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ON ζ_n-WEYL ALGEBRA $W_r(\zeta_n, Z)$

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0.

Weyl algebra is an associative algebra generated by two elements $\hat{a}$ and a over $\boldsymbol{R}$ such that the generating relation is given by

$$\hat{a}a - a\hat{a} = 1,$$

which is isomorphic to the algebra of differential operators

$$\boldsymbol{R}\left[z, \frac{d}{dz}\right].$$

q analog of Weyl algebra $W_1(q, \boldsymbol{R})$ is an associative algebra with two generators $\hat{a}$ and a such that the generating relations is

$$\hat{a}a - qa\hat{a} = 1.$$

If q is not a root of unity of finite degree, q-analog $W_1(q, \boldsymbol{R})$ is isomorphic to the algebra of q-Differential operators

$$\boldsymbol{R}[z, D_q],$$

where

$$D_q(f(z)) = \frac{f(z) - f(qz)}{z(1-q)}.$$

q-analog of Weyl algebra is sometimes called q-quatisation by physisist ([2], [3]).

Exceptional case $q = a$ primitive n-th root of unity ζ_n, $W_1(\zeta_n, Z)$ has quite beautiful properties; standard elements $\hat{a}^n$, a^n, $\hat{a}a - a\hat{a}$ play important part of role.

§1.

We mean by ζ_n a primitive n-th root of unity, and define ζ_n-analog of Weyl algebra $Z[z, d/dz]$ as follows;

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ζ_n-Weyl algebra $W_1(\zeta_n, \boldsymbol{Z})$ is a $\boldsymbol{Z}[\zeta_n]$-algebra generated by two elements $\hat{a}$ and a such that

$$\hat{a}a - \zeta_n a\hat{a} = 1 \tag{1}$$

is the generator of relations between $\hat{a}$ and a.

Putting

$$\hat{u} = \hat{a}^n\,, \quad u = a^n\,, \quad c = \hat{a}a - a\hat{a} \tag{2}$$

we shall show that i) $\boldsymbol{Z}[\zeta_n, \hat{u}, u]$ is the center of $W_1(\zeta_n, \boldsymbol{Z})$, i.e. $W_1(\zeta_n, \boldsymbol{Z})$ is a central $\boldsymbol{Z}[\zeta_n, \hat{u}, u]$-algebra generated by $\hat{a}$ and a such that i) $\hat{a}a - \zeta_n a\hat{a} = 1$, $\hat{a}^n = \hat{u}$, $a^n = u$ and $\hat{u}$ and u are independent variables, ii) $c^n = 1 - (1 - \zeta_n)^n u\hat{u}$, iii) $W_1(\zeta_n, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}[\zeta_n, \hat{u}, u]} \boldsymbol{Q}(\zeta_n, \hat{u}, u)$ is a central division algebra over $\boldsymbol{Q}(\zeta_n, \hat{u}, u)$, which is given by the factor system

$$(\boldsymbol{Q}(\zeta_n, \hat{u}, u, c)/\boldsymbol{Q}(\zeta_n, \hat{u}, u),\ a^{-1}ca = \zeta_n c,\ a^n = u)\,.$$

We shall also generalize these results to the $\boldsymbol{Z}[\zeta_n]$-algebra $W_r(\zeta_n, \boldsymbol{Z})$ generated $\hat{a}_1, \cdots, \hat{a}_r, a_1, \cdots, a_r$ such that the generators of relations are given by

$$\hat{a}_i a_i - \zeta_n a_i \hat{a}_i = 1 \qquad (1 \leqq i \leqq r)$$

$$\hat{a}_i\hat{a}_j - \hat{a}_j\hat{a}_i = \hat{a}_i a_j - a_j\hat{a}_i = a_i a_j - a_j a_i \qquad (i \neq j)\,.$$

§ 2.

Let us prove some lemmas.

