V. Barucci
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A LIPMAN'S TYPE CONSTRUCTION, GLUEINGS AND COMPLETE INTEGRAL CLOSURE

VALENTINA BARUCCI

§ 0. Introduction

Given a semilocal 1-dimensional Cohen-Macauly ring A, J. Lipman in [10] gives an algorithm to obtain the integral closure $\bar{A}$ of A, in terms of prime ideals of A. More precisely, he shows that there exists a sequence of rings $A = A_0 \subset A_1 \subset \cdots \subset A_i \subset \cdots$, where, for each i, $i \geq 0$, A_{i+1} is the ring obtained from A_i by "blowing-up" the Jacobson radical $\mathscr{R}_i$ of A_i, i.e. $A_{i+1} = \bigcup_n (\mathscr{R}_i^n : \mathscr{R}_i^n)$. It turns out that $\bigcup\{A_i;\, i \geq 0\} = \bar{A}$ (cf. [10, proof of Theorem 4.6]) and, if $\bar{A}$ is a finitely generated A-module, the sequence $\{A_i;\, i \geq 0\}$ is stationary for some m and $A_m = \bar{A}$, so that

$$(+) \qquad A = A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_m = \bar{A}\,.$$

In [15] G. Tamone studies when in the Lipman's sequence (+) A_i is a "glueing of primary ideals of A_{i+1} over a prime ideal of A" (see [14] for definition). She shows in particular that A_i is not always a glueing of primary ideals of A_{i+1}.

In this paper we give an algorithmic construction, for a Noetherian domain A of any dimension, such that $\bar{A}$ is a finitely generated A-module, defining a new sequence $\{A_i;\, i \geq 0\}$ of overrings of A; A_{i+1} is obtained from A_i, taking the dual of a distinguished radical ideal of A_i. We show that such a sequence is stationary for some m, $A_m = \bar{A}$ (cf. Theorem 1.8), and A_i is always a glueing of primary ideals of A_{i+1} (cf. Proposition 2.7 and Remark 2.2, a)).

A similar sequence was considered in [17] by K. Yoshida in the case of a Noetherian ring satisfying the S_1-condition. As a matter of fact, the intermediate rings of the Yoshida sequence are defined in a rather different way, but the prime ideals occuring in their definition are linked to those that we use in our sequence (cf. for more details Remark 1.7).

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However our result holds in a more general situation which turns out to be its natural context, that is A is just a Mori domain. We recall that a Mori domain is a domain such that the ascending chain condition holds for integral divisorial ideals (e.g. Noetherian and Krull domains are Mori; for other examples and further properties of these domains cf. [11, 12, 13, 2, 4]). In this case the sequence of overrings of A is stationary at A^*, the complete integral closure of A (for a Noetherian domain, it coincides with $\bar{A}$, the integral closure of A).

In Section 2 we study the general procedure in order to descend along the sequence $\{A_i;\ i \geq 0\}$ constructed above. This procedure consists in a "contraction of ideals of A_{i+1} over prime ideals of A_i" (cf. Definition 2.1), that, in the Noetherian case, coincides with the glueing of primary ideals, as defined by G. Tamone in [14].

With the additional hypothesis that in our sequence $\{A_i;\ i \geq 0\}$ the conductor of A_i in A_{i+1} is a radical ideal of A_{i+1}, for each i (cf. Section 3), we show that the "contraction" coincides exactly with the glueing (of prime ideals), as defined by F. Ischebeck in [9]. Under this particular hypothesis, in the Noetherian case, we get a new characterization of seminormal domains (cf. Theorem 3.8); an analogous characterization, involving conductor ideals, was given by K. Yoshida, using his sequence (cf. [17, Theorem 2.2]). On the other hand, if the domain A is not Noetherian, but Mori, we obtain a natural extension of the notion of seminormal domain (not in the integral closure but) in its complete integral closure: similarly to Traverso's result for Noetherian seminormal rings, (cf. [16, Theorem 2.1]) such a domain A is obtained from its complete integral closure A^* (that is a Krull domain) with a finite number of glueings over prime ideals of A of a certain type (cf. Corollary 3.7). The paper ends with some examples of Mori, non-Noetherian domains of this kind.

Throughout the paper, if $\mathfrak{J}$ is an ideal of an integral domain A, we denote, as usual, $A:(A:\mathfrak{J})$ by $\mathfrak{J}_v$. An ideal $\mathfrak{J}$ is called *divisorial* if $\mathfrak{J} = \mathfrak{J}_v$, *strong* if $(A:\mathfrak{J}) = (\mathfrak{J}:\mathfrak{J})$ (cf. [3]), *strongly divisorial* if it is strong and divisorial (cf. [11]).

§ 1. The algorithmic construction

We begin by showing that any non-zero intersection of strongly divisorial prime ideals is a strongly divisorial ideal. We need the following:

LEMMA 1.1. *Let $\mathfrak{P}$ be a prime ideal containing a radical ideal $\mathfrak{J}$ of an integral domain A. Then* $(\mathfrak{P}:\mathfrak{P}) \subset (\mathfrak{J}:\mathfrak{J})$.

Proof. Let $\mathfrak{J} = \cap\{\mathfrak{P}_\lambda;\ \lambda \in \Lambda\}$, where, for each λ, $\mathfrak{P}_\lambda$ is a minimal prime of $\mathfrak{J}$. Since $\mathfrak{J} \subset \mathfrak{P}$, we have $\mathfrak{J}(\mathfrak{P}:\mathfrak{P}) \subset \mathfrak{P}$. But, for each $\mathfrak{P}_\lambda$, we have $\mathfrak{J}(\mathfrak{P}:\mathfrak{P}) \subset \mathfrak{P}_\lambda(\mathfrak{P}:\mathfrak{P}) \subset \mathfrak{P}_\lambda(A:\mathfrak{P}) \subset (\mathfrak{P}_\lambda:\mathfrak{P})$. Notice that, for each $\mathfrak{P}_\lambda$ with $\mathfrak{P}_\lambda \neq \mathfrak{P}$, we have $(\mathfrak{P}_\lambda:\mathfrak{P}) \cap A = \mathfrak{P}_\lambda$, because if $x \in A$ and $x\mathfrak{P} \subset \mathfrak{P}_\lambda$, then, since $\mathfrak{P} \not\subset \mathfrak{P}_\lambda$, $x \in \mathfrak{P}_\lambda$. Thus we have $\mathfrak{J}(\mathfrak{P}:\mathfrak{P}) \subset \mathfrak{P} \cap \{(\mathfrak{P}_\lambda:\mathfrak{P});\ \mathfrak{P}_\lambda \neq \mathfrak{P}\} \subset \mathfrak{P} \cap \{\mathfrak{P}_\lambda;\ \mathfrak{P}_\lambda \neq \mathfrak{P}\} = \mathfrak{J}$, that is $(\mathfrak{P}:\mathfrak{P}) \subset (\mathfrak{J}:\mathfrak{J})$.

PROPOSITION 1.2. *Let $\mathfrak{J} = \cap\{\mathfrak{P}_\lambda;\ \lambda \in \Lambda\}$, where for each $\lambda \in \Lambda$, $\mathfrak{P}_\lambda$ is a strongly divisorial prime ideal of an integral domain A. If $\mathfrak{J} \neq (0)$, then $\mathfrak{J}$ is a strongly divisorial ideal of A.*

Proof. It is enough to show that $\mathfrak{J} = A:(\mathfrak{J}:\mathfrak{J})$ (cf. [3, Proposition 6]). It is obvious that $\mathfrak{J} \subset A:(\mathfrak{J}:\mathfrak{J})$. For the opposite inclusion, since, by Lemma 1.1, $(\mathfrak{P}_\lambda:\mathfrak{P}_\lambda) \subset (\mathfrak{J}:\mathfrak{J})$ for each $\lambda \in \Lambda$, we have $\mathfrak{P}_\lambda = A:(A:\mathfrak{P}_\lambda) = A:(\mathfrak{P}_\lambda:\mathfrak{P}_\lambda) \supset A:(\mathfrak{J}:\mathfrak{J})$. Thus $\cap\{\mathfrak{P}_\lambda;\ \lambda \in \Lambda\} = \mathfrak{J} \supset A:(\mathfrak{J}:\mathfrak{J})$.

For a Mori domain, a "converse" for Proposition 1.2 holds:

PROPOSITION 1.3. *Let A be a Mori domain and let $\mathfrak{J}$ be a strongly divisorial ideal of A. If $\mathfrak{P}$ is a prime ideal minimal over $\mathfrak{J}$, then $\mathfrak{P}$ is strongly divisorial.*

Proof. Consider the localization $A_\mathfrak{P}$. Since $(\mathfrak{J}A_\mathfrak{P})_v = \mathfrak{J}_v A_\mathfrak{P} = \mathfrak{J}A_\mathfrak{P}$ and $(A_\mathfrak{P}:\mathfrak{J}A_\mathfrak{P}) = A_\mathfrak{P}(A:\mathfrak{J}) = A_\mathfrak{P}(\mathfrak{J}:\mathfrak{J}) = (\mathfrak{J}A_\mathfrak{P}:\mathfrak{J}A_\mathfrak{P})$ (cf. for example [11], proof of Theorem 2), $\mathfrak{J}A_\mathfrak{P}$ is a strongly divisorial ideal of $A_\mathfrak{P}$. Therefore $\mathfrak{J}A_\mathfrak{P}$ is contained in at least one strong maximal divisorial ideal of $A_\mathfrak{P}$ (cf. [5, Proposition (1.7)]), that is $\mathfrak{P}A_\mathfrak{P}$ is strongly divisorial. By [11, Lemma 4], we conclude that $\mathfrak{P}$ is a strongly divisorial ideal of A.

As usual, we denote by A^* the complete integral closure of A. We consider in the following results mainly the case where the conductor of A in A^*, $(A:A^*)$ is different from (0). This hypothesis is equivalent for a Noetherian domain A to suppose that the integral closure of A, $\bar{A} = A^*$ is a finitely generated A-module.

LEMMA 1.4. *Let A be a Mori domain such that $(A:A^*) \neq 0$. Then any decreasing chain of strongly divisorial ideals of A is stationary.*

Proof. Let $\{\mathfrak{J}_n;\ n \geq 0\}$ be a strictly decreasing chain of strongly divisorial ideals of A. Since A is a Mori domain, $\cap\{\mathfrak{J}_n;\ n \geq 0\} = (0)$ (cf. [12, I, Theorem 1]). On the other hand, since $(A: A^*) \neq (0)$, $\cap\{\mathfrak{J}_n; n \geq 0\} \neq (0)$ (cf. [3, Proposition 16]), a contradiction.

