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SYMBOLIC POWERS OF PRIME IDEALS WITH APPLICATIONS TO HYPERSURFACE RINGS

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Introduction

Let R be a commutative Noetherian ring and suppose q is a prime ideal of R. A fundamental problem is to decide when powers q^n of q are primary (that is q^n is its own primary decomposition). If q is generated by a regular sequence then powers of q are always primary, because $G(q, R)$ (the associated graded ring of R with respect to q) is an integral domain (see [12 page 98] and also [5 (2.1)]). Let $q^{(n)}$ denote the n^{th} symbolic power of q-defined by $q^{(n)} = \{r \in R \mid \text{there exists } s \in R \setminus q \text{ such that } sr \in q^n\}$. Then q^n is primary if and only if $q^n = q^{(n)}$. If q is generated by a regular sequence then we call it a complete intersection prime ideal, so if q is a complete intersection prime ideal then $q^n = q^{(n)}$ for all $n \geq 1$. If q is not a complete intersection then powers need not be primary. If R is a three-dimensional regular local ring and q is a non-complete intersection height two prime ideal for example, then Huneke showed [11 Corollary (2.5)] that $q^n \neq q^{(n)}$ for all $n \geq 2$. Thus, for such a prime q it is impossible for q^n to occur in the primary decomposition of any ideal. This phenomena increases the difficulty in finding a primary decomposition for an ideal having q as an associated prime.

One objective in the present article is to compare powers and symbolic powers, for prime ideals in hypersurface rings. Recall that T is said to be regular if its localizations at prime ideals are regular local rings. We call R a hypersurface ring if $R = T/fT$ where T is a regular ring and f is a non-unit element of T. If R is a hypersurface ring, then any prime ideal q of R has the form $q = P/fT$ where P is a prime ideal of T. If we assume that $P^n = P^{(n)}$ for all $n \geq 1$ then one might hope that this propery is preserved in q—that is $q^n = q^{(n)}$ for all $n \geq 1$.

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Simple examples show that this is asking too much; e.g. Let

$$R = K[U, V, W]/(V^2 - UW)K[U, V, W]$$

and

$$q = P/(V^2 - UW)K[U, V, W]$$

where $P = (U, V)K[U, V, W]$. Then $P^n = P^{(n)}$ for all $n \geq 1$, but $q^n \neq q^{(n)}$ for all $n \geq 2$. A more subtle question can be constructed by employing a condition on the analytic spread $l(qR_Q)$ of qR_Q for $Q > q$ (see Section One for the definition). We will prove that if P is a complete intersection prime ideal and $l(qR_Q) <$ height (Q) for all prime ideals Q properly containing q, then $q^n = q^{(n)}$ for all $n \geq 1$ (see Corollary 2.2). In fact, we obtain a more general result (Theorem 2.1) which does not assume that R is a hypersurface ring, but instead assumes only that R is Noetherian and $G(q, R)$ is Cohen-Macaulay (a property which is forced in the hypersurface case when P is a complete intersection). The condition that $l(qR_Q) <$ height (Q) for all Q properly containing q has been investigated frequently during the past (see for example [9], [17] or [13 Chapter 4]). It forces the so-called linear equivalence of the q-adic and q-symbolic topologies in the case that R is locally unmixed (see [17 Corollary 1]). Recall that the q-adic and q-symbolic topologies are said to be linearly equivalent if there exists an integer $c \geq 0$ such that $q^{(n+c)} \subset q^n$ for all $n \geq 0$. As a corollary to Theorem 2.1 we establish that in the hypersurface case with P a complete intersection and $q = P/fT$, the q-adic and q-symbolic topologies are linearly equivalent if and only if $q^n = q^{(n)}$ for all $n \geq 1$.

In Section One we review definitions and notation, as well as giving some background material.

Section Two is devoted to proving our main result (Theorem 2.1) from which some example producing corollaries follow, and to discussing several illustrative examples.

