homogeneous of non-zero degree λ. By Theorem 2′ and [9, Hilfssatz 2] there are an analytic coordinate change $z(x)$ and a unit $u(x) \in \mathcal{O}_n$ such that $g(x) = f(z(x)) \cdot u(x)$ is weighted homogeneous of degree λ. Since $\lambda \neq 0$ this implies (ii) with suitably modified $z(x)$.

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