LEMMA 1.

$$c = 1 - (1 - \zeta_n)a\hat{a} \tag{3}$$

$$\hat{a}c = \zeta_n c\hat{a}\,, \qquad ca = \zeta_n ac\,. \tag{4}$$

Proof. (3) is a direct consequence of (1) and (2). (3) implies (4) as follows,

$$\begin{aligned} ca - \zeta_n ac &= (1 - (1 - \zeta_n)a\hat{a})a - \zeta_n a(1 - (1 - \zeta_n)a\hat{a}) \\ &= a - (1 - \zeta_n)a(1 + \zeta_n a\hat{a}) - \zeta_n a + \zeta_n(1 - \zeta_n)a^2\hat{a} = 0\,, \\ \hat{a}c - \zeta_n c\hat{a} &= \hat{a}(1 - (1 - \zeta_n)a\hat{a}) - \zeta_n(1 - (1 - \zeta_n)a\hat{a})\hat{a} \\ &= \hat{a} - (1 - \zeta_n)(1 + \zeta_n a\hat{a})\hat{a} - \zeta_n\hat{a} + \zeta_n(1 - \zeta_n)a\hat{a}^2 = 0\,. \end{aligned}$$

LEMMA 2.

$$\hat{a}a^{\ell} = (1-\zeta_n)^{-1}(1-\zeta_n^{\ell})a^{\ell-1} + \zeta_n^{\ell}a^{\ell}\hat{a}\,, \tag{5}$$

$$\hat{a}^{\ell}a = (1-\zeta_n)^{-1}(1-\zeta_n^{\ell})\hat{a}^{\ell-1} + \zeta_n^{\ell}a\hat{a}^{\ell}\,. \tag{6}$$

Proof. For $\ell = 1$, (5) and (6) are nothing else than (1). Assuming (5) for ℓ, we have

$$\begin{aligned}\hat{a}a^{\ell+1} &= (1-\zeta_n)^{-1}(1-\zeta_n^{\ell})a^{\ell} + \zeta_n^{\ell}a^{\ell}\hat{a}a \\ &= (1-\zeta_n)^{-1}(1-\zeta_n^{\ell})a^{\ell} + \zeta_n^{\ell}a^{\ell}(1+\zeta_n a\hat{a}) \\ &= ((1-\zeta_n)^{-1}(1-\zeta_n^{\ell}) + \zeta_n^{\ell})a^{\ell} + \zeta_n^{\ell+1}a^{\ell+1}\hat{a} \\ &= (1-\zeta_n)^{-1}(1-\zeta_n^{\ell+1})a^{\ell} + \zeta_n^{\ell+1}a^{\ell+1}\hat{u}\,.\end{aligned}$$

Similarly (6) can be proved.

LEMMA 3. *$\hat{u}$ and $\hat{u}$ belong to the center of $W_1(\zeta_n, Z)$.*

Proof. From $\zeta_n^n = 1$, it follows

$$\begin{aligned}\hat{u}a &= \hat{a}^n a = (1-\zeta_n^{-1})(1-\zeta_n^n)\hat{a}^{n-1} - \zeta_n^n a\hat{a}^n \\ &= a\hat{a}^n = a\hat{u}\,, \\ \hat{a}u &= \hat{a}a^n = (1-\zeta_n^{-1})(1-\zeta_n^n)a^{n-1} + \zeta_n^n a^n\hat{a} \\ &= a^n\hat{a} = u\hat{a}\,.\end{aligned}$$

PROPOSITION 1. *$Z[\zeta_n, \hat{u}, u]$ is the center of $W_1(\zeta_n, Z)$.*

Proof. Since (1) is the generator of the relations between $\hat{a}$ and a, the set $\{a^{\ell}\hat{a}^h,\ 0 \leqq \ell,\ h \leqq n-1\}$ is a basis of $W_1(\zeta_n, Z)$ over $Z[\zeta_n, \hat{u}, u]$. From $\hat{a}c = \zeta_n c\hat{a}$ and $ca = \zeta_n ac$, it follows

$$ca^{\ell}\hat{a}^h = \zeta_n^{\ell-h}a^{\ell}\hat{a}^h c\,.$$

This means that any element in the center must be written as follows,

$$\alpha_0 + \sum_{\ell=1}^{n-1}\alpha_{\ell}a^{\ell}\hat{a}^{\ell} \qquad (\alpha_{\ell} \in Z[\zeta_n, \hat{u}, u])\,.$$