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1. Background

Throughout the paper we assume that all rings are commutative and Noetherian, and contain a unit element. Let R be such a ring and let I be an ideal in R. The associated graded ring of R with respect to I, denoted by $G(I, R)$, is defined by

$$G(I, R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots .$$

If (R, M) is local with maximal ideal M, and R/M is infinite, then we define the analytic spread of I by

$$l(I) = \dim (R/M \oplus I/MI \oplus I^2/MI^2 \oplus \cdots)$$

(Northcott and Rees introduced and studied analytic spread in [14]). The analytic spread of I roughly measures the growth of the number of generators of I^n as n increases. One always has the inequalities

$$\text{height}\,(I) \leq l(I) \leq \dim\,(R)\,.$$

If q is a prime ideal in a Noetherian ring R, then the condition $l(qR_Q) = \text{height}\,(Q)$ for some prime ideal Q properly containing q is relevant in discussing symbolic powers. According to [13 Proposition 4.1] (see also [9 Theorem 2.1]) followed by [13 Proposition 3.17] (see also [15]), if $l(qR_Q) = \text{height}\,(Q)$ for some prime ideal Q properly containing q, then $q^n \neq q^{(n)}$ for all large n. Thus if (R, M) is local and q is a prime ideal of R such that $\text{height}(q) = \dim\,(R) - 1$ and $l(q) \neq \text{height}\,(q)$, then $q^n \neq q^{(n)}$ for all large n. Notice that Huneke's result (mentioned in the introduction) considerably sharpens this in the special case when q is a height two prime ideal, not a complete intersection, and R is a three-dimensional regular local ring (by [2 Theorem 1], $l(q) = 3$ for such q).

In light of the above comments, when searching for prime ideals in a ring R whose powers are primary, one must investigate the class of primes q for which $l(qR_Q) < \text{height}\,(Q)$ for every Q properly containing q, or perhaps even the smaller class of primes q for which $l(qR_Q) = \text{height}\,(q)$. These conditions are satisfied by complete intersection prime ideals. In general it can happen that $l(qR_Q) = \text{height}\,(q)$ for all $Q > q$ while $q^n \neq q^{(n)}$ for all large n. A relatively simple example appears in [7 Example 1.5] (e.g. Let $R = K[x^2, x^3y, y^3, y^5]$, K a field, and let $q = (x^2, x^3y)R$. Then $l(qR_M) = \text{height}\,(q) = 1$ where $M = (x^2, x^3y, y^3, y^5)R$, but $q^n \neq q^{(n)}$ for all $n \geq 3$.).

Suppose R is a Noetherian ring, let q be a prime ideal of R, and assume $G(q, R)$ is a Cohen-Macaulay ring. We will show below that if $l(qR_Q) < \text{height}(Q)$ for every prime ideal Q properly containing q, then the powers of q are all primary. The proof involves the use of various graded rings, so we take a moment to settle on notation and recall some technical points.

If R is an arbitrary Noetherian ring and I is an ideal in R, the Rees ring of R with respect to I (also known as the Rees algebra) is defined and denoted by $R[It] = \bigoplus_{n\geq 0} I^n t^n$, where t is an indeterminate.

A useful isomorphism connecting the Rees ring with the associated graded ring is that $G(I, R) \cong R[It]/IR[It]$. If R is local with maximal ideal M, another frequently used isomorphism is

$$\bigoplus_{n\geq 0} I^n/MI^n \cong R[It]/MR[It]\,.$$

Notice that $l(I)$ can thus be computed by calculating $\dim(R[It]/MR[It])$. If $a \in R$ we denote by a' the leading form of a in $G(I, R)$. Recall that if a' is a regular element (not a zero-divisor) in $G(I, R)$, then

$$G((I, aR)/aR, R/aR) \cong G(I, R)/a'G(I, R)$$

([12 page 118]). For terminology not otherwise explained, we direct the reader to [12].

Much work has been done involving symbolic powers. For a sample see [5], [6], [8], [9], [10], [11], [2], [1], [13], [15], [16] and [17].

2. Main results and examples

Throughout this section we will assume that all local rings have infinite residue fields. Although we have an eye on hypersurface rings, the result we are aiming for is true more generally. Theorem 2.1 below illustrates this point, and the corollaries which follow are applications of the theorem.

THEOREM 2.1. *Let R be a Noetherian ring and suppose q is a prime ideal of R. Assume that $G(q, R)$ is a Cohen-Macaulay ring and q satisfies the property that $l(qR_Q) <$ height (Q) for every prime ideal Q properly containing q. Then $q^n = q^{(n)}$ for every $n \geq 1$.*