We denote, as in [4] by $D_m(A)$ the set of maximal divisorial ideals of a Mori domain A. The elements of $D_m(A)$ are prime ideals of A and, if $\mathfrak{P} \in D_m(A)$, either $A_{\mathfrak{P}}$ is a DVR or $\mathfrak{P}$ is strong, i.e. strongly divisorial (cf. [4, Proposition (2.1) and Theorem (2.5)]). The set $\mathscr{S}(A) = \{\mathfrak{P} \in D_m(A) \mid \mathfrak{P}$ is strong$\}$ is nearly related to A^*, as we shall see later. At the moment we prove:

PROPOSITION 1.5. *Let A be a Mori domain such that $(A: A^*) \neq (0)$. Then $\mathscr{S}(A)$ is empty or finite.*

Proof. The first case, $\mathscr{S}(A) = \varnothing$, occurs if and only if A is a Krull domain. In fact, if A is a Krull domain, it is well known that $A_{\mathfrak{P}}$ is a DVR, for each $\mathfrak{P} \in D_m(A)$ and, conversely, if $\mathscr{S}(A) = \varnothing$, A is a Krull domain (cf. [4, Theorem (3.3)]). Suppose that $\mathscr{S}(A)$ is non empty. If $\mathscr{S}(A)$ is not finite, consider a countable set $\{\mathfrak{P}_1, \cdots \mathfrak{P}_n, \cdots\}$ of elements of $\mathscr{S}(A)$, with $\mathfrak{P}_i \neq \mathfrak{P}_j$, for $i \neq j$. We can consider the decreasing chain $\{\mathfrak{J}_n; n \geq 1\}$, where $\mathfrak{J}_n = \cap\{\mathfrak{P}_i; 1 \leq i \leq n\}$. For each n, $\mathfrak{J}_n$ is a strongly divisorial ideal by Proposition 1.2. Moreover the chain $\{\mathfrak{J}_n; n \geq 1\}$ is strictly decreasing because, if $\mathfrak{J}_n = \mathfrak{J}_{n+1}$, then $\mathfrak{P}_1 \cdots \mathfrak{P}_n \subset \mathfrak{J}_n = \mathfrak{J}_{n+1} \subset \mathfrak{P}_{n+1}$, thus $\mathfrak{P}_i \subset \mathfrak{P}_{n+1}$ for some i, $1 \leq i \leq n$, which is clearly impossible. By Lemma 1.4 we get a contradiction.

COROLLARY 1.6. *Let A be a Mori domain such that $(A: A^*) \neq (0)$. Then the set of strongly divisorial prime ideals of A is empty or finite.*

Proof. Let $\mathscr{P}$ be the set of strongly divisorial prime ideals of A. $\mathscr{P} = \varnothing$ if and only if A is a Krull domain (cf. [3, Corollary 14]). If $\mathscr{P} \neq \varnothing$, notice that the set of the maximal elements of $\mathscr{P}$ is exactly $\mathscr{S}(A)$. In fact, trivially, if $\mathfrak{P} \in \mathscr{S}(A)$, $\mathfrak{P}$ is a maximal element of $\mathscr{P}$. Conversely, let $\mathfrak{P}$ be a maximal element of $\mathscr{P}$. Since $\mathfrak{P}$ is divisorial, $\mathfrak{P} \subset \mathfrak{M}$ for some $\mathfrak{M} \in D_m(A)$. But $\mathfrak{P}A_{\mathfrak{M}}$ is a strongly divisorial ideal of $A_{\mathfrak{M}}$, thus $A_{\mathfrak{M}}$ is not a DVR and $\mathfrak{M} \in \mathscr{S}(A) \subset \mathscr{P}$. For the maximality of $\mathfrak{P}$, $\mathfrak{P} = \mathfrak{M} \in \mathscr{S}(A)$. Therefore, by Proposition 1.5, the maximal elements of $\mathscr{P}$ are a finite number: $\mathfrak{P}_1, \cdots, \mathfrak{P}_s$. Arguing as in the proof of Proposition 1.5, we can show that $\mathscr{P} \setminus \{\mathfrak{P}_1, \cdots, \mathfrak{P}_s\}$ has a finite number of maximal elements $\mathfrak{P}'_1, \cdots, \mathfrak{P}'_t$ and trivially, for each i, $1 \leq i \leq t$, $\mathfrak{P}'_i \subsetneq \mathfrak{P}_j$ for some j, $1 \leq j \leq s$. To conclude the proof it is enough to observe that any decreasing chain of elements of $\mathscr{P}$ is finite (cf. Lemma 1.4).

REMARK 1.7. Let A be a Noetherian ring satisfying the S_1-condition and let R, $R \subset \overline{A}$, be a finite overring of A. In this case K. Yoshida [17] considers a sequence of intermediate rings between A and R (related to a sequence that we are going to introduce) and a set of distinguished prime ideals of A, $D(A, R)$ (cf. [17, Proposition-Definition 1.1]). We notice that, if A is a Noetherian domain and $R = \overline{A}$, the set $D(A, \overline{A})$ of [17] coincides with the set of strongly divisorial prime ideals of A.

In fact, if $\mathfrak{P} \in \operatorname{Spec} A$ and $\operatorname{ht} P = 1$, then $\mathfrak{P} \in D(A, \overline{A})$ if only if $A_{\mathfrak{P}}$ is not integrally closed (cf. [17, p. 54]), i.e. if and only if $\mathfrak{P}A_{\mathfrak{P}}$ is not principal (cf. for example [1, Proposition 9.2]). It is easy to prove that the previous statement is equivalent to assume that $\mathfrak{P}$ is a strong ideal of A. Since in this case ($\operatorname{ht} \mathfrak{P} = 1$) $\mathfrak{P}$ is always divisorial (cf. for example [11, Proposition 1]), we have that $\mathfrak{P} \in D(A, \overline{A})$ if and only if $\mathfrak{P}$ is strongly divisorial. On the other hand, if $\mathfrak{P} \in \operatorname{Spec} A$ and $\operatorname{ht} \mathfrak{P} > 1$, then $\mathfrak{P} \in D(A, \overline{A})$ if and only if $\mathfrak{P}$ is divisorial (cf. [17, Proposition 1.10, (vi) $\Leftrightarrow$ (xi)]). Since in this case ($\operatorname{ht} \mathfrak{P} > 1$) $\mathfrak{P}$ is always strong (if not $\mathfrak{P}A_{\mathfrak{P}}$ would be a principal ideal of the Mori domain $A_{\mathfrak{P}}$, a contradiction with [11, Lemma 3]), we have that $\mathfrak{P} \in D(A, \overline{A})$ if and only if $\mathfrak{P}$ is strongly divisorial.

We notice in particular that Corollary 1.6 generalizes Yoshida's result on the finiteness of the set $\{\mathfrak{P} \in \operatorname{Spec} A \mid \operatorname{ht} \mathfrak{P} > 1 \text{ and depth } A_{\mathfrak{P}} = 1\}$ (cf. [17, Proposition 1.10 and Corollary 1.12]).

We recall that if A is a Mori domain and $\mathfrak{J}$ is a strongly divisorial ideal of A, then $(A\colon \mathfrak{J}) = (\mathfrak{J}\colon \mathfrak{J})$ is a Mori overring of A (cf. [13, p. 11] or [3, Corollary 11]). If, moreover, A is a Mori domain such that $(A\colon A^*) \neq (0)$, then also $(A\colon \mathfrak{J})$ has the same property, that is $((A\colon \mathfrak{J})\colon (A\colon \mathfrak{J})^*) \neq (0)$, because $(A\colon \mathfrak{J})^* = A^*$. Thus, under the preceding hypothesis, we can construct a sequence of Mori overrings of A

$$A = A_0 \subset A_1 \subset \cdots \subset A_m \subset \cdots$$

setting for each $i \geq 0$, $A_{i+1} = (A_i\colon \mathscr{R}_i)$, where $\mathscr{R}_i = \cap\{\mathfrak{P};\, \mathfrak{P} \in \mathscr{S}(A_i)\}$, if $\mathscr{S}(A_i) \neq \varnothing$ and $A_{i+1} = A_i$, if $\mathscr{S}(A_i) = \varnothing$.

Notice that, in the first case, $\mathscr{R}_i \neq (0)$, by Proposition 1.5, and that $\mathscr{R}_i$ is a strongly divisorial ideal of A_i, by Proposition 1.2; thus, if $\mathscr{S}(A_i) \neq \varnothing$, $A_i \subsetneqq A_{i+1}$. Conversely, if $\mathscr{S}(A_i) = \varnothing$, $A_i = A_j$, for each $j \geq i$.

THEOREM 1.8. *Let A be a Mori domain such that* $(A\colon A_*) \neq (0)$. *Then* *the sequence of overrings of A considered above is stationary for some $m \geq 0$ and $A_m = A^*$.*

Proof. For any i, $i \geq 0$ it is easy to see that A_i is an overring of the type $\mathfrak{J}_i^{-1}$ for some ideal $\mathfrak{J}_i$ of A, that is A_i is a (fractional) divisorial ideal of A. In correspondence with the sequence $\{A_i;\ i \geq 0\}$ of overrings of A, we get the decreasing sequence of strongly divisorial ideals of A, $\{(A\colon A_i);\ i \geq 0\}$. This is stationary by Lemma 1.4, thus the sequence of overrings $\{A_i;\ i \geq 0\}$ is stationary too (cf. [3, Corollary 8]).

Therefore there exists an $m \geq 0$ such that $A_m = A_{m+1}$. Thus $\mathscr{S}(A_m) = \emptyset$ i.e. A_m is a Krull domain (cf. [4, Theorem (3.3)]). However $A^* = (A_m)^*$, because $(A\colon A_m) \neq (0)$ i.e. A and A_m have a nonzero ideal in common. On the other hand A_m is completely integrally closed, that is $(A_m)^* = A_m$, thus $A^* = A_m$.