It is sufficient to prove $\alpha_{\ell} = 0$ $(1 \leqq \ell \leqq n-1)$. Assume α_{ℓ_0} be the first non-zero one in $\{\alpha_1, \cdots, \alpha_{n-1}\}$, Then we have

$$\begin{aligned}0 &= a\left(\sum_{\ell=1}^{n-1}\alpha_{\ell}a^{\ell}\hat{a}^{\ell}\right) - \left(\sum_{\ell=1}^{n-1}\alpha_{\ell}a^{\ell}\hat{a}^{\ell}\right)a \\ &= \sum_{\ell=1}^{n-1}\alpha_{\ell}a^{\ell+1}\hat{a} - \sum_{\ell=1}^{n-1}\alpha_{\ell}\{(1 \quad \zeta_n)^{-1}(1 \quad \zeta_n^{\ell})a^{\ell}\hat{a}^{\ell-1}\zeta_n^{\ell}a^{\ell+1}\hat{a}^{\ell} \\ &= -\alpha_{\ell_0}(1-\zeta_n)^{-1}(1-\zeta_n^{\ell})a^{\ell_0}\hat{a}^{\ell_0-1} + \sum_{\ell=\ell_0-1}^{n-1}\beta_{\ell}a^{\ell-1}\hat{a}^{\ell} + \beta_{n-1}u\hat{a}^{n-1}\end{aligned}$$

with $\beta_1, \cdots, \beta_{n-1} \in Z[\zeta_n, \hat{u}, u]$. This means $\alpha_{\ell_0} = 0$, therefore $\alpha_\ell = 0$ ($1 \leqq \ell \leqq n-1$).

LEMMA 4.

$$(7) \qquad (a\hat{a})^\ell = \zeta_n^{\ell(\ell-1)/2} a^\ell \hat{a}^\ell + \sum_{h=1}^{\ell-1} \alpha_{\ell,h} a^h \hat{a}^h$$

with $\alpha_{\ell,h} \in Z[\zeta_n, \hat{u}, u]$.

Proof. For $\ell = 1$ (7) is the identity $a\hat{a} = a\hat{a}$. Assuming (7) for ℓ, we have

$$\begin{aligned}(a\hat{a})^{\ell+1} &= \zeta_n^{\ell(\ell-1)/2} a^\ell \hat{a}^\ell a\hat{a} + \sum_{h-1}^{\ell-1} \alpha_{\ell,h} a^h \hat{a}^h a\hat{a} \\ &= \zeta^{\ell(\ell-1)/2} a^\ell \{(1-\zeta_n)^{-1}(1-\zeta_n^\ell)\hat{a}^{\ell-1} + \zeta_n^\ell a\hat{a}^\ell\}\hat{a} \\ &\quad + \sum_{h=1}^{\ell-1} \alpha_{\ell,h} a^h \{(1-\zeta_n)^{-1}(1-\zeta_n^h)\hat{a}^{h-1} + \zeta_n^h a\hat{a}^h\}\hat{a} \\ &= (\zeta_n^{\ell(\ell-1)/2}\zeta_n^\ell) a^{\ell+1}\hat{a}^{\ell+1} + \sum_{h=1}^{\ell} \alpha_{\ell+1,h} a^h \hat{a}^h \\ &= \zeta_n^{\ell(\ell-1)/2} a^{\ell+1}\hat{a}^{\ell+1} + \sum_{h=1}^{\ell} \alpha_{\ell+1,h} a^h \hat{a}^h\end{aligned}$$

with $\alpha_{\ell+1,1}, \cdots, \alpha_{\ell+1,\ell}$ in $Z[\zeta_n, \hat{u}, u]$.

PROPOSITION 2.

$$(8) \qquad c^n = 1 - (1-\zeta_n)^n u\hat{u}\,.$$

Proof. From $\zeta_n^n = 1$ it follows $\hat{a}c^n = c^n\hat{a}$ and $ac^n = c^n a$, i.e. c^n belongs to the center $Z[\zeta_n, \hat{u}, u]$. By virtue of (3) and (7)

$$\begin{aligned}c^n &= (1-(1-\zeta_n)a\hat{a})^n = 1 + (-1)^n(1-\zeta_n)^n(a\hat{a})^n \\ &\quad + \sum_{\ell=1}^{n-1}(-1)^\ell\binom{n}{\ell}(1-\zeta_n)^\ell(a\hat{a})^\ell \\ &= 1 + (-1)^n(1-\zeta_n)^n\zeta_n^{n(n-1)/2}a^n\hat{a}^n + \sum_{\ell=1}^{n-1}\gamma_\ell a^\ell\hat{a}^\ell\,.\end{aligned}$$

Since c^n belongs to $Z[\zeta_n, \hat{u}, u]$,

$$c^n = 1 + (1-\zeta_n)^n(-1)^n\zeta_n^{n(n-1)/2}u\hat{u} = 1 - (1-\zeta_n)^n u\hat{u}\,.$$

§3.