Proof. Suppose $q^n \neq q^{(n)}$ for some positive integer n. Then there exist $s \in R\backslash q$ and $z \in R\backslash q^n$ such that $sz \in q^n$. Hence $s'z' = 0$ in $G(q, R)$

so that s' is a zero-divisor in $G(q, R)$. Since $G(q, R) \cong R[qt]/qR[qt]$ with the isomorphism sending s' to $\bar{s}$ where "$-$" denotes image modulo $qR[qt]$, it follows that $s \in \cup \operatorname{Ass}(R[qt]/qR[qt])$, say $s \in P \in \operatorname{Ass}(R[qt]/qR[qt])$. Clearly $q \subset P$ and since $s \notin q$, q is properly contained in $Q = P \cap R$. Since $G(q, R)$ is Cohen-Macaulay, the localization $(R[qt]/qR[qt])_S$ where $S = R \setminus Q$ is also Cohen-Macaulay. One can routinely verify that

$$(R[qt]/qR[qt])_S \cong R_Q[qR_Qt]/qR_Q[qR_Qt] .$$

Furthermore, P is an associated prime of $qR_Q[qR_qt]$ since $S \cap P = \varnothing$. By the Cohen-Macaulay property on $R_Q[qR_Qt]/qR_Q[qR_Qt]$ it follows that $\dim(R_Q[qR_Qt]/qR_Q[qR_Qt]) = \dim(R_Q[qR_Qt]/P_S)$. Since $qR_Q[qR_Qt] \subset QR_Q[qR_Qt] \subset P_S$ it is therefore true that $\dim(R_Q[qR_Qt]/qR_Q[qR_Qt]) = \dim(R_Q[qR_Qt]/QR_Q[qR_Qt])$. By using the property that $\dim(G(qR_Q, R_Q)) = \dim(R_Q)$, we obtain $\operatorname{height}(Q) = \dim(R_Q) = \dim G(qR_Q, R_Q) = \dim(R_Q[qR_Qt]/QR_Q[qR_Qt]) = l(qR_Q)$. This contradiction completes the proof of Theorem 2.1.

Remark. Theorem 2.1 can actually be strengthened by assuming only that the zero ideal in $G(q, R)$ is unmixed and locally equidimensional, instead of the full Cohen-Macaulay assumption.

The following corollary yields a wide class of examples.

COROLLARY 2.2. *Let T be a regular ring and P a complete intersection prime ideal of T. Let $f \in P$, set $R = T/fT$ and let $q = P/fT$. If $l(qR_Q) <$ height Q for every prime ideal Q properly containing q, then $q^n = q^{(n)}$ for all $n \geq 1$.*

Proof. Since P is a complete intersection prime ideal of T it follows that T/P is Cohen-Macaulay and that $G(P, T)$ is a polynomial ring over T/P. Thus $G(P, T)$ is Cohen-Macaulay domain. Therefore $G(q, R) \cong G(P, T)/f'G(P, T)$ is Cohen-Macaulay and the result follows from Theorem 2.1.

For the next two corollaries we recall some notation from [13]. For R a Noetherian ring and I an ideal of R we define $A^*(I) = \{P/P$ is a prime ideal of R such that $P \in \operatorname{Ass}(R/I^n)$ for all large $n\}$ and $\bar{A}^*(I) = \{P/P$ is a prime ideal of R such that $P \in \operatorname{Ass}(R/(I^n)')$ for all large $n\}$, where for an ideal J of R, J' denotes the integral closure of J (see [13, page 3 and Chapter 3]).

COROLLARY 2.3. *Let T be a regular ring and P a complete intersection prime ideal of T. Let $f \in P$, set $R = T/fT$ and let $q = P/fT$. Then the*

q-adic and q-symbolic topologies are linearly equivalent if and only if $q^n = q^{(n)}$ for all $n \geq 1$.

Proof. If $q^n = q^{(n)}$ for all $n \geq 1$ then the two topologies are clearly linearly equivalent. If the topologies are linearly equivalent then by [17 Corollary 1] we have that $l(qR_Q) < \text{height}(Q)$ for every prime ideal $Q \in A^*(q) \setminus \{q\}$. In fact we claim that $l(qR_Q) < \text{height}\, Q$ for every prime ideal Q properly containing P. If $l(qR_Q) = \text{height}(Q)$ then $Q \in \bar{A}^*(q)$ by [13 Proposition 4.1], hence $Q \in A^*(q)$ by [13 Proposition 3.17]. Thus the only possible prime ideals Q for which $l(qR_Q) = \text{height}(Q)$ are those in $A^*(q)$ and we have observed that $l(qR_Q) < \text{height}(Q)$ for those primes. Now by Corollary 2.2 $q^n = q^{(n)}$ for all $n \geq 1$.