EXAMPLES 1.9. a) Let $A = k[\![t^3, t^5]\!]$, where k is a field. A is a 1-dimensional Noetherian (in particular Mori) local domain and its maximal ideal $\mathscr{M} = (t^3, t^5)$ is strongly divisorial. In this case $\mathscr{R}_0 = \mathfrak{M}$ and $A_1 = (A\colon \mathscr{R}_0) = k[\![t^3, t^5, t^7]\!]$; $\mathscr{R}_1 = (t^3, t^5, t^7)$ and $A_2 = (A_1\colon \mathscr{R}_2) = k[\![t^2, t^3]\!]$; $\mathscr{R}_2 = (t^2, t^3)$ and $A_3 = (A_2\colon \mathscr{R}_2) = k[\![t]\!]$.

Observe that in this example our sequence of overrings of A is different from the sequence constructed by J. Lipman (cf. [10, p. 661]). As a matter of fact, in this case the steps in the Lipman sequence are $k[\![t^3, t^5]\!] \subset k[\![t^2, t^3]\!] \subset k[\![t]\!]$.

b) Let $A = k + XK[X] + YK[X, Y, Z]$, where $k \subsetneqq K$ are fields. A is a Mori (possibly non-Noetherian) domain, because $A = K[X, Y, Z] \cap B_1 \cap B_2$ where $B_1 = k + (X, Y, Z)K[X, Y, Z]_{(X,Y,Z)}$ and $B_2 = K(X) + YK[X, Y, Z]_{(Y)}$ are Mori domains (cf. [12, I, Theorem 2] and [2, Proposition 3.4]). In this case $\mathscr{R}_0 = XK[X] + YK[X, Y, Z]$, $A_1 = (A\colon \mathscr{R}_0) = K[X] + YK[X, Y, Z]$, $\mathscr{R}_1 = YK[X, Y, Z]$ and finally $A_2 = (A_1\colon \mathscr{R}_1) = K[X, Y, Z]$.

We recall that if A is a domain, $\mathfrak{J}$ is a strongly divisorial ideal of A and $C = (A\colon \mathfrak{J})$, then Spec A and Spec C are closely related. More precisely the canonical map associated to the inclusion $i\colon A \to C$, ${}^a i\colon \operatorname{Spec} C \to \operatorname{Spec} A$ gives a one-to-one correspondence between $\{\mathfrak{Q} \in \operatorname{Spec} C \mid \mathfrak{Q} \not\supset \mathfrak{J}\}$ and $\{\mathfrak{P} \in \operatorname{Spec} A \mid \mathfrak{P} \not\supset \mathfrak{J}\}$; moreover, if $\mathfrak{Q} \in \operatorname{Spec} C$, $\mathfrak{Q} \not\supset \mathfrak{J}$ and $\mathfrak{P} = \mathfrak{Q} \cap A$, then $C_{\mathfrak{Q}} = A_{\mathfrak{P}}$ (cf. for instance [7, Theorem 1.4, c)]). We notice also that for any $\mathfrak{P} \in \operatorname{Spec} A$, $\mathfrak{P} \not\supset \mathfrak{J}$, the unique $\mathfrak{Q} \in \operatorname{Spec} C$ above $\mathfrak{P}$ is $(\mathfrak{P}\colon \mathfrak{J})$. Actually $(\mathfrak{P}\colon \mathfrak{J})$ is a prime ideal of C, because if $ab \in (\mathfrak{P}\colon \mathfrak{J})$ and $a \notin (\mathfrak{P}\colon \mathfrak{J})$, with $a, b \in C = (A : \mathfrak{J})$, then $ab \in (\mathfrak{P} : \mathfrak{J}^2)$ i.e. $a\mathfrak{J}b\mathfrak{J} \subset \mathfrak{P}$, so, since $a\mathfrak{J} \subset A$, $b\mathfrak{J} \subset A$ and $a\mathfrak{J} \not\subset \mathfrak{P}$, we have $b\mathfrak{J} \subset \mathfrak{P}$, that is $b \in (\mathfrak{P} : \mathfrak{J})$. Moreover $(\mathfrak{P} : \mathfrak{J}) \cap A = \mathfrak{P}$, because if $x \in A$ is such that $x\mathfrak{J} \subset \mathfrak{P}$, then, since $\mathfrak{J} \not\subset \mathfrak{P}$, $x \in \mathfrak{P}$, and, on the other hand, it is trivial that $\mathfrak{P} \subset (\mathfrak{P} : \mathfrak{J}) \cap A$.

We want to show that, if A is a Mori domain, in the previous one-to-one correspondence, strongly divisorial primes of C correspond to strongly divisorial primes of A.

PROPOSITION 1.10. *Let A be a Mori domain, $\mathfrak{J}$ a strongly divisorial ideal of A and $C = (A : \mathfrak{J})$. If $\mathfrak{P} \in \operatorname{Spec} A$, $\mathfrak{P} \not\supset \mathfrak{J}$ and $\mathfrak{Q} = (\mathfrak{P} : \mathfrak{J})$ (i.e. $\mathfrak{Q} \cap A = \mathfrak{P}$), then $\mathfrak{P}$ is a strongly divisorial ideal of A if and only if $\mathfrak{Q}$ is a strongly divisorial ideal of C. Moreover if $\mathfrak{P} \in \mathscr{S}(A)$, then $\mathfrak{Q} \in \mathscr{S}(C)$.*

Proof. We know that C is a Mori domain and that, if $\mathfrak{P} \in \operatorname{Spec} A$, $\mathfrak{P} \not\supset \mathfrak{J}$, is a strongly divisorial ideal of A, then $\mathfrak{Q} = (\mathfrak{P} : \mathfrak{J})$ is a divisorial ideal of C (cf. [13, p. 11]). We want to prove that $\mathfrak{Q}$ is strong.

Denote by F the quotient field of A (and of C). If $\mathfrak{Q}$ is not strong, there exists $x \in F$ such that $x\mathfrak{Q} \subset C$ and $x\mathfrak{Q} \not\subset \mathfrak{Q}$. Thus $x\mathfrak{Q}C_{\mathfrak{Q}} = C_{\mathfrak{Q}}$ and $\mathfrak{Q}C_{\mathfrak{Q}} = x^{-1}C_{\mathfrak{Q}}$ is principal. But $C_{\mathfrak{Q}}$ is a Mori domain (cf. [11, Corollary 3]) and so if ht $\mathfrak{Q} \geq 2$, we have a contradiction with [11, Lemma 2]. On the other hand, if ht $\mathfrak{Q} = 1$, $C_{\mathfrak{Q}} = A_{\mathfrak{P}}$ is a DVR (cf. [13, Theorem A-4]). This also is a contradiction because $\mathfrak{P}$ (and consequently $\mathfrak{P}A_{\mathfrak{P}}$) is strong.

Conversely, let $\mathfrak{Q} = (\mathfrak{P} : \mathfrak{J})$ be a strongly divisorial ideal of C, with $\mathfrak{P} \in \operatorname{Spec} A$, $\mathfrak{P} \not\supset \mathfrak{J}$. As noted before, $\mathfrak{P} = \mathfrak{Q} \cap A$, thus it is easy to see that $\mathfrak{P}$ is a divisorial ideal of A. In fact, since $\mathfrak{Q} = \bigcap\{xC;\ x \in F$ and $xC \supset \mathfrak{Q}\}$, $\mathfrak{P} = \bigcap\{x(A : \mathfrak{J});\ x \in F$ and $xC \supset \mathfrak{Q}\} \cap A$ is an intersection of divisorial ideals of A. We want to prove now that $\mathfrak{P}$ is strong, i.e. that $(A : \mathfrak{P}) = (\mathfrak{P} : \mathfrak{P})$. Actually we have $(A : \mathfrak{P}) \subset (A : \mathfrak{J}\mathfrak{Q}) = ((A : \mathfrak{J}) : \mathfrak{Q}) = (C : \mathfrak{Q}) = (\mathfrak{Q} : \mathfrak{Q})$. Thus if $x \in (A : \mathfrak{P})$, $x\mathfrak{P} \subset x\mathfrak{Q} \subset \mathfrak{Q}$. From $x\mathfrak{P} \subset A$ and $x\mathfrak{P} \subset \mathfrak{Q}$, we get $x\mathfrak{P} \subset A \cap \mathfrak{Q} = \mathfrak{P}$, so $x \in (\mathfrak{P} : \mathfrak{P})$.

For the last part of Proposition notice that if $\mathfrak{P} \in D_m(A)$ and $\mathfrak{Q} = (\mathfrak{P} : \mathfrak{J}) \subset \mathfrak{M} \in D_m(C)$, then $\mathfrak{M} \cap A$ is a divisorial ideal of A. Thus $\mathfrak{M} \cap A = \mathfrak{P}$ and, for the one-to-one correspondence, $\mathfrak{Q} = \mathfrak{M}$.

Given a Mori domain A such that $(A : A^*) \neq (0)$, we have associated to A a sequence of Mori overrings:

$$(*) \qquad A = A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_m = A^*.$$

From the previous Proposition we get the following:

COROLLARY 1.11. *Let A be a Mori domain such that $(A: A^*) \neq (0)$ and let (*) be the associated sequence. Then* $m \geq \sup$ *{lengths of chains of strongly divisorial primes of A}.*

Proof. Let $l_i = \sup$ {lengths of chains of strongly divisorial primes of A_i} and let $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_{l_i}$ be a chain of strongly divisorial primes of A_i. Then necessarily $\mathfrak{P}_{l_i} \in \mathscr{S}(A_i)$ and $\mathfrak{P}_0, \cdots, \mathfrak{P}_{l_i-1} \not\supset \mathscr{R}_i = \cap\{\mathfrak{P}; \mathfrak{P} \in \mathscr{S}(A_i)\}$. So, by Proposition 1.10, there exists in $A_{i+1} = (A_i: \mathscr{R}_i)$ a chain of strongly divisorial primes of length at least $l_i - 1$. Recalling that A_m is the only ring in the sequence (*) which does not have strongly divisorial primes, the conclusion follows easily.