We mean by $Q(\zeta_n, \hat{u}, u)$ and $Q(\zeta_n, \hat{u}, u, c)$ the quotient fields of $Z[\zeta_n, \hat{u}, u]$ and $Z[\zeta_n, \hat{u}, u, c]$, respectively.

THEOREM 1. *$W_1(\zeta_n, Z) \otimes_{Z[\zeta_n,\hat{u},u]} Q(\zeta_n, \hat{u}, u)$ is a central division algebra over $Q(\zeta_n, \hat{u}, u)$, which is given by the n-cyclic factor system.*

$$(9)\qquad (\boldsymbol{Q}(\zeta_n, \hat{u}, u, c)/\boldsymbol{Q}(\zeta_n, \hat{u}, u), aca^{-1} = c^\sigma, a^n = u),$$

where $\boldsymbol{Q}(\zeta_n, \hat{u}, u, c)/\boldsymbol{Q}(\zeta_n, \hat{u}, u)$ *is the Kummer extension with galois group* $\langle \sigma \mid \sigma^n = 1\rangle$ *such that* $c^\sigma = \zeta_n c$.

Proof. From (4) and (8)

$$W_1(\zeta_n, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}[\zeta_n, \hat{u}, u]} \boldsymbol{Q}(\zeta_n, \hat{u}, u)$$

is given by the factor system (9), and thus it is a central simple $\boldsymbol{Q}(\zeta_n, \hat{u}, n)$-algebra. On the other hand $\hat{u}$ and u are independent variables over $\boldsymbol{Q}(\zeta_n)$, and thus u is not a norm from any subfield of $\boldsymbol{Q}(\zeta_n, \hat{u}, u, \sqrt[n]{1-(1-\zeta_n)^n u\hat{u}})$ to $\boldsymbol{Q}(\zeta_n, \hat{u}, u)$. This means that the algebra is a division algebra.

EXAMPLE. — 1-Weyl algebra $W_1(-1, \boldsymbol{Z})$ is a $\boldsymbol{Z}[\hat{u}, u]$-algebra generated by two elements $\hat{a}$ and a such that

$$(10)\qquad \hat{a}a + a\hat{a} = 1, \quad \hat{a}^2 = \hat{u}, \quad a^2 = u,$$

where $\hat{u}$ and u are independent commutative variables over $\boldsymbol{Z}$.

PROPOSITION 3. $W_1(-1, \boldsymbol{Z})$ *is isomorphic to the* $\boldsymbol{Z}$*-algebra generated by two* 2×2*-matrices*

$$(11)\qquad \rho(\hat{a}) = \begin{pmatrix} 0 & \dfrac{1+\sqrt{1-4u\hat{u}}}{2} \\ \dfrac{1-\sqrt{1-4u\hat{u}}}{2u} & 0 \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix},$$

where $\hat{u}$ *and* u *are independent commutative variables over* $\boldsymbol{Z}$.

Proof. By calculation we have

$$\rho(\hat{a})\rho(a) + \rho(a)\rho(\hat{a}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho(\hat{a})^2 = \begin{pmatrix} \hat{u} & 0 \\ 0 & \hat{u} \end{pmatrix}, \quad \rho(a)^2 = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.$$

From $(\hat{a}a - a\hat{a})^2 = 1 - 4u\hat{u}$, it follows that

$$W_1(-1, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}[\hat{u}, u]} \boldsymbol{Q}(\hat{u}, u)$$

is given by the factor system

$$\boldsymbol{Q}(\hat{u}, u\sqrt{1-4u\hat{u}})/\boldsymbol{Q}(\hat{u}, u), a^{-1}\sqrt{1-4u\hat{u}}\,a = -\sqrt{1-4u\hat{u}}, a^2 = u).$$

§ 4.