COROLLARY 2.4. *Let T be a regular local ring and P a complete intersection prime ideal of T with* $\text{height}(P) = \dim T - 1$. *Let $f \in P$, $R = T/fT$ and $q = P/fT$. Then* $l(q) = \text{height}(q)$ *if and only if $q^n = q^{(n)}$ for all $n \geq 1$.*

Proof. If $l(q) = \text{height}(q)$ then $q^n = q^{(n)}$ for all $n \geq 1$ by Corollary 2.2. Conversely, if $q^n = q^{(n)}$ for all $n \geq 1$ then $A^*(q) = \{q\}$, hence $\bar{A}^*(q) = \{q\}$ by [13 Proposition 3.17]. Therefore $l(q) = l(q_M) < \text{height}(M) = \dim R = \text{height}(q) + 1$ (where M is the maximal ideal of R) by [13 Proposition 4.1]. Thus $\text{height}(q) \leq l(q) < \text{height}(q) + 1$ so it follows that $\text{height}(q) = l(q)$.

We now present a few examples to illustrate the Theorem and its Corollaries. Let K denote an infinite field, and let X, Y, Z, U, V and W be indeterminates in what follows.

EXAMPLE 2.5. Let $R = K[\![X^a, X^bY^c, Y^d]\!]$ where a, b, c and d are positive integers, and let $q = (X^a, X^bY^c)R$. If $l(q) = 1$ then $q^n = q^{(n)}$ for all $n \geq 1$.

Proof. Since R is a subring of $K[\![X, Y]\!]$ over which $K[\![X, Y]\!]$ is integral, $\dim(R) = \dim(K[\![X, Y]\!]) = 2$. Thus, R is a hypersurface ring by considering $K[\![U, V, W]\!] \to R$ given by $U \to X^a$, $V \to X^bY^c$ and $W \to Y^d$. Furthermore, q is the image of $(U, V)K[\![U, V, W]\!]$ in R. By Corollary 2.4 $q^n = q^{(n)}$ or all $n \geq 1$.

We use Corollary 2.2 on the next example for the sake of application, but observe that $G(q, R)$ is actually a domain so that Theorem 2.1 could be bypassed.

EXAMPLE 2.6. Let $T = K[X, Y, Z, W]$ and $f = XY^a - ZW^b$ where a and b are positive integers. Set $R = T/fT$ and let $q = (X, Z)T/fT$. Then $q^n = q^{(n)}$ for all $n \geq 1$.

Proof. Since q is generated by two elements we must have $l(qR_M) \leq 2 < 3 = \dim(R) = \text{height}(M)$ where M is any maximal ideal of R. If Q is a height two prime ideal of R which contains q, then Q is the image of a height three prime ideal P_1 of T such that either $Y \notin P_1$ or $W \notin P_1$. If $Y \notin P_1$ then qR_Q is principal generated by the image of Z in R_Q, and if $W \notin P_1$ then qR_Q is principal generated by the image of X in R_Q. Therefore $l(qR_Q) \leq 1 < \text{height}(Q)$. By Corollary 2.2 $q^n = q^{(n)}$ for all $n \geq 1$.

It is surprising to note that Corollary 2.2 is false if R is merely a complete intersection (a homomorphic image of a regular ring by a regular sequence). The next example shows this. It is lifted from [7 Example 1.6] where is appears in more detail.

EXAMPLE 2.7. Let

$$R = K[U, V, W, Z]/(W^7 - U^{35}Z^2, V^3 - WZ)K[U, V, W, Z]$$

and let q be the image of $(U, V, W)K[U, V, W, Z]$ in R. Then R is a two-dimensional domain isomorphic to $K[X^3, X^5Y^3, X^{15}Y^2, Y^7]$, and q is a prime ideal of R satisfying $l(qR_Q) < \text{height}(Q)$ for all Q properly containing q, while $q^n \neq q^{(n)}$ for all $n \geq 2$.

Proof. See [7 Example 1.6].

Remark. For another example which achieves the same conclusion as Example 2.7, see [4 Example 1].