Other informations about the relationship between strongly divisorial primes of two consecutive rings of the sequence (*) are given in the following:

PROPOSITION 1.12. *Let A be a Mori domain such that $(A: A^*) \neq (0)$ and let B, $C = (B: \mathscr{R})$ be consecutive (Mori) domains of the associated sequence (*), where $\mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$ and $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_n\} = \mathscr{S}(B)$. If $\mathfrak{Q}$ is a strongly divisorial prime ideal of C such that $\mathfrak{Q} \supset \mathscr{R}$, then $\mathfrak{Q} \cap B = \mathfrak{P}_j$ for some j, $j = 1, \cdots, n$.*

Proof. As in the proof of Proposition 1.10 it is easy to see that $\mathfrak{Q} \cap B$ is a divisorial ideal of B. But, since $\mathfrak{Q} \supset \mathscr{R}$ and $B \supset \mathscr{R}$, $\mathfrak{P} = \mathfrak{Q} \cap B \supset \mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n \supset \mathfrak{P}_1 \cdots \mathfrak{P}_n$. Since $\mathfrak{P}$ is a prime ideal, $\mathfrak{P} \supset \mathfrak{P}_j$ for some j, $j = 1, \cdots, n$. Thus $\mathfrak{P} = \mathfrak{P}_j$, becasue $\mathfrak{P}$ is divisorial and $\mathfrak{P}_j$ is maximal divisorial in B.

For an example of the situation described in Proposition 1.12, look at Example 1.9 a). A_1 (resp. A_2) has a strongly divisorial prime, $\mathscr{R}_1$ (resp. $\mathscr{R}_2$), above $\mathscr{R}_0 \in \mathscr{S}(A)$ (resp. $\mathscr{R}_1 \in \mathscr{S}(A_1)$).

Clearly in this case, if (*) is the associated sequence of overrings of A, $m > \sup$ {lengths of chains of strongly divisorial primes of A}.

PROPOSITION 1.13. *Let A be a Mori domain and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n \in \mathscr{S}(A)$. If $\mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$ and $C = (A: \mathscr{R})$, then $A = C \cap A_{\mathfrak{P}_1} \cap \cdots \cap A_{\mathfrak{P}_n}$.*

Proof. The inclusion $A \subset C \cap A_{\mathfrak{P}_1} \cap \cdots \cap A_{\mathfrak{P}_n}$ is trivial. For the opposite inclusion we recall that if A is a Mori domain, $A = \cap\{A_{\mathfrak{P}}; \mathfrak{P} \in D_m(A)\}$ (cf. [4, Proposition (2.2) b)]). Thus it is enough to show that $C \subset A_{\mathfrak{P}}$, for any maximal divisorial ideal $\mathfrak{P}$ of A, $\mathfrak{P} \neq \mathfrak{P}_1, \cdots, \mathfrak{P}_n$. Actually for such maximal divisorial ideal $\mathfrak{P}$ of A, $\mathfrak{P} \not\supset \mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$, thus there is exactly one $\mathfrak{Q} \in \operatorname{Spec} C$ above $\mathfrak{P}$ and $A_{\mathfrak{P}} = C_{\mathfrak{Q}}$ (cf. [7, Theorem 1.4, c)]). Therefore it is clear that $C \subset A_{\mathfrak{P}}$.

Next we study in greater detail the generic step $A_i \subset A_{i+1}$ in the sequence (*). Putting $A_i = B$ and $A_{i+1} = C$ and using the notation of Proposition 1.12, we describe the extension $B \subset C$ in n steps, in correspondence with the n prime ideals $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$.

We shall denote by $\mathscr{D}(A)$ the set of divisorial ideals of a domain A.

Let $B_0 = B$ and $\alpha_0\colon \mathscr{D}(B) \to \mathscr{D}(B)$ the identity map. Define, for $1 \leq j \leq n$, the pair (B_j, α_j) in the following way:

$$\begin{aligned} &B_j = B_{j-1}\colon (\alpha_{j-1} \circ \cdots \circ \alpha_0(\mathfrak{P}_j)) \\ &\alpha_j\colon \mathscr{D}(B_{j-1}) \longrightarrow \mathscr{D}(B_j) \\ &H \longrightarrow H\colon (\alpha_{j-1} \circ \cdots \circ \alpha_0(\mathfrak{P}_j)) \end{aligned}$$

Denote, for simplicity, the map $(\alpha_{j-1} \circ \cdots \circ \alpha_0)\colon \mathscr{D}(B) \to \mathscr{D}(B_{j-1})$ by Ψ_{j-1}.

Observe that, for each j, $j = 1, \cdots, n$, $\Psi_{j-1}(\mathfrak{P}_j) \in \mathscr{S}(B_{j-1})$. In fact, if $j = 1$, $\Psi_0(\mathfrak{P}_1) = \mathfrak{P}_1 \in \mathscr{S}(B_0)$. If $j \geq 2$, applying Proposition 1.10, we get that $\Psi_k(\mathfrak{P}_j) \in \mathscr{S}(B_k)$ and $\Psi_k(\mathfrak{P}_j) \not\supset \Psi_k(\mathfrak{P}_{k+1})$ for any k, $k = 0, 1, \cdots, j-2$. So, again by Proposition 1.10, $\Psi_{j-1}(\mathfrak{P}_j) \in \mathscr{S}(B_{j-1})$.

Therefore we have a sequence of Mori overrings of B, $B = B_0 \subset B_1 \subset \cdots \subset B_n$ (cf. again [13, p. 11]). We can prove:

Proposition 1.14. *Preserving the notation introduced above, the integral domain B_n coincides with C.*

Proof. Observe first that for each j, $j = 1, \cdots, n$, $\Psi_{j-1}(\mathfrak{P}_j)$ is a fractional ideal of B and that

$$\begin{aligned} B_n &= (B_{n-1}\colon \Psi_{n-1}(\mathfrak{P}_n)) = (B_{n-2}\colon \Psi_{n-2}(\mathfrak{P}_{n-1}))\colon (\Psi_{n-1}(\mathfrak{P}_n)) \\ &= B_{n-2}\colon (\Psi_{n-2}(\mathfrak{P}_{n-1})\Psi_{n-1}(\mathfrak{P}_n)) = \cdots = B\colon (\Psi_0(\mathfrak{P}_1) \cdots \Psi_{n-1}(\mathfrak{P}_n)). \end{aligned}$$

Observe secondly that, since for each j, $j = 1, \cdots, n$, $\mathfrak{P}_j B_{\mathfrak{P}_j} = (\mathfrak{P}_j B_{\mathfrak{P}_j})_v = (\mathfrak{P}_1 \cdots \mathfrak{P}_n B_{\mathfrak{P}_j})_v$, we have $\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n = \mathfrak{P}_1 B_{\mathfrak{P}_1} \cap \cdots \cap \mathfrak{P}_n B_{\mathfrak{P}_n} \cap B = \mathfrak{P}_1 B_{\mathfrak{P}_1} \cap \cdots \cap \mathfrak{P}_n B_{\mathfrak{P}_n} \cap \{B_{\mathfrak{P}}; \mathfrak{P} \in D_m(B), \mathfrak{P} \neq \mathfrak{P}_j\} = (\mathfrak{P}_1 \cdots \mathfrak{P}_n B_{\mathfrak{P}_1})_v \cap \cdots \cap (\mathfrak{P}_1 \cdots \mathfrak{P}_n B_{\mathfrak{P}_n})_v \cap \{(\mathfrak{P}_1 \cdots \mathfrak{P}_n B_{\mathfrak{P}})_v; \mathfrak{P} \in D_m(B), \mathfrak{P} \neq \mathfrak{P}_j\} = (\mathfrak{P}_1 \cdots \mathfrak{P}_n)_v$ (cf. [4, Proposition (2.2), c)]).

Thus we have $C = (B\colon \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n) = (B\colon (\mathfrak{P}_1 \cdots \mathfrak{P}_n)_v) = (B\colon \mathfrak{P}_1 \cdots \mathfrak{P}_n)$. Now, since for each j, $j = 1, \cdots, n$, $\mathfrak{P}_j \subset \Psi_{j-1}(\mathfrak{P}_j)$, we have $\mathfrak{P}_1 \cdots \mathfrak{P}_n \subset \Psi_0(\mathfrak{P}_1) \cdots \Psi_{n-1}(\mathfrak{P}_n)$ and so $C \supset B_n$. For the opposite inclusion it is enough to show by induction that $\Psi_0(\mathfrak{P}_1)\cdots\Psi_{n-1}(\mathfrak{P}_n)\subset\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_n$. Trivially $\Psi_0(\mathfrak{P}_1)=\mathfrak{P}_1\subset\mathfrak{P}_1$. Suppose that $\Psi_0(\mathfrak{P}_1)\cdots\Psi_{n-2}(\mathfrak{P}_{n-1})\subset\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_{n-1}$ $(n\geq 2)$. Since $\Psi_{n-1}(\mathfrak{P}_n)\subset B_{n-1}$ and $\Psi_{n-2}(\mathfrak{P}_{n-1})$ is an ideal of B_{n-1}, we have that $\Psi_{n-2}(\mathfrak{P}_{n-1})\Psi_{n-1}(\mathfrak{P}_n)\subset\Psi_{n-2}(\mathfrak{P}_{n-1})$, thus $\Psi_0(\mathfrak{P}_1)\cdots\Psi_{n-1}(\mathfrak{P}_n)\subset\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_{n-1}$.

Moreover, since by definition $\Psi_{n-1}(\mathfrak{P}_n)=(\Psi_{n-2}(\mathfrak{P}_n):\Psi_{n-2}(\mathfrak{P}_{n-1}))$, it is clear that $\Psi_{n-1}(\mathfrak{P}_n)\Psi_{n-2}(\mathfrak{P}_{n-1})\subset\Psi_{n-2}(\mathfrak{P}_n)$. So $\Psi_0(\mathfrak{P}_1)\cdots\Psi_{n-1}(\mathfrak{P}_n)\subset\Psi_{n-2}(\mathfrak{P}_n)\cap B=\mathfrak{P}_n$ and $\Psi_0(\mathfrak{P}_1)\cdots\Psi_{n-1}(\mathfrak{P}_n)\subset\mathfrak{P}_1\cap\cdots\cap\mathfrak{P}_{n-1}\cap\mathfrak{P}_n$.

§ 2. Contraction of ideals and glueings

To descend in the sequence (*) associated to a Mori domain, defined in Section 1, we need some further definitions.

DEFINITION 2.1. Let $A\subset B$ be two rings and let $\mathfrak{J}$ be an integral ideal of B such that $\mathfrak{J}\cap A=\mathfrak{p}\in\operatorname{Spec}A$. Let $S=A\backslash\mathfrak{p}$. S is a multiplicative set of A and of B. Denote by ϕ the composition of canonical maps $B\to S^{-1}B\to S^{-1}B/S^{-1}\mathfrak{J}$ and by $k(\mathfrak{p})$ the residue field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Let $k(\mathfrak{p})\to S^{-1}B/S^{-1}\mathfrak{J}$ be the canonical immersion. Then the *ring obtained from B by contracting* $\mathfrak{J}$ *over* $\mathfrak{p}$ is the pullback $\phi^{-1}(k(\mathfrak{p}))=B\times_{S^{-1}B/S^{-1}\mathfrak{J}}k(\mathfrak{p})$.