For a natural number r, ζ_n-Weyl algebra $W_r(\zeta_n, \boldsymbol{Z})$ is defined as a $\boldsymbol{Z}[\zeta_n]$-algebra generated by $\hat{a}_1, \cdots, \hat{a}_r, a_1, \cdots, a_r$ such that

$$\hat{a}_i a_i - \zeta_n a_i \hat{a}_i = 1 \qquad (1 \leqq i \leqq r) \tag{12}$$

$$\hat{a}_i \hat{a}_j - \hat{a}_j \hat{a}_i = \hat{a}_i a_j - a_j \hat{a}_i = a_i a_j - a_j a_i = 0 \qquad (i \neq j) \tag{13}$$

are the generators of relations between $\hat{a}_1, \cdots, \hat{a}_r, a_1, \cdots, a_r$.

PROPOSITION 4. *Put* $\hat{a}_i^n = \hat{u}_i$, $a_i^n = u_i$ $(1 \leqq i \leqq r)$. *Then* $\boldsymbol{Z}[\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r]$ *is the center of* $W_r(\zeta_n, \boldsymbol{Z})$.

This is proved similarly as for $W_1(\zeta_n, \boldsymbol{Z})$.

PROPOSITION 5. *Denoting* $c_i = \hat{a}_i a_i - a_i \hat{a}_i$ $(1 \leqq i \leqq r)$, *we have*

$$\begin{cases} c_i = 1 - (1 - \zeta_n) a_i \hat{a}_i \\ \hat{a}_i c_i = \zeta_n c_i \hat{a}_i\,, \qquad c_i a_i = \zeta_n a_i c_i \\ c_i^n = 1 - (1 - \zeta_n)^n u_i \hat{u}_i \end{cases} \tag{14}$$

$$c_i \hat{a}_j = \hat{a}_j c_i\,, \quad c_i a_j = a_j c_i \quad (i \neq j)\,. \tag{15}$$

Proof. (14) is proved similarly as for $W_1(\zeta_n, \boldsymbol{Z})$ and (15) is a direct consequence of the relation (13).

Similarly as $W_1(\zeta_n, \boldsymbol{Z})$, we have

PROPOSITION 6. $W_r(\zeta_n, \boldsymbol{Z})$ *is a central* $\boldsymbol{Z}[\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r]$-*algebra generated by* $\hat{a}_1, \cdots, \hat{a}_r, a_1, \cdots, a_r$ *such that*

$$\begin{aligned} &\hat{a}_i a_i - \zeta_n a_i \hat{a}_i = 1 && (1 \leqq i \leqq r)\,, \\ &\hat{a}_i \hat{a}_j - \hat{a}_j \hat{a}_i = \hat{a}_i a_j - a_j \hat{a}_i = a_i a_j - a_j a_i = 0 && (i \neq j)\,, \\ &\hat{a}_i^n = \hat{u}_i\,, \quad a_i^n = u_i && (1 \leqq i \leqq r)\,, \end{aligned}$$

where $\hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r$ *are independent commutative variables over* $\boldsymbol{Z}$.

Similarly as $W_1(\zeta_n, \boldsymbol{Z})$, we have

THEOREM 2. $W_r(\zeta_n, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}[\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r]} \boldsymbol{Q}(\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r)$ *is a central division algebra over* $\boldsymbol{Q}(\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r)$, *which is given by the factor system*

$$\begin{gathered}(\boldsymbol{Q}(\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r, c_1, \cdots, c_r)/\boldsymbol{Q}(\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1 \cdots, u_r)\,; \\ a_1^{-1} c_1 a_1 = c_1^{\sigma_1}, \cdots, a_r^{-1} c_r a_r = c_r^{\sigma_r}; a_1^n = u_1, \cdots, a_r^n = u_r)\,,\end{gathered}$$

where $\boldsymbol{Q}(\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r, c_1, \cdots, c_r)/\boldsymbol{Q}(\zeta_n, \hat{u}_1, \cdots, \hat{u}_r, u_1, \cdots, u_r)$ *is the Kummer extension with galois group* $\langle \sigma_1, \cdots, \sigma_r \mid \sigma_1^n = \cdots = \sigma_r^n = 1 \rangle$ *such that* $c_1^{\sigma_1} = \zeta_n c_1, \cdots, c_r^{\sigma_r} = \zeta_n c_r, c_i^{\sigma_j} = c_i \ (i \neq j)$.

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