Another interesting problem arises upon viewing more closely the nature of the prime ideals to which Corollary 2.4 applies. Assume T is a regular local ring and P is a prime ideal of T such that $\text{height}(P) = \dim(T) - 1$. If $f \in P$ then the key hypothesis for Corollary 2.4 is that $l(P/fT) = \text{height}(P/fT) = l(P) - 1$ (that is the analytic spread behaves the same as the height and drops by exactly one). A simple example where the analytic spread drops is given by taking $R = K[\![U, V, W]\!]/(v^6 - u^9w^2)K[\![U, V, W]\!] = T/fT$, and $P = (U, V)K[\![U, V, W]\!]$. Then $l(P/fT) = 1$. On the other hand, if we let $P = (V, W)K[\![U, V, W]\!]$ then $l(P/fT) = 2$ so that the analytic spread does not drop. In order to find non complete intersection prime ideals P which satisfy the condition that

$l(P/fT) = \text{height}\,(P/fT)$ it is necessary that $l(P/fT)$ drop by more than one. We will give examples to show that this can in fact happen (Examples 2.8 and 2.10). In addition, Example 2.8 will give an application of Theorem 2.1 (see Observation 2.9), while Example 2.10 will show how Theorem 2.1 can be used to test whether or hot $G(q, R)$ is Cohen-Macaulay. The examples themselves were discovered by studying height two primes in a three-dimensional regular local ring rather than studying specific hypersurface rings. Huneke's paper [10] was quite instrumental in supplying the techniques for finding these examples. In particular Propositions 2.2 and 2.3 of [10] were very useful.

EXAMPLE 2.8. Let $T = K[\![U, V, W]\!]$, suppose t is an indeterminate, and let P be the kernel of the homomorphism $T \to K[\![t^3, t^4, t^5]\!]$ given by $U \to t^3$, $V \to t^4$, $W \to t^5$. Then there exists $f \in P$ such that $l(q) = 1$ if $q = P/fT$ (note that $l(P) = 3$ since P is not a complete intersection).

Proof. By Herzog's work (see [20, page 137]) it follows that $P = (a, b, c)T$ where a, b and c are the alternating 2×2 minors of the matrix

$$A = \begin{pmatrix} W & U^2 & V \\ V & W & U \end{pmatrix}.$$

By the Hilbert-Burch Theorem [19], a minimal resolution of P is given by

$$0 \longrightarrow T^2 \xrightarrow{A} T^3 \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} P \longrightarrow 0.$$

Therefore, the relations on a, b, c are generated by the equations

$$aW + bU^2 + cV = 0$$
$$aV + bW + cU = 0.$$

By eliminating W in the above equations one gets

$$U(b^2U - ac) = V(a^2 - bc).$$

Since $\{U, V\}$ is a T-regular sequence, it follows that there exists $f \in T$ such that

$$Uf = a^2 - bc$$
$$Vf = b^2U - ac.$$

Furthermore, by eliminating V instead of W above one can show that

$$Wf = c^2 - abU.$$

Now consider the prime ideal $q = P/fT$ in the hypersurface ring $R = T/fT$ (notice that $f \in P$ since it multiplies elements outside P into P). Let a_1, b_1 and c_1 denote the images of a, b, and c modulo (fT). Then $q^2 = b_1 q$, hence $l(q) = 1$. Explicitly, f is given by

$$\begin{aligned} f &= U^2a + UVb + Wc \\ &= U^5 - 3U^2VW + UV^3 + W^3. \end{aligned}$$

OBSERVATION 2.9. Set $R = T/fT$ and $q = P/ft$ where P, f and T are the same as those defined in Example 2.8. Then $G(q, R)$ is Cohen-Macaulay. Consequently, $q^n = q^{(n)}$ for all $n \geq 1$ by Theorem 2.1.

Proof. By the computation above we have that $l(q) = \text{height}\,(q)$ and $q^2 = b_1 q$. Furthermore R/q is a one-dimensional local domain, hence is Cohen-Macaulay. Thus by [21 Proposition 7.4], $G(q, R)$ is Cohen-Macaulay.

Remark. Observation 2.9 shows that Example 2.8 is an application of Theorem 2.1 for the case where q is not the image of a complete intersection prime ideal of T and R is the hypersurface T/fT.

Our next example shows how Theorem 2.1 may be used as a test for deciding when $G(q, R)$ is Cohen-Macaulay. It also provides an example of a prime ideal in a local hypersurface ring whose adic and symbolic topologies are linearly equivalent but whose powers are not primary.

EXAMPLE 2.10. Let $T = K[U, V, W]$ localized at $(U, V, W)K[U, V, W]$, suppose t is an indeterminate, and let P be the kernel of the homomorphism $T \to K[t^4, t^5, t^7]_{(t^4, t^5, t^7)}$. Then there exists $g \in P$ such that $l(q) = 1$ and $q^2 \neq q^{(2)}$ where $q = P/gT$ in $R = T/gT$. Consequently $G(q, R)$ is not Cohen-Macaulay by Theorem 2.1.