Remark 2.2. a) In Definition 2.1, if $\mathfrak{J}$ is an intersection of a family $\{\mathfrak{Q}_\lambda;\lambda\in\Lambda\}$ of primary ideals of B, such that $\mathfrak{Q}_\lambda\cap A=\mathfrak{p}$, for each $\lambda\in\Lambda$, then the ring obtained from B by contracting $\mathfrak{J}$ over $\mathfrak{p}$ coincides with the ring obtained from B by glueing the primary ideals $\{\mathfrak{Q}_\lambda;\lambda\in\Lambda\}$ over $\mathfrak{p}$, as defined in [14] (cf. [14, Proposition 1.5]).

b) If we suppose that $\mathfrak{J}=\sqrt{\mathfrak{p}B}$, that is if $\mathfrak{J}$ is an intersection of a family $\{\mathfrak{P}_\lambda;\lambda\in\Lambda\}$ of prime ideals of B, then the ring obtained from B by contracting $\mathfrak{J}$ over $\mathfrak{p}$, defined in 2.1, coincides with the ring obtained from B by glueing over $\mathfrak{p}$, as defined in [9]. In particular, if B is integral and finite over A (and $\mathfrak{J}=\sqrt{\mathfrak{p}B}$), then the family $\{\mathfrak{P}_\lambda;\lambda\in\Lambda\}$ is finite and, locally, for each λ, $S^{-1}\mathfrak{P}_\lambda$ is a maximal ideal of $S^{-1}B$. Thus, in this case, the pullback diagram is of the following form:

$$\begin{tikzcd} \phi^{-1}(k(\mathfrak{p})) \arrow[r] \arrow[d, hook] & k(\mathfrak{p}) \arrow[d, hook] \\ B \arrow[r, "\phi"'] & k(\mathfrak{P}_1)\times\cdots\times k(\mathfrak{P}_n) \end{tikzcd}$$

and we obtain the "classical" definition of the ring obtained from B by glueing over $\mathfrak{p}$, as defined in [16].

c) Notice that to define properly the ring obtained from B by glueing over $\mathfrak{p} \in \operatorname{Spec} A$ (i.e. by contracting $\mathfrak{J} = \sqrt{\mathfrak{p}B}$ over $\mathfrak{p}$) or the ring obtained from B by contracting $\mathfrak{J} = \mathfrak{p}B$ over $\mathfrak{p}$, it is necessary that one of the following equivalent conditions holds:

i) the canonical map $A/\mathfrak{p} \to B/\mathfrak{p}B$ is injective (cf. Iscebeck's definition);

ii) $\mathfrak{p}B$ is over $\mathfrak{p}$, that is $\mathfrak{p}B \cap A = \mathfrak{p}$;

iii) $\mathfrak{p}S^{-1}B \neq S^{-1}B$ (with $S = A \backslash \mathfrak{p}$);

iv) there exists a prime ideal $\mathfrak{Q}$ of B over $\mathfrak{p}$;

v) $\sqrt{\mathfrak{p}B}$ is over $\mathfrak{p}$.

Using the hypotheses and notation of Definition 2.1, we can show that:

PROPOSITION 2.3. *The ring obtained from B by contracting $\mathfrak{J}$ over $\mathfrak{p}$ is the largest subring A' of B such that*

i) *$\mathfrak{J} = \mathfrak{p}'$ is a prime ideal of A';*

ii) *the canonical homomorphism $k(\mathfrak{p}) \to k(\mathfrak{p}')$ is an isomorphism.*

Proof. Notice that in our hypotheses, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "g"] \arrow[dd, hook] & & A/\mathfrak{p} \arrow[dr, "\bar{g}"] \arrow[dd, hook, "\bar{h}" near start] & \\
& S^{-1}A \arrow[rr, crossing over] & & k(\mathfrak{p}) \arrow[dd, hook] \\
B \arrow[rr, "h" near end] \arrow[dr, "g"] & & B/\mathfrak{J} \arrow[dr, "\bar{g}"] & \\
& S^{-1}B \arrow[rr, "\bar{h}"'] \arrow[from=uu, hook, crossing over] & & S^{-1}B/S^{-1}\mathfrak{J}
\end{tikzcd}$$

Observe moreover that $S^{-1}B/S^{-1}\mathfrak{J} = \bar{S}^{-1}(B/\mathfrak{J})$, where $\bar{S} = h(S) = \{s + \mathfrak{J};\ s \in S\}$ is a multiplicative part of $B/\mathfrak{J}$. Since in $\bar{S}$ there are not zero-divisors (in fact $(s_1 + \mathfrak{J})(s_2 + \mathfrak{J}) = \mathfrak{J}$, with $s_1, s_2 \in S$, implies $s_1 s_2 \in \mathfrak{p}$ and so $s_1 \in \mathfrak{p}$ (and $(s_1 + \mathfrak{J}) = \mathfrak{J}$) or $s_2 \in \mathfrak{p}$ (and $(s_2 + \mathfrak{J}) = \mathfrak{J}$)) the homomorphism $\bar{g}$ is injective.

Let C be the ring obtained from B by contracting $\mathfrak{J}$ over $\mathfrak{p}$. By definition $C = \phi^{-}(k(\mathfrak{p}))$, where $\phi = \bar{h} \circ g = \bar{g} \circ h$. Thus, considering the injection $\bar{g}$ as an inclusion, C is the pullback of the diagram

$$\begin{tikzcd} C = h^{-1}((B/\mathfrak{J}) \cap k(\mathfrak{p})) \arrow[r] \arrow[d, hook] & B/\mathfrak{J} \cap k(\mathfrak{p}) \arrow[d, hook] \\ B \arrow[r, "h"] & B/\mathfrak{J} \end{tikzcd}$$

where the intersection is in $S^{-1}B/S^{-1}\mathfrak{J}$.

Since $C/\mathfrak{J} = B/\mathfrak{J} \cap k(\mathfrak{p})$ is an integral domain, $\mathfrak{J} = \mathfrak{p}'$ is a prime ideal of C. Therefore C is a ring that contains A and has a prime ideal $\mathfrak{p}'$ over $\mathfrak{p}$ and hence we have the canonical monomorphism $k(\mathfrak{p}) \to k(\mathfrak{p}')$. However $k(\mathfrak{p}')$ is the quotient field of $C/\mathfrak{p}' = B/\mathfrak{J} \cap k(\mathfrak{p})$, thus it is contained in $k(\mathfrak{p})$ and so $k(\mathfrak{p}) \cong k(\mathfrak{p}')$.

Now, we want to show that C is maximal with respect to the properties i) and ii). A subring of B with properties i) and ii) is in fact a pullback of the type $B \times_{B/\mathfrak{J}} D$ where D is a domain contained in $B/\mathfrak{J}$ and containing $A/\mathfrak{p}$ and with quotient field isomorphic to $k(\mathfrak{p})$. The largest ring of this kind is clearly C, constructed in correspondence with the largest $D = B/\mathfrak{J} \cap k(\mathfrak{p})$ with the described properties.

Remark 2.4. Observe that if C is the ring obtained from B by contracting $\mathfrak{J}$ over $\mathfrak{p} \in \operatorname{Spec} A$, then:

a) C may have also other primes over $\mathfrak{p}$ (cf. [14, Oss. 1, p. 5]).

b) $A + \mathfrak{J} \subset C$ and, with an analogous argument to [14, Proposition 1.7], it can be shown that $A + \mathfrak{J} = C$ if and only if $A/\mathfrak{p} = C/\mathfrak{J}$ $(= B/\mathfrak{J} \cap k(\mathfrak{p}))$.

The following example shows that it may be $A \subsetneqq A + \mathfrak{J} \subsetneqq C$.

EXAMPLE 2.5. Let $A = D + ZK[Z]$, where D is a domain, K its quotient field. Let $B = K[Y, Z]$ and $\mathfrak{J} = ZK[Y, Z]$. Clearly $\mathfrak{J} \cap A = \mathfrak{p} = ZK[Z]$. In this case the ring obtained from B by contracting $\mathfrak{J}$ over $\mathfrak{p}$ is the pullback of the diagram:

$$\begin{tikzcd} & K \arrow[d, hook] \\ B = K[Y, Z] \arrow[r] & K[Y] \end{tikzcd}$$

Thus it is $C = K + ZK[Y, Z]$ and $A = D + ZK[Z] \subsetneqq A + \mathfrak{J} = D + ZK[Y, Z] \subsetneqq C$.

We extend Definition 2.1 to finitely many prime ideals:

DEFINITION 2.6. Let $A \subset B$ be two rings and let $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ be integral ideals of B such that $\mathfrak{J}_j \cap A = \mathfrak{p}_j \in \operatorname{Spec} A$, $j = 1, \cdots, n$. We call the ring $B_1 \cap \cdots \cap B_n$ the *ring obtained from* B *by contracting* $\mathfrak{J}_1$ over $\mathfrak{p}_1, \cdots, \mathfrak{J}_n$ *over* $\mathfrak{p}_n$, where for each j, $j = 1, \cdots, n$, B_j is the ring obtained from B by contracting $\mathfrak{J}_j$ over $\mathfrak{p}_j$.

PROPOSITION 2.7. *Let* A *be a Mori domain and let* $\mathfrak{P}_1, \cdots, \mathfrak{P}_n \in \mathscr{S}(A)$. *If* $\mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$ *and* $C = (A : \mathscr{R})$, *then* A *is the ring obtained from* C *by contracting* $\mathfrak{P}_1 C$ *over* $\mathfrak{P}_1$, $\mathfrak{P}_2 C$ *over* $\mathfrak{P}_2, \cdots, \mathfrak{P}_n C$ *over* $\mathfrak{P}_n$.

Proof. By Proposition 1.13, we have $A = C \cap A_{\mathfrak{P}_1} \cap \cdots \cap A_{\mathfrak{P}_n}$. Thus it is enough to show that for each j, $j = 1, \cdots, n$, $C \cap A_{\mathfrak{P}_j}$ is the ring obtained from C by contracting $\mathfrak{P}_j C$ over $\mathfrak{P}_j$. If $S_j = A \backslash \mathfrak{P}_j$ first observe that $S_j^{-1}C = S_j^{-1}(A : \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n) = (S_j^{-1}A : (S_j^{-1}\mathfrak{P}_1 \cap \cdots \cap S_j^{-1}\mathfrak{P}_j \cap \cdots \cap S_j^{-1}\mathfrak{P}_n))$ (cf. for example [11, proof of Theorem 2] for the first equality and [1, Proposition 3.11 v),] for the second). Thus $S_j^{-1}C = (S_j^{-1}A : S_j^{-1}\mathfrak{P}_j) = S_j^{-1}(A : \mathfrak{P}_j)$. Using this equality, it is not difficult to see that the following diagram

$$\begin{array}{ccc} A_{\mathfrak{P}_j} & \longrightarrow & k(\mathfrak{P}_j) = A_{\mathfrak{P}_j}/\mathfrak{P}_j A_{\mathfrak{P}_j} \\ \cap\downarrow & & \cap\downarrow \\ S_j^{-1}C & \longrightarrow & S_j^{-1}C/S_j^{-1}\mathfrak{P}_j \end{array}$$

is a pullback. Recalling now that C is a domain and so the canonical map $g: C \to S_j^{-1}C$ is injective, we can see that $C \cap A_{\mathfrak{P}_j}$ coincides with the pullback of the diagram

$$\begin{array}{ccc} & & k(\mathfrak{P}_j) \\ & & \cap\downarrow \\ C & \longrightarrow & S_j^{-1}C/S_j^{-1}\mathfrak{P}_j\,. \end{array}$$

That is, $C \cap A_{\mathfrak{P}_j}$ is the ring obtained from C contracting $\mathfrak{P}_j C$ over $\mathfrak{P}_j$.

COROLLARY 2.8. *Let* A *be a Mori domain such that* $(A : A^*) \neq (0)$ *and let* B, $C = (B : \mathscr{R})$ *be two consecutive* (*Mori*) *domains of the associated sequence* (*) *of Section* 1, *where* $\mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$ *and* $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ *are the strong maximal divisorial ideals of* B. *Then* B *is exactly the ring obtained from* C *by contracting* $\mathfrak{P}_1 C$ *over* $\mathfrak{P}_1$, $\mathfrak{P}_2 C$ *over* $\mathfrak{P}_2, \cdots, \mathfrak{P}_n C$ *over* $\mathfrak{P}_n$.

§ 3. The "seminormal" case

Let A be a Mori domain such that $(A : A^*) \neq (0)$. Let

$$(*) \qquad A = A_0 \subsetneqq A \subsetneqq \cdots \subsetneqq A_m = A^*$$

be the sequence of overrings of A constructed in Section 1.

Section 3 is devoted to study the particular case where $\mathscr{R}_i = (A_i: A_{i+1})$ is a radical ideal of A_{i+1}, for each i, $i = 0, \cdots, m-1$. As we shall see, this case is closely related to Traverso's seminormalization.

It is convenient to define the *strong dimension* of an integral domain A, $\dim_s A$, to be the supremum of the lengths of all chains of strongly divisorial prime ideals in A. If A contains no proper strongly divisorial prime ideal, we say that A has strong dimension -1; thus, if A is completely integrally closed, then $\dim_s A = -1$ (cf. for example [3, Corollary 13]).

In our hypothesis, by Corollary 1.6, $\dim_s A$ is finite and, by [3, Corollary 14], A is completely integrally closed if and only if $\dim_s A = -1$.

LEMMA 3.1. *Let $\mathfrak{J}$ be a strongly divisorial ideal of a domain A and let $B = (A: \mathfrak{J})$. If $\mathfrak{J}$ is a radical ideal of B and if $\mathfrak{J} \subset \mathfrak{Q} \in \operatorname{Spec} B$, then $\mathfrak{Q}$ is not a strongly divisorial ideal of B.*

Proof. Let $\mathfrak{J} \subset \mathfrak{Q} \in \operatorname{Spec} B$. Restrict $\mathfrak{Q}$ to a minimal prime $\mathfrak{P}$ of $\mathfrak{J}$. By Lemma 1.1 $(\mathfrak{P}:\mathfrak{P}) \subset (\mathfrak{J}:\mathfrak{J})$ and, by [8, Lemma 3.7] $(\mathfrak{Q}:\mathfrak{Q}) \subset (\mathfrak{P}:\mathfrak{P})$. Since $(\mathfrak{J}:\mathfrak{J}) = (A:\mathfrak{J}) = B$, we have $(\mathfrak{Q}:\mathfrak{Q}) = B$. If $\mathfrak{Q}$ is strong, then $(B:\mathfrak{Q}) = (\mathfrak{Q}:\mathfrak{Q}) = B$ and $\mathfrak{Q}_v = B$, thus $\mathfrak{Q}$ is not divisorial.

PROPOSITION 3.2. *Let A be a Mori domain such that $(A: A^*) \neq (0)$ and let (*) be the associated sequence. If, for each i, $i = 0, \cdots, m-1$, $\mathscr{R}_i = (A_i: A_{i+1})$ is a radical ideal of A_{i+1}, then:*

1) *no strongly divisorial prime ideal of A_{i+1} contains $\mathscr{R}_i$, for each i, $i = 0, \cdots, m-1$;*

2) $\dim_s A_i = m - i - 1$, *for each i, $i = 0, \cdots, m$. In particular* $\dim_s A = m - 1$;

3) *$(A: A_i)$ is a radical ideal of A_i, for each i, $i = 1, \cdots, m$.*

Proof. Recall that by construction $A_{i+1} = (A_i: \mathscr{R}_i)$, for $i = 0, \cdots, m-1$, and $\mathscr{R}_i$ is a strongly divisorial ideal of A_i. Thus to prove 1) it is enough to apply Lemma 3.1. To prove 2) observe that, by 1) and Proposition 1.10, $\dim_s A_{i+1} = \dim_s A_i - 1$, for each i, $i = 0, \cdots, m-1$. Recalling moreover that A_m does not have strongly divisorial prime ideals, i.e. $\dim_s A_m = -1$, we get $\dim_s A_i = -1 + (m - i) = m - i - 1$. In particular $\dim_s A = \dim_s A_0 = m - 1$. To prove 3), we show that A contains the radical of $(A: A_i)$ in A_i for each i, $i = 1, \cdots, m$. Let $x \in A_i$

and $x^n \in (A: A_i)$, for some $n \in N$. We want to prove that $x \in A$. It is enough to prove that $x \in A_{i-1}$ and $x^n \in (A: A_{i-1})$. We have $(A: A_i) \subset (A_{i-1}: A_i) = \mathscr{R}_{i-1}$, thus, since $\mathscr{R}_i$ is a radical ideal of A_i, $x \in \mathscr{R}_{i-1} \subset A_{i-1}$. Moreover, trivially, $x^n \in (A: A_i) \subset (A: A_{i-1})$.

If A is a Noetherian domain such that $\overline{A} = A^*$ is an A-module of finite type (i.e. $(A: \overline{A}) \neq (0)$), we shall prove that the particular case considered above (i.e. $\mathscr{R}_i$ radical ideal of A_{i+1} in the sequence (*)) corresponds to seminormal case.

Recall that, given two rings $A \subset B$, B integral over A, the *seminormalization* of A in B is the ring

$$A_B^+ = \{b \in B \mid b/1 \in A_{\mathfrak{P}} + \mathrm{Rad}\,(S^{-1}B),\ \forall\, \mathfrak{P} \in \mathrm{Spec}\, A\}$$

where $S = A \backslash \mathfrak{P}$ and $\mathrm{Rad}\,(S^{-1}B)$ is the Jacobson radical of $S^{-1}B$ (cf. [16]). It is well known that A_B^+ is the largest subring A' of B such that

i) for each $\mathfrak{P} \in \mathrm{Spec}\, A$, there is exactly one $\mathfrak{Q} \in \mathrm{Spec}\, A'$ above $\mathfrak{P}$;

ii) the canoncal homomorphism $k(\mathfrak{P}) \to k(\mathfrak{Q})$ is an isomorphism. (cf. [16, (1.1)]).

PROPOSITION 3.3. *Let A be a Mori domain and let $\mathfrak{P}_1, \cdots, \mathfrak{P}_n \in \mathscr{S}(A)$. If $\mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$ and $C = (A: \mathscr{R})$, then the following conditions are equivalent*:

1) *$\mathscr{R}$ is a radical ideal of C;*

2) *$S_j^{-1}\mathfrak{P}_j = \mathfrak{P}_j A_{\mathfrak{P}_j}$ is a radical ideal of $S_j^{-1}C$ (where $S_j = A \backslash \mathfrak{P}_j$), for each j, $j = 1, \cdots, n$;*

3) *A is the ring obtained from C by glueing over $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$.*

Moreover, if A is Noetherian, then the following are equivalent to each other and to the above conditions:

4) *A is seminormal in C;*

5) *$S_j^{-1}A = A_{\mathfrak{P}_j}$ is seminormal in $S_j^{-1}C$ (where $S_j = A \backslash \mathfrak{P}_j$), for each j, $j = 1, \cdots, m$.*