Proof. By using Herzog and Hilbert-Burch once again we find that $P = (a, b, c)T$ where a, b and c are the alternating 2×2 minors of the matrix

$$A = \begin{pmatrix} W & U & V \\ V^2 & W & U^2 \end{pmatrix}$$

and the relations on a, b and c are generated by

$$(1) \qquad aW + bU + cV = 0$$

$$(2) \qquad aV^2 + bW + cU^2 = 0\,.$$

By eliminating W and using that $\{U, V\}$ is a T-regular sequence (as in Example 2.8), we find an element $f \in T$ such that

$$(3) \qquad Uf = a^2V - bc$$

$$(4) \qquad Vf = b^2 - acU\,.$$

Furthermore, elimination of V yields

$$(5) \qquad Wf = c^2U - abV\,.$$

The relations produced by going modulo (fT) in (3), (4) and (5) are not enough to force $l(P/fT) = 1$, hence we construct another candidate. If we multiply (5) by c and use (1) to eliminate cV in the resulting equation, we get $Wcf = c^3U + a^2bW + ab^2U$. Thus $W(cf - a^2b) = U(c^3 + ab^2)$, and since $\{U, W\}$ is a T-regular sequence there exists $g \in T$ such that

$$(6) \qquad Ug = cf - a^2b$$

$$(7) \qquad Wg = c^3 + ab^2\,.$$

Furthermore, if we multiply (5) by b and use (1) to eliminate bU we get $Wbf = -ac^2W - c^3V - ab^2V$ which implies (using that $\{V, W\}$ is a T-regular sequence) that (8) $Vg = -bf - ac^2$. Multiplying (6) by U and using (3) yields

$$(9) \qquad U^2g = ca^2V - bc^2 - a^2bU\,.$$

Similarly we get

$$(10) \qquad UVg = b^2c - ac^2U - a^2bV$$

$$(11) \qquad V^2g = -b^3 + abcU - ac^2V\,.$$

Now let, a_1, b_1 and c_1 denote the images of a, b, and c modulo (gT), and set $q = P/gT$ in the hypersurface ring $R = T/gT$. Using the relations (7), (9), (10) and (11), it follows routinely that $q^3 = a_1q^2$. Therefore $l(q) = 1$. We have left to show that q^2 is not primary. To do this we first grade T by setting $\deg U = 4$, $\deg V = 5$ and $\deg W = 7$. Then P becomes homogeneous of degree 12 with $\deg a = 12$, $\deg b = 15$, $\deg c = 14$, $\deg f = 25$ and $\deg g = 35$. By using (3), (4) or (5) it follows that $f \in P^{(2)}$, hence

$f_1 \in q^{(2)}$ where f_1 is the image of f modulo (gT). We claim that $f_1 \notin q^2$. If $f_1 \in q^2$ then $f \in (P^2, g)$. A straight computation shows however that P^2 cannot contain any element of degree 25, thus (P^2, g) cannot contain any element of degree 25. It follows that $f_1 \notin q^2$, therefore $q^2 \neq q^{(2)}$.

Explicitly,

$$f = U^2Va + V^2b + UWc = U^5V - 3U^2V^2W + V^5 + UW^3$$

and

$$\begin{aligned} g &= Wc^2 - 2U^2V^3a + U^4Wa - UV^2Wc - V^4b \\ &= W^5 - 3UV^2W^3 + 5U^2V^4W - 2U^5V^3 + U^7W - V^7 - U^4VW^2 . \end{aligned}$$

Remark. In fact it is true that $q^n \neq q^{(n)}$ for all $n \geq 2$ in Example 2.10.

Proof. We have seen above that $f_1 \in q^{(2)} \setminus q^2$. Since $a_1 \in q$ it follows that $a_1^{n-2} f_1 \in q^{(n)}$ for all $n > 2$. Since R is Cohen-Macaulay q contains a regular element, and since $a_1 q^2 = q^3$, a_1 itself must be a regular element of R. Furthermore, $q^{m+1} = a_1 q^m = a_1^{m-1} q^2$ for all $m \geq 2$. Suppose $a_1^{n-2} f_1 \in q^n$ for some $n > 2$. Then $a_1^{n-2} f_1 \in a_1^{n-2} q^2$ which implies $f_1 \in q^2$, a contradiction.

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