Proof. 1) $\Rightarrow$ 2): since $\mathscr{R}$ is an ideal of C, $S_j^{-1}\mathscr{R} = S_j^{-1}(\mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n) = S_j^{-1}\mathfrak{P}_1 \cap \cdots \cap S_j^{-1}\mathfrak{P}_n = S_j^{-1}\mathfrak{P}_j$ is an ideal of $S_j^{-1}C$; since $\mathscr{R}$ is radical in C, $S_j^{-1}\mathfrak{P}_j$ is a radical ideal of $S_j^{-1}C$. 2) $\Rightarrow$ 1): $\mathscr{R} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n = \mathfrak{P}_1 A_{\mathfrak{P}_1} \cap \cdots \cap \mathfrak{P}_n A_{\mathfrak{P}_n} \cap A$. By Proposition 1.13, $A = C \cap A_{\mathfrak{P}_1} \cap \cdots \cap A_{\mathfrak{P}_n}$, thus $\mathscr{R} = \mathfrak{P}_1 A_{\mathfrak{P}_1} \cap \cdots \cap \mathfrak{P}_n A_{\mathfrak{P}_n} \cap C$. Since $S_j^{-1}\mathfrak{P}_j$ is a radical ideal of $S_j^{-1}C$, $S_j^{-1}\mathfrak{P}_j \cap C$ is a radical ideal of C for each j, $j = 1, \cdots, n$, therefore $\mathscr{R}$ is a radical ideal of C. 2) $\Leftrightarrow$ 3): by Proposition 2.7, A is the ring obtained from C

contracting $\mathfrak{P}_1C$ over $\mathfrak{P}_1$, $\mathfrak{P}_2C$ over $\mathfrak{P}_2, \cdots, \mathfrak{P}_nC$ over $\mathfrak{P}_n$. Thus A is obtained by glueing over $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$ if and only if $\mathfrak{P}_1C, \cdots, \mathfrak{P}_nC$ are radical ideals of C. This happens if and only if for each j, $j = 1, \cdots, n$, $S_j^{-1}\mathfrak{P}_jC = S_j^{-1}\mathfrak{P}_j$ is a radical ideal of $S_j^{-1}C$. 2) $\Rightarrow$ 5): if $S_j^{-1}\mathfrak{P}_j$ is a radical ideal of $S_j^{-1}C$, necessarily $S_j^{-1}\mathfrak{P}_j = \mathrm{Rad}(S_j^{-1}C)$, the Jacobson radical of $S_j^{-1}C$, thus $S_j^{-1}A + \mathrm{Rad}(S_j^{-1}C) = S_j^{-1}A$ and $S_j^{-1}A$ is seminormal in $S_j^{-1}C$. 5) $\Rightarrow$ 4): observe that for each j, $j = 1, \cdots, n$, the seminormalization of A in C is contained in the seminormalization of $S_j^{-1}A$ in $S_j^{-1}C$, as it follows by definition. Therefore we have $A_C^+ \subset C \cap A_{\mathfrak{P}_1} \cap \cdots \cap A_{\mathfrak{P}_n}$. By Proposition 1.13, $C \cap A_{\mathfrak{P}_1} \cap \cdots \cap A_{\mathfrak{P}_n} = A$, thus A is seminormal in C. 4) $\Rightarrow$ 1): by [16, Lemma 1.3], because $\mathscr{R}$ is the conductor of A in C.

Remark 3.4. Let A be a Noetherian domain such that $\bar{A}$ is an A-module of finite type and let B, C be two consecutive (Noetherian) domains of the associated sequence (*). Proposition 3.3 gives, in particular, equivalent conditions in order that B is seminormal in C.

LEMMA 3.5. *Let $A_1 \subset A_2 \subset B$ be domains and let $A_2 = (A_1 : \mathfrak{J})$, where $\mathfrak{J}$ is a strongly divisorial ideal of A_1. If $\mathfrak{P} \in \mathrm{Spec}\, A_2$, $\mathfrak{P} \not\supset \mathfrak{J}$, $\mathfrak{p} = \mathfrak{P} \cap A_1$, $T_1 = A_1 \setminus \mathfrak{p}$ and $T_2 = A_2 \setminus \mathfrak{P}$, then $T_1^{-1}B = T_2^{-1}B$ and the ring obtained from B by glueing over $\mathfrak{p} \in \mathrm{Spec}\, A_1$ coincides with the ring obtained from B by glueing over $\mathfrak{P} \in \mathrm{Spec}\, A_2$.*

Proof. Let's prove first that $T_1^{-1}B = T_2^{-1}B$. Let $x = bs^{-1} \in T_2^{-1}B$, with $b \in B$, $s \in T_2$. If $0 \neq i \in \mathfrak{J} \setminus \mathfrak{P}$, $bs^{-1} = (ib)(is)^{-1} \in T^{-1}B$, because $ib \in B$, $is \in \mathfrak{J} \subset A_1$ and $i \in A_2 \setminus \mathfrak{P}$, $s \in A_2 \setminus \mathfrak{P}$ so $is \notin \mathfrak{P} \cap A_1 = \mathfrak{p}$. Thus $T_1^{-1}B \supset T_2^{-1}B$. The opposite inclusion is trivial. Let's prove now that $T_1^{-1}\mathfrak{p}B = T_2^{-1}\mathfrak{P}B$. Let $x = qbs^{-1}$, with $q \in \mathfrak{P}$, $b \in B$, $s \in T_2$. Pick as before an element $i \in \mathfrak{J} \setminus \mathfrak{P}$. We have $x = bqi(si)^{-1} \in T_1^{-1}\mathfrak{p}B$ because $qi \in \mathfrak{p}$ and $si \in A_1 \setminus \mathfrak{p}$. Thus $T_1^{-1}\mathfrak{p}B \supset T_2^{-1}\mathfrak{P}B$. The opposite inclusion is trivial. Therefore $T_1^{-1}\sqrt{\mathfrak{p}B} = \sqrt{T_1^{-1}\mathfrak{p}B} = \sqrt{T_2^{-1}\mathfrak{P}B} = T_2^{-1}\sqrt{\mathfrak{P}B}$. Recalling now that $(A_1)_\mathfrak{p} = (A_2)_\mathfrak{P}$ (cf. [7, 1.4, c)]), we have that $k(\mathfrak{p}) = k(\mathfrak{P})$ and, by definition of glueing, the conclusion.

PROPOSITION 3.6. *Let A be a Mori domain such that $(A : A^*) \neq (0)$ and let (*) be the associated sequence. If, for each i, $i = 0, \cdots, m-1$, $\mathscr{R}_i = (A_i : A_{i+1})$ is a radical ideal of A_{i+1} and if $\mathscr{S}(A_i) = \{\mathfrak{P}_{i1}, \cdots, \mathfrak{P}_{in(i)}\}$, then A_i is the ring obtained from A_{i+1} by glueing over $\mathfrak{p}_{i1} = \mathfrak{P}_{i1} \cap A, \cdots, \mathfrak{p}_{in(i)} = \mathfrak{P}_{in(i)} \cap A$.*

Proof. We already know according to Proposition 3.3, 1) $\Rightarrow$ 3), that A_i is the ring obtained from A_{i+1} by glueing over $\mathfrak{P}_{i1}, \cdots, \mathfrak{P}_{in(i)}$. Observing that for each j, $j = 1, \cdots, n(i)$, $\mathfrak{P}_{ij} \not\supset (A: A_i)$ (cf. Lemma 3.1), and applying Lemma 3.5 we arrive at the conclusion.

COROLLARY 3.7. *Let A be a Mori domain such that $(A: A^*) \neq (0)$ and let (*) be the associated sequence. If, for each i, $i = 0, \cdots, m-1$, $\mathscr{R}_i = (A_i: A_{i+1})$ is a radical ideal of A_{i+1}, then A is obtained from A^* by a finite number of glueings over all the strongly divisorial prime ideals of A.*

Proof. The Corollary follows immediately from Proposition 3.6. We have just to prove that the set $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} = \mathfrak{P} \cap A$ for some i, $i = 0, \cdots, m-1$, and some $\mathfrak{P} \in \mathscr{S}(A_i)\}$ is the set of the strongly divisorial prime ideals of A. If $\mathfrak{P} \in \mathscr{S}(A_i)$ for some i, by Proposition 3.2, 3), $(A: A_i)$ is a radical ideal of A_i and so, by Lemma 3.1, $\mathfrak{P} \not\supset (A: A_i)$. Thus we can apply Proposition 1.10 and conclude that $\mathfrak{p} = \mathfrak{P} \cap A$ is a strongly divisorial ideal of A. On the other hand, let $\mathfrak{p}$ be a strongly divisorial prime ideal of A. If $\mathfrak{p} \notin \mathscr{S}(A)$, then $\mathfrak{p} \not\supset \mathscr{R}_0 = \bigcap \{\mathfrak{Q}; \mathfrak{Q} \in \mathscr{S}(A)\} = (A: A_1)$ and thus, again by Proposition 1.10 there exists in A_1 a strongly divisorial prime ideal $\mathfrak{p}_1$ over $\mathfrak{p}$. If $\mathfrak{p}_1 \notin \mathscr{S}(A_1)$, then $\mathfrak{p}_1 \not\supset \mathscr{R}_1 = (A_1: A_2)$, thus there exists in A_2 a strongly divisorial prime ideal $\mathfrak{p}_2$ over $\mathfrak{p}_1$ (therefore over $\mathfrak{p}$) and so on. Since in A_m there are not strongly divisorial prime ideals at all, there exist i and $\mathfrak{P} \in \mathscr{S}(A_i)$ such that $\mathfrak{P} \cap A = \mathfrak{p}$.

THEOREM 3.8. *Let A be a Noetherian domain such that $\overline{A}$ is an A-module of finite type and let (*) be the associated sequence. Then A is seminormal if and only if $\mathscr{R}_i = (A_i: A_{i+1})$ is a radical ideal of A_{i+1}, for each i, $i = 0, \cdots, m-1$.*

Proof. If $\mathscr{R}_i$ is a radical ideal of A_{i+1} for each i, $i = 0, \cdots, m-1$, then, by Proposition 3.3 and Remark 3.4. A_i is seminormal in A_{i+1}. Thus, by [16, Lemma 1.2], we have that $A = A_0$ is seminormal in $\overline{A} = A_m$.

Conversely, let A be seminormal (in $A_m = \overline{A}$). We want to prove that A_{m-1} is seminormal in A_m. By Proposition 3.3 (and Remark 3.4), it is enough to show that, if $\mathfrak{P} \in \mathscr{S}(A_{m-1})$, then $\mathfrak{P}(A_{m-1})_{\mathfrak{P}}$ is a radical ideal of $S^{-1}A_m$ (where $S = A_{m-1} \setminus \mathfrak{P}$). Since, trivially, A is seminormal in A_{m-1}, $(A: A_{m-1})$ is a radical ideal of A_{m-1} (cf. [16, Lemma 1.3]), so, by Lemma 3.1, $\mathfrak{P} \not\supset (A: A_{m-1})$. Therefore we can apply Lemma 3.5 and, if $\mathfrak{p} = \mathfrak{P} \cap A$ and $T = A \setminus \mathfrak{p}$, we have $T^{-1}A_m = S^{-1}A_m$. Moreover $A_{\mathfrak{p}} = (A_{m-1})_{\mathfrak{P}}$ and so

$\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{P}(A_{m-1})_{\mathfrak{P}}$. Thus we have to show that $\mathfrak{p}A_{\mathfrak{p}}$ is a radical ideal of $T^{-1}A_m$. Observe now that, if $\mathfrak{J} = (A\colon A_m)$, since $\mathfrak{P} \supset (A_{m-1}\colon A_m) \supset \mathfrak{J}$, $\mathfrak{p} \supset \mathfrak{J}$. We claim that $\mathfrak{p}$ is a minimal over $\mathfrak{J}$. If not, we have $\mathfrak{J} \subset \mathfrak{q} \subsetneq \mathfrak{p}$, where $\mathfrak{q}$ is a strongly divisorial prime of A (cf. Proposition 1.3). If this is the case, since $\mathfrak{q} \not\supset (A\colon A_{m-1})$, by Proposition 1.10, there is in A_{m-1} a strongly divisorial prime ideal $\mathfrak{Q} \subsetneq \mathfrak{P}$ and this is a contradiction, because $\dim_s A_{m-1} = 0$ (cf. Proposition 3.2, 2)). Thus $T^{-1}\mathfrak{J} = T^{-1}\mathfrak{p}$. Since $\mathfrak{J}$ is a radical ideal of A_m (cf. again [16, Lemma 1.3]), $T^{-1}\mathfrak{J} = T^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$ is a radical ideal of $T^{-1}A_m$.

Remark 3.9. As we recalled, if A is seminormal, $(A\colon \overline{A})$ is a radical ideal of $\overline{A}$ (cf. [16, Lemma 1.3]). Observe that Theorem 3.8 provides, for a Noetherian domain A such that $\overline{A}$ is an A-module of finite type, a kind of converse of this result. In order that A is seminormal, it is not sufficient in general that the conductor $(A\colon \overline{A})$ is radical in $\overline{A}$, but it is sufficient (and necessary) that all the conductors $\mathscr{R}_i = (A_i\colon A_{i+1})$, $i = 0, \cdots, m-1$, of our sequence are radical in A_{i+1}. Trivially, if $m = 1$ in the sequence (*), the two conditions ($(A\colon \overline{A})$ radical in $\overline{A}$ and $\mathscr{R}_i$ radical in A_{i+1}, for each i) are equivalent. A more general result in this spirit is the following:

PROPOSITION 3.10. *Let A be a Mori domain such that $(A\colon A^*) \neq (0)$ and let (*) be the associated sequence. If $(A\colon A^*)$ is a radical ideal of A and if $\dim_s A = 0$, then $m = 1$, i.e. the sequence (*) is simply $A = A_0 \subset A_1 = A^*$.*

Proof. Since $(A\colon A^*)$ is radical, $(A\colon A^*) = \cap \{\mathfrak{P}_\lambda; \lambda \in \Lambda\}$, where taking only the minimal primes over $(A\colon A^*)$, we can assume, by Proposition 1.3, that all the $\mathfrak{P}_\lambda$ are strongly divisorial primes of A. Since $(A\colon A^*)$ is the minimum strongly divisorial ideal of A (cf. [3. Proposition 16]) and any intersection of strongly divisorial primes is a strongly divisorial ideal (cf. Proposition 1.2), it turns out that $(A\colon A^*)$ is the intersection of all the strongly divisorial primes of A. However, since by hypothesis there are not in A non trivial chains of strongly divisorial primes, the set $\{\mathfrak{P}_\lambda; \lambda \in \Lambda\}$ coincides with the set of all the strong maximal divisorial ideals of A, $\mathscr{S}(A)$ which, by Corollary 1.5 and since $\dim_s A = 0$, is finite: $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_n\}$. Thus $(A\colon A^*) = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n = \mathscr{R}_0$ and $A_1 = (A\colon \mathscr{R}_0) = A^*$.

Remark 3.11. a) Notice that in Proposition 3.10 the hypothesis that $(A\colon A^*)$ is radical in A is necessary, as Example 1.9, a) shows.

b) If A is a Mori domain such that $(A: A^*) \neq 0$, if (*) is the associated sequence, and if $\dim_s A = 0$, we deduce easily from Proposition 3.10 that the following conditions are equivalent:

i) $\mathscr{R}_i = (A_i: A_{i+1})$ is a radical ideal of A_{i+1}, for each i, $i = 0, \cdots, m-1$;

ii) $(A: A^*)$ is a radical ideal of A^*.

In fact i) $\Rightarrow$ ii) is an easy consequence of Proposition 3.2, 3) (recalling that $A_m = A^*$) and ii) $\Rightarrow$ i) is an easy consequence of Proposition 3.10, noticing that, if $(A: A^*)$ is radical in A^*, it is radical in A.

c) If A is Noetherian, the equivalence of conditions i) and ii) above gives in particular the following known result: if A is a Noetherian domain (with $A \neq (A: \bar{A}) \neq (0)$) which satisfies condition (S_2) (depth $A_{\mathfrak{P}} \geq \inf(2, \operatorname{ht} \mathfrak{P})$, for all $\mathfrak{P} \in \operatorname{Spec} A$), then A is seminormal if and only if $(A: \bar{A})$ is a radical ideal of $\bar{A}$ (cf. [6 Proposition 7.12]). In fact (S_2) holds in the Noetherian domain A if and only if each $(0) \neq \mathfrak{P} \in \operatorname{Spec} A$, such that depth $A_{\mathfrak{P}} = 1$, is of height 1, i.e., by [17, Proposition 1.10, i) $\Leftrightarrow$ vi)], if and only if each divisorial prime of A is of height 1. However there is in A at least one strongly divisorial prime, because A ($\neq \bar{A}$) is not a Krull domain (cf. [3, Corollary 14]), thus, if (S_2) holds in A, $\dim_s A = 0$. Moreover, if A is Noetherian, condition i) above means that A is seminormal (cf. Theorem 3.8).

Finally we point out that in the Mori, non-Noetherian case, the glueings over the strongly divisorial prime ideals of A (of Corollary 3.7) do not request any algebraic or finiteness condition on the extension $k(\mathfrak{p}) \subset S^{-1}B/S^{-1}\mathfrak{J}$ (cf. Definition 2.1), as the simple following examples show:

EXAMPLES 3.12. a) Let $A = k + Xk[X, Y]$ where k is a field and X, Y indeterminates over k, then A is a Mori domain (cf. [4, Example (4.6), b)]). The associated sequence (*) is simply $A = A_0 \subset A_1 = A^* = k[X, Y]$ and $(A_0: A_1) = Xk[X, Y]$ is a radical (in fact prime) ideal of A^*. A is obtained from A^* by glueing over $\mathfrak{p} = Xk[X, Y]$. The transcendence degree 1 of the extension $k \subset k[Y]$ in the diagram

$$\begin{array}{ccc} A = \phi^{-1}(k) & \longrightarrow & k \\ \cap\downarrow & & \cap\downarrow \\ A^* = k[X, Y] & \xrightarrow[\phi]{} & k[Y] \end{array}$$

corresponds to the contraction of the affine line of generic point $Xk[X, Y] \in \operatorname{Spec} A^*$ to the point $\mathfrak{p} = Xk[X, Y] \in \operatorname{Spec} A$. Outside of $\mathfrak{p}$, in the complement open set, Spec A and Spec A^* are scheme theoretically isomorphic.

b) Let $A = k[Z] + XYk[X, Y, Z]$, where k is a field and X, Y, Z indeterminates over k. Then A is a Mori domain, because $A = C \cap B_1 \cap B_2$, where $C = k[X, Y, Z]$, $B_1 = k(Z) + Xk[X, Y, Z]_{(X)}$ and $B_2 = k(Z) + Yk[X, Y, Z]_{(Y)}$ are Mori domains (cf. [12, I, Theorem 2] and [2, Proposition 3.4]). The associated sequence (*) is simply $A = A_0 \subset A_1 = A^* = k[X, Y, Z]$ and $(A_0 : A_1) = XYk[X, Y, Z]$ is a radical (non prime) ideal of A^* (in fact $XYk[X, Y, Z] = Xk[X, Y, Z] \cap Yk[X, Y, Z]$). The domain A is obtained from A^* by glueing over $\mathfrak{p} = XYk[X, Y, Z]$.

The two affine planes of generic points $\mathfrak{P}_1 = Xk[X, Y, Z]$ and $\mathfrak{P}_2 = Yk[X, Y, Z]$ of Spec A^* are identified in Spec A in the affine line of generic point $\mathfrak{p}$. Outside of $\mathfrak{p}$, in the complement open set, Spec A and Spec A^* are scheme theoretically isomorphic.

c) Let $A = k + Xk[X] + XYk[X, Y, Z]$, where k is a field and X, Y, Z indeterminates over k. Then A is a Mori domain, because it is not difficult to show that $A = C \cap B_1 \cap B_2$, where $C = k[X, Y, Z]$, $B_1 = k(Z) + Xk[X, Y, Z]_{(X)}$ and $B_2 = k(X) + Yk[X, Y, Z]_{(Y)}$ are Mori domains (cf. [12, 1, Theorem 2] and [2, Proposition 3.4]). Since $\mathfrak{p}_1 = Xk[X, Y, Z]_{(X)} \cap A = Xk[X] + XYk[X, Y, Z] \supset \mathfrak{p}_2 = Yk[X, Y, Z]_{(Y)} \cap A = XYk[X, Y, Z]$, by [4, Theorem (4.3)], $\{\mathfrak{p}_1\} = \mathscr{S}(A)$, and the associated sequence (*) is $A = A_0 \subset A_1 = k[X] + Yk[X, Y, Z] \subset A_2 = A^* = k[X, Y, Z]$. $(A_0 : A_1) = Xk[X] + XYk[X, Y, Z]$ is a prime ideal of A_1 and $(A_1 : A_2) = Yk[X, Y, Z]$ is a prime ideal of A^*. Thus A is obtained from A^* by glueing over the strongly divisorial prime ideals of A, $\mathfrak{p}_1$ and $\mathfrak{p}_2$. The affine plane of generic point $\mathfrak{P}_1 = Xk[X, Y, Z]$ of Spec A^* is contracted in Spec A into the point $\mathfrak{p}_1$; the affine plane of generic point $\mathfrak{P}_2 = Yk[X, Y, Z]$ of Spec A^* is contracted in Spec A into the affine line of generic point $\mathfrak{p}_2$. Since $(A : A^*) = \mathfrak{p}_2$, outside of $\mathfrak{p}_2$, in the complement open set, Spec A and Spec A^* are scheme theoretically isomorphic.

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Dipartimento di Matematica
Istituto "G. Castelnuovo"
Università di Roma "La Sapienza"
Piazzale A. Moro 5, 00185 Roma
